X : M = manifold, $X \in \mathfrak{X}(M)$ 1

$M \ni p \longmapsto X_p \in T_pM$ "smooth in p"

GLOBALLY: $L_X : C^\infty(M) \longrightarrow C^\infty(M)$, $f \longmapsto L_X(f)$ $\Big\}$ $L_X(f)(p) = \partial_{X_p}(f) = (df^x)_{x(p)}(X_p^x)$ if (U,x) = chart around p

if $X_p = \frac{d\gamma}{dt}(0) \longrightarrow = \frac{d}{dt} f(\gamma(t))$

LOCALLY, w.r.t. chart $x : U \xrightarrow{\sim} \Omega \subseteq \mathbb{R}^n$

$$X_p = \sum_i F_i^x(x(p)) \left(\frac{\partial}{\partial x_i}\right)_p \quad \text{(for all } p \in U\text{)}$$

$$\left(\text{OR}: x_*(F) = \sum_i F_i^x \cdot \frac{\partial}{\partial x_i}\right) \quad \text{with } F_i^x \in C^\infty(\Omega).$$

INTEGRAL CURVE of X: any $\gamma : I \longrightarrow M$ ($I \subseteq \mathbb{R}$ open interval) satisfying

$$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad (\forall)\, t \in I \qquad (1)$$

INITIAL CONDITIONS may require

$$\gamma(0) = p \quad (p \in M \text{ given}) \qquad (2)$$

4

$F^x = (F^x_{\dots}) : \Omega \to \mathbb{R}^m$

Ex1: $X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} \Rightarrow$ to sol

$p = (a,b) \in \mathbb{R}^2$ th

Locally, (1) ② $\Longleftrightarrow \dfrac{d\gamma^x}{dt}(t) = F^x(\gamma(t))$ where $F^x = (F_1^x, \dots) : \Omega \to \mathbb{R}^m$

$\Rightarrow$ can use Cauchy $\exists!$ in $\mathbb{R}^m \Rightarrow$ local $\exists$ & ! for (1) & (2)

. . . $\Rightarrow$

$\Rightarrow$ MAXIMAL INTEGRAL CURVES OF X: $\gamma_p : I_p \to M$ maximal s.t. $\begin{cases} \dfrac{d\gamma_p}{dt}(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = o \end{cases}$

COMPLETE X: if $I_p = \mathbb{R}$ $(\forall)$ $p \in M$.

FLOWS OF X: Change viewpoint/notations: $\gamma_p(t) =: \varphi_X(p,t) =: \varphi_X^t(p)$

In the complete case $\Rightarrow \begin{cases} \varphi_X : M \times \mathbb{R} \to M & \text{the FLOW OF X} \\ \varphi_X^t : M \to M & \text{the FLOW OF X at time } t \end{cases}$

diffeomorphisms

Basic properties: $\begin{cases} \varphi_X = \text{smooth},\ \varphi_X^t = \text{diffeomorphisms} \\ \varphi_X^t \circ \varphi_X^s = \varphi_X^{t+s},\ \varphi_X^0 = Id_M. \end{cases}$

$\begin{cases} \dfrac{d\gamma_p}{dt}(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = p \end{cases}$ | $\begin{cases} \dfrac{d}{dt}\varphi_X(p,t) = X_{\varphi_X(p,t)} \\ \varphi_X(p,0) = p. \end{cases}$ | $\begin{cases} \dfrac{d}{dt}\varphi_X^t(p) = X_{\varphi_X^t(p)} \\ \varphi_X^0 = Id_M \end{cases}$

Ex1: $X = \dfrac{\partial}{\partial x} + \lambda \dfrac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases} \dot{x} = 1 \\ \dot{y} = \lambda \end{cases}$ ③

$\Rightarrow$ for each $p = (a,b) \in \mathbb{R}^2$ the solution $\gamma_p(t) = (t$

$\Rightarrow X$ is complete and $\varphi_X^t : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi_X^t(a,b) = (a+t, b$

Ex2: $X = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases} \dots \Rightarrow$

$\Rightarrow$ for each $p = (a,b) \in \mathbb{R}^2$ the sol $\gamma_p(t) = (a\cos t + b\sin t, b\cos t - a\sin$

Ex3: $X = -x^2\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y} \Rightarrow \dots \Rightarrow$

$\Rightarrow$ for each $p = (a,b) \in \mathbb{R}^2$ the solution $\gamma_p(t) = \left(\dfrac{a}{at+1}, be^{-t}\right)$

Hence: X is not complete: $\gamma_p(t)$ defined only when $at+1>0$

where $F^\lambda=(F^\lambda_1,\dots):\Omega\to\mathbb{R}^m$

cal $\exists$ & ! for (1) & (2)

$\gamma_p: I_p\to M$ maximal s.t. M.

$$\begin{cases}\frac{d\gamma_p}{dt}(t)=X_{\gamma_p(t)}\\ \gamma_p(0)=0\end{cases}$$

ions: $\gamma_p(t)=:\varphi_X(p,t)$
$=:\varphi_X^t(p)$

M the FLOW OF X

the FLOW OF X at time t

φ^t_X = diffeomorphisms

, $\varphi^0_X = Id_M$

$$\begin{cases}\frac{d}{dt}\varphi^t_X(p)=X_{\varphi^t_X(p)}\\ \varphi^0_X=Id_M\end{cases}$$

<u>Ex1</u>: $X=\frac{\partial}{\partial x}+\lambda\frac{\partial}{\partial y}$ $\Rightarrow$ to solve $\begin{cases}\dot{x}=1\\ \dot{y}=\lambda\end{cases}$ 3

$\Rightarrow$ for each $p=(a,b)\in\mathbb{R}^2$ the solution $\gamma_p(t)=(t+a,\ t+b)$

$\Rightarrow$ X is complete and $\varphi^t_X:\mathbb{R}^2\to\mathbb{R}^2$, $\varphi^t_X(a,b)=(a+t,b+t)$

<u>Ex2</u>: $X=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$ $\Rightarrow$ to solve $\begin{cases}\dot{x}=y\\ \dot{y}=-x\end{cases}$ $\dots\Rightarrow$

$\Rightarrow$ for each $p=(a,b)\in\mathbb{R}^2$ the sol $\gamma_p(t)=\boxed{(a\cos t+b\sin t,\ b\cos t-a\sin t)=\varphi^t_X(a,b)}$

<u>Ex3</u>: $X=-x^2\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}$ $\Rightarrow\dots\Rightarrow$

$\Rightarrow$ for each $p=(a,b)\in\mathbb{R}^2$ the solution $\gamma_p(t)=\left(\frac{a}{at+1},\ be^{-t}\right)$

Hence: X is not complete: $\gamma_p(t)$ defined only when $at+1>0$.

4

$X \in \mathfrak{X}(M)$ arbitrary:

$$\mathcal{D}(X) := \{(p,t) \in M \times \mathbb{R} \;/\; \gamma_p(t) \text{ is defined} \Leftrightarrow t \in I_p\} \subseteq M \times \mathbb{R}$$

$$\varphi_X : \mathcal{D}(X) \to M, \quad \varphi_X(p,t) = \gamma_p(t).$$

$$\mathcal{D}^t(X) := \{p \in M \;/\; t \in I_p\}$$

$$\varphi_X^t : \mathcal{D}^t(X) \to M, \quad \varphi_X^t(p) = \gamma_p(t).$$

Prop: (1) $\mathcal{D}(X) \subseteq M \times \mathbb{R}$ open and φ_X is smooth

(2) each $\mathcal{D}^t(X) \subseteq M$ open, φ_X^t is a diffeom

$$\varphi_X^t : \mathcal{D}^t(X) \to \mathcal{D}^{-t}(X)$$

(3) $\varphi_X^t(\varphi_X^s(p)) = \varphi_X^{t+s}(p)$ $\qquad (*)$

i.e. $\forall\, p,t,s$ s.t. $\begin{cases} \varphi_X^s(p) \text{ makes sense} \\ \varphi_X^t(\varphi_X^s(p)) \text{ -- " --} \end{cases}$ $\Rightarrow$ also $\varphi_X^{t+s}(p)$ makes sense & $(*)$ holds

OR: $\forall\, (p,s) \in \mathcal{D}(X)$ s.t. $(\varphi_X^s(p), t) \in \mathcal{D}(X)$ $\Rightarrow$ $(p, t+s) \in \mathcal{D}(X)$ & $(*)$ holding

in p"   if (U,x) = chart around p

$\partial_{x_p}(f) = (df^x)_{x(p)}(X_p^x)$

$= \frac{d}{dt}\Big|_{t=0} f(\gamma(t))$

all $p \in U$)

$F_i^x \in C^\infty(\Omega)$.

$\in \mathbb{R}$ open interval) satisfying

$=1$ (1)

en)   (2)

$\Leftrightarrow \frac{d\gamma^i}{dt}(t) = F^i(\gamma(t))$ where $F^i = (F_1^i, \dots) : \Omega \to \mathbb{R}^m$

$\exists!$ in $\mathbb{R}^m \Rightarrow$ local $\exists\, \gamma\,!$ for (1) & (2)

INTEGRAL CURVES OF X: $\gamma_p : I_p \to M$ maximal s.t. $\begin{cases} \frac{d\gamma_p}{dt}(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = p \end{cases}$, $I_p \subseteq \mathbb{R}$

if $I_p = \mathbb{R}$ $\forall p \in M$.

Change viewpoint/notations: $\boxed{\gamma_p(t) =: \varphi_X(p,t) =: \varphi_X^t(p)}$

case $\Rightarrow \begin{cases} \varphi_X : M \times \mathbb{R} \to M & \text{the FLOW OF X} \\ \varphi_X^t : M \to M & \text{the FLOW OF X at time } t \end{cases}$

$\begin{cases} \varphi_X = \text{smooth},\ \varphi_X^t = \text{diffeomorphisms} \\ \varphi_X^t \circ \varphi_X^s = \varphi_X^{t+s},\ \varphi_X^0 = Id_M \end{cases}$

$\begin{cases} \frac{d}{dt}\varphi_X(p,t) = X_{\varphi_X(p,t)} \\ \varphi_X(p,0) = p \end{cases}$ $\begin{cases} \frac{d}{dt}\varphi_X^t(p) = X_{\varphi_X^t(p)} \\ \varphi_X^0 = Id_M \end{cases}$

3'

Ex1: $X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases} \dot{x} = 1 \\ \dot{y} = \lambda \end{cases}$

$\Rightarrow$ for each $p = (a,b) \in \mathbb{R}^2$ the solution $\gamma_p(t) = (t+a, \lambda t + b)$

$\Rightarrow X$ is complete and $\varphi_X^t : \mathbb{R}^2 \to \mathbb{R}^2$, $\varphi_X^t(a,b) = (a+t, b+\lambda t)$

Ex2: $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ $\dots \Rightarrow$

$\Rightarrow$ for each $p = (a,b) \in \mathbb{R}^2$ the sol $\boxed{\gamma_p(t) = (a\cos t + b\sin t,\ b\cos t - a\sin t) = \varphi_X^t(a,b)}$

Ex3: $X = -x^2\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \Rightarrow \dots \Rightarrow$

$\Rightarrow$ for each $p = (a,b) \in \mathbb{R}^2$ the solution $\gamma_p(t) = \left(\frac{a}{at+1}, be^{-t}\right)$ $\begin{cases} p = (a,b) & a > 0 \\ I_p = \left(-\frac{1}{a}, \infty\right) \end{cases}$

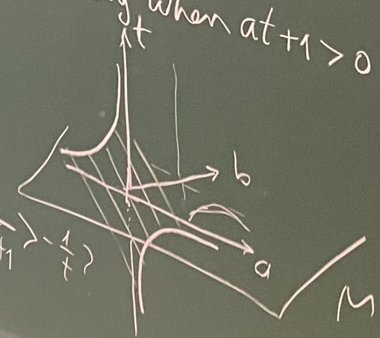
Hence: X is not complete: $\gamma_p(t)$ defined only when $at+1 > 0$.

$\mathcal{D}(X) = \{(p,t) : at+1 > 0\}$, $p = (a,b)$

$\mathcal{D}^t(X) = \{(a,b) : a > -\frac{1}{t}\}$ for $t > 0$

$\varphi_X^t(a,b) = \left(\frac{a}{at+1}, be^{-t}\right)$, $(a,b) \in \mathcal{D}^t$, $\underbrace{\left(\frac{a}{at+1}, be^{-t}\right)}_{\mathcal{D}^{-t}}$

$\frac{a}{at+1} > -\frac{1}{t}$?



5

$M \times \bigcap I_p$

Thm: $M = \text{cpct} \Rightarrow$ all $X \in \mathfrak{X}(M)$ are complete.

Pf: Let $X \in \mathfrak{X}(M)$ be arbitrary. Look at $M \times \{0\} \subseteq \mathcal{D}(X) \underset{\text{open}}{\subseteq} M \times \mathbb{R}$

Tube Lemma: $M = \text{compact} \Rightarrow \exists\, \varepsilon > 0 :$ $\boxed{M \times [-\varepsilon, \varepsilon] \subseteq \mathcal{D}(X)}$

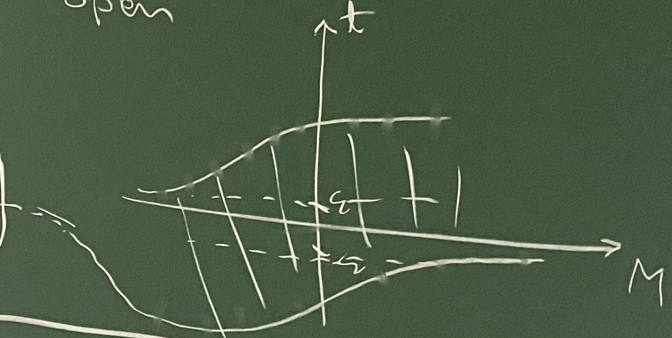
$\Rightarrow \varphi_X^t(p)$ defined $\forall\, t \in [-\varepsilon, \varepsilon], \forall\, p \in M$

$\xrightarrow[(3)]{\text{Prop}} \varphi_X^t(p)$ def $\forall\, t \in [-2\varepsilon, 2\varepsilon], \forall\, p \Rightarrow \boxed{M \times [-2\varepsilon, 2\varepsilon] \subseteq \mathcal{D}(X)}$

$\Downarrow$

...

$M \times \mathbb{R} \subseteq \mathcal{D}(X) \Rightarrow \boxed{\mathcal{D}(X) = M \times \mathbb{R}}$!



(7)

Example: When "the natural objects ξ" is a functions $f \in C^\infty(M)$:

Meaning of $\varphi^*(f)$ when $\varphi: M \to M$ a diffeom ?

Answer: $\varphi: M \longrightarrow N \xrightarrow{f} \mathbb{R}$

$\Rightarrow \varphi^*: C^\infty(N) \to C^\infty(M)$, $\varphi^*(f) = f \circ \varphi$.

Hence, the Slogan formula reads, at a point p:

$$\boxed{\mathcal{L}_X(f)(p) := \frac{d}{dt}\Big|_{t=0} f(\underbrace{\varphi_X^t(p)}_{\gamma(t)})}$$

$\mathcal{L}_X(f): M \to \mathbb{R}$

$\dot\gamma(0) =$

$= L_X(f)$

$\Rightarrow$ We recover L_X !

Example: When "objects ξ" are vector fields $Y \in \mathfrak{X}(M)$.

Meaning of $\varphi^*(Y)$ when $\varphi: M \to M$ diffeo, $Y \in \mathfrak{X}(M)$?

if (U,x) = chart around 1

$(df^x)_{x(p)}(X_p^x)$

$\frac{d}{dt}\Big|_{t=0} f(\gamma(t))$

$\in \mathfrak{X}(M)$

l object"

ξ ALONG X

nterval) sat

(1)

(2)

(7)

Example: When "the natural objects ξ" is a functions $f \in C^\infty(M)$:

Meaning of $\varphi^*(f)$ when $\varphi: M \to M$ a diffeo ?

Answer $\varphi: M \longrightarrow N \xrightarrow{f} \mathbb{R}$

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$$\boxed{\mathcal{L}_X(f)(p) := \frac{d}{dt}\Big|_{t=0} f(\underbrace{\varphi_X^t(p)}_{\gamma(t)})}$$

$\mathcal{L}_X(f): M \to \mathbb{R}$ $\quad \dot\gamma(0) =$

$= L_X(f)$

$\Rightarrow$ we recover L_X !

Example: When "objects ξ" are vector fields $Y \in \mathfrak{X}(M)$

Meaning of $\varphi^*(Y)$ when $\varphi: M \to M$ diffeo, $Y \in \mathfrak{X}(M)$?

Answer: $\varphi : M \to N$ diffeom $\Rightarrow$ 8

$\Rightarrow \varphi^* : \mathfrak{X}(N) \longrightarrow \mathfrak{X}(M)$, $\varphi^*(Y)_p = (d\varphi^{-1})_{\varphi(p)}(Y_{\varphi(p)})$

Hence, the slogan reads, at $p \in M$

$$\mathcal{L}_X(Y)_p = \frac{d}{dt}\Big|_{t=0}\Bigg(\underbrace{(d\varphi_X^{-t})_{\varphi_X^t(p)}\big(Y_{\underbrace{\varphi_X^t(p)}_{q}}\big)}_{\in T_pM}\Bigg) \qquad Y_q = \frac{d}{ds}\Big|_{s=0}\varphi_Y^s(q)$$

[**Prop**: This is precisely $[X,Y]_p$!

i.e. $\mathcal{L}_X(Y) = [X,Y]$.

Pf: lecture notes.

Rk: $[X,Y]_p = \mathcal{L}_X(Y)_p = \frac{d}{dt}\Big|_{t=0}(d\varphi_X^{-t})_{\varphi_X^t(p)}\left(\frac{d}{ds}\Big|_{s=0}\varphi_Y^s\big(\overbrace{\varphi_X^t(p)}^{q}\big)\right)$

$= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\left(\varphi_X^{-t}\left(\varphi_Y^s\left(\varphi_X^t(p)\right)\right)\right) = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\left(\varphi_X^{-t}\circ\varphi_Y^s\circ\varphi_X^t\right)(p)$

[**Corollary**: X, Y = complete, if their flows commute, i.e.

$\left.\varphi_X^t \circ \varphi_Y^s = \varphi_Y^s \circ \varphi_X^t \quad \forall t, s\right) \Leftrightarrow [X,Y] = 0$

ons $f \in C^\infty(M)$:

ffom ?

$= f \circ \varphi$

at p:

$\in \mathfrak{X}(M)$

$Y \in \mathfrak{X}(M)$?

8

Covectors, 1-forms:

→ Dual of a vector space V. $V^* = \{ \xi : V \to \mathbb{R} \ / \ \xi = \text{linear} \}$

Vectorspace: $\xi_1 + \xi_2 \in V^*$ by $(\xi_1 + \xi_2)(v) := \xi_1(v) + \xi_2(v)$

Basis? Assume $\{e_1, \dots, e_m\}$-basis of V ⇒

⇒ define $e^1, \dots, e^m \in V^*$ by $e^i(e_j) := \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Lemma: $\{e^1, \dots, e^m\}$ – basis of V^*. $\left(e^i\left(\sum v^j e_j\right) = v^i \right)$

→ Cotangent spaces: M = manifold, $p \in M$: the cotangent space of M at p.

$T_p^* M = (T_p M)^* = \{ \xi_p : T_p M \to \mathbb{R} / \xi_p = \text{linear} \}$

ξ_p called cotangent vectors $\xi_p(v_p) \in \mathbb{R}$

Ex: $f \in C^\infty(M) \Rightarrow (df)_p : T_p M \to \mathbb{R} \Rightarrow (df)_p \in T_p^* M$

$$(df)_p\left(\frac{d\gamma}{dt}(0)\right) = \frac{d\, f \circ \gamma}{dt}(0)$$

Ex: (U, X)-chart of M around p ⇒ $\left(\frac{\partial}{\partial x}\right)$ ...

9

$$(df)_p\left(\frac{d\gamma}{dt}(0)\right) = \frac{d\, f\circ\gamma}{dt}(0).$$

Ex. (U, x)-chart of M around $p \Rightarrow \left(\frac{\partial}{\partial x_1}\right)_p, \dots$ 9 $\in T_pM$ basis

Its dual basis is denoted

$$(dx_1)_p, \dots, (dx_m)_p \in T_p^*M$$

basis!

Rk: compatible with previous ex.: $x_i : U \to \mathbb{R}$, $U \subseteq M$

$\Rightarrow (dx_i)_p \in T_p^*M$ in 1st sense:

$$(d\underbrace{x_i}_{f})_p\left(\left(\frac{\partial}{\partial x_j}\right)_p\right) = (d\underbrace{x_i^x}_{pr_i})_{x(p)}\left(\underbrace{\left(\frac{\partial}{\partial x_j}\right)_p^x}_{e_j}\right) = \delta^i_j.$$

~~$f: M \to \mathbb{R}$ ⇝ $(df)_p : T_pM \to \mathbb{R}$~~

(hence

$$(dx_i)_p\left(\left(\frac{\partial}{\partial x_j}\right)_p\right) = \delta^i_j \;)$$