1

(Differential) 1-forms:

- V = vector space $\rightsquigarrow$ its dual $V^* = \{\xi : V \to \mathbb{R} \,/\, \xi = \text{linear}\}$
- $\{e_1, \dots, e_m\}$ - basis of V $\rightsquigarrow$ the dual basis $\{e^1, \dots, e^m\}$ of V^*: $e^i(e_j) = \delta^i_j$.
- $M, p \in M$ $\rightsquigarrow$ the cotangent space of M at p: $T_p^*M = (T_pM)^*$

  Its elements: cotangent vectors, or 1-forms at p.

  (they eat tangent vectors & spit out numbers).
- ex: for $f \in C^\infty(M)$, $p \in M$: $(df)_p \in T_p^*M$
- ex: (U, x) - chart $\Rightarrow$

  $(dx_1)_p, \dots, (dx_m)_p$ - basis of T_p^*M

  the dual basis to $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ of T_pM.

Hence $(dx_i)_p : T_pM \to \mathbb{R}$

$$X_p = \sum_j \text{coeff}_j(p) \left(\frac{\partial}{\partial x_j}\right)_p \longmapsto \text{coeff}_i(p).$$

der:

- for any M, $p \in M$: the tangent space T_pM

1

- any chart (U, x) around p gives rise to

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \text{ – basis of } T_pM$$

- vector field X on M:

$$M \ni p \longmapsto X_p \in T_pM$$

which is smooth: $(\forall)(U, x)$, writing $X_p = \sum \underbrace{\text{coeff}_i(p)}_{\text{all smooth}} \left(\frac{\partial}{\partial x_i}\right)_p$

$\mathcal{X}(M)$ = the collection of such vector fields.

- $\mathcal{X}(M)$ is a vector space: $(X+Y)_p := X_p + Y_p$
- $\mathcal{X}(M)$ is a $C^\infty(M)$-module: $(f \cdot X)_p := f(p) X_p$
- globally, any $X \in \mathcal{X}(M)$ induces $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$

& we have

$$\mathcal{X}(M) \xleftrightarrow{1-1} \left\{ \begin{array}{c} L : C^\infty(M) \to C^\infty(M) \\ \text{derivation} \end{array} \right\}$$

& we have $\mathfrak{X}(M) \overset{1-1}{\longleftrightarrow} \{ \ldots$ derivation $\}$

2

- $\mathfrak{X}(M)$ = Lie algebra: for $X, Y \in \mathfrak{X}(M) \Rightarrow [X,Y] \in \mathfrak{X}(M)$

$$\left(\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X\right)$$

- in $\mathbb{R}^L$: $X \in \mathfrak{X}(\mathbb{R}^L)$:

$$X = \sum_{i=1}^{L} F_i \cdot \frac{\partial}{\partial x_i} \quad \text{with } F_i \in C^{\infty}(\mathbb{R}^L)$$

(& in general, for $X \in \mathfrak{X}(M)$, what we see locally, w.r.t. a chart (U, x), are m functions (on U))

- Submanifolds $M \subseteq \mathbb{R}^L$: can use the ambient $\mathbb{R}^L$, and any $X \in \mathfrak{X}(M)$ can be written

$$X_p = \sum_{i=1}^{L} F_i(p) \left(\frac{\partial}{\partial x_i}\right)_p \quad (\text{for } p \in M)$$

but not all such expressions are automatically tangent to M!

...inder:

- for any $M, p \in M$: <u>the cotangent space</u> $T_p^* M$
- any Chart (U, x) around p gives rise to $(dx_1)_p, \ldots, (dx_m)_p$ — basis of $T_p^* M$

(2')

- 1-forms ω on M:

$$M \ni p \longmapsto \omega_p \in T_p^* M$$

which is smooth: $(\forall)(U, x)$, writing $\omega_p = \sum \underbrace{\text{coeff}_i(p)}_{\text{all smooth}} (dx_i)_p$

$\Omega^1(M)$ = the collection of such 1-forms.

- $\Omega^1(M)$ is a vector space: $(\omega + \theta)_p := \omega_p + \theta_p$
- $\Omega^1(M)$ is a $C^\infty(M)$-module: $(f \cdot \omega)_p := f(p)\, \omega_p$
- globally, any $X \in \mathfrak{X}(M)$ induces $\alpha_X : C^\infty(M) \to C^\infty(M)$

& we have

$$\mathfrak{X}(M) \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} L : C^\infty(M) \to C^\infty(M) \\ \text{derivation} \end{array} \right\}$$

- For $\omega \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$ we can talk about $\omega(X) \in C^\infty(M)$ (the function $p \longmapsto \omega_p(X_p)$)

Differential 1-forms:

- V = vector space ⇝ its dual $V^* = \{ \ldots \}$
- $\{e_1, \ldots, e_m\}$ – basis of V ⇝ the dual basis ...
- M, $p \in M$ ⇝ the cotangent space of M at p: $T_p^* M = (T_p M)^*$
- ex: $f \in C^\infty(M)$, $p \in M$: $(df)_p \in T_p^* M$
- ex: (U, x) – chart ⇒ $(dx_1)_p, \ldots, (dx_m)_p$ – basis of $T_p^* M$

the dual basis to $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p$ of $T_p M$

Hence $(dx_i)_p : T_p M \to \mathbb{R}$

$v_p = \sum \text{coeff}_j \left(\frac{\partial}{\partial x_j}\right)_p \longmapsto \text{coeff}_i$

$\mathfrak{X}(M)$ = Lie algebra: for $X, Y \in \mathfrak{X}(M) \Rightarrow [X,Y] \in \mathfrak{X}(M)$

$(\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X)$

in $\mathbb{R}^L$: $X \in \mathfrak{X}(\mathbb{R}^L)$:

$$X = \sum_{i=1}^{L} F_i \cdot \frac{\partial}{\partial x_i} \quad \text{with } F_i \in C^\infty(\mathbb{R}^L)$$

in general, for $X \in \mathfrak{X}(M)$, what we see locally, w.r.t. a chart (U, χ), are m functions (on U)

- Submanifolds $M \subseteq \mathbb{R}^L$: can use the ambient $\mathbb{R}^L$, and any $X \in \mathfrak{X}(M)$ can be written

$$X_p = \sum_{i=1}^{L} F_i(p) \left(\frac{\partial}{\partial x_i}\right)_p \quad (\text{for } p \in M)$$

but not all such expressions are automatically tangent to M!

- Operation $d: C^\infty(M) \longrightarrow \Omega^1(M)$, $f \longmapsto df$ $\left(p \longmapsto (df)_p\right)$ 3

Rk: $df(X) = \mathcal{L}_X(f)$

- in $\mathbb{R}^L$: $\omega \in \Omega^1(\mathbb{R}^L)$: $\boxed{\omega = \sum_{i=1}^{L} F_i \cdot dx_i}$ (*) with $F_i \in C^\infty(\mathbb{R}^L)$.

Ex: For $\omega = df$ with $f \in C^\infty(\mathbb{R}^L)$: $df = \sum_{i=1}^{L} \frac{\partial f}{\partial x_i} \cdot dx_i$.

- submanifolds $M \subseteq \mathbb{R}^L$: $\Rightarrow$ any expression (*) can be restricted to M.

$(\sum F_i \, dx_i)|_M \in \Omega^1(M)$ also called: $\sum F_i \, dx_i$ on M

OR $\sum F_i \, dx_i$ as a 1-form on M.

Be aware: Some different looking formulas on $\mathbb{R}^L$ may become equal as 1-forms on M!

Ex: $[\omega = \cancel{x}\,dx + y\,dy + z\,dz] \in \Omega^1(\mathbb{R}^3)$ $\quad \eta = (\cancel{x}\,dx + y\,dy + z\,dz)$

$\omega|_{S^2} = x\,dx + y\,dy + z\,dz$ as a 1-form on S^2 is 0!!

Pf: $p = (x,y,z) \in S^2$, $v_p = a\left(\frac{\partial}{\partial x}\right)_p + b\left(\frac{\partial}{\partial y}\right)_p + c\left(\frac{\partial}{\partial z}\right)_p \in T_p\mathbb{R}^3$

T_pS^2 i.e. $ax + by + cz = 0$

$\omega_p(v_p) = x \cdot a + y \cdot b + z \cdot c = 0$.

Restriction

- for $W \subseteq$ …
- Use it f…
- Apply i…

• $\{e_1, \dots, e_m\}$ – basis of V ⇝ the dual basis $\{e^1, \dots, e^m\}$ of V^*: $e^i(e_j) = \delta^i_j$.

) 3

m be

form on M.

equal

forms on M !

. $ax+by+cz=0$

Restrictions: $M \subseteq N$, $p \in M$ 4

• for $W \subseteq V$ vector subspace ⇒ restriction $V^* \longrightarrow W^*$, $\xi \longmapsto \xi|_W$

• Use it for $T_pM \subseteq T_pN$ ⇒ — " — $T_p^*N \longrightarrow T_p^*M$

• Apply it to all $p \in M$ ⇒ — " — $\Omega^1(N) \longrightarrow \Omega^1(M)$

denoted $\omega \longmapsto \omega|_M$ (with $\omega \in \Omega^1(N)$, $\omega|_M \in \Omega^1(M)$)

(Hence $\underset{p \in M}{(\omega|_M)_p}\Big(\underset{T_pM \subseteq T_pN}{X_p}\Big) := \omega_p(X_p)$.)

(2')

tangent space T_p^*M

... rise to

... basis of T_p^*M

$\longrightarrow \omega_p \in T_p^*M$

$x)$, writing $\omega_p = \sum \underbrace{\text{coeff}_i(p)}(dx^i)_p$

f such 1-forms, all smooth

ace: $(\omega+\theta)_p := \omega_p + \theta_p$

$(f\cdot\omega)_p := f(p)\,\omega_p$

induces $\alpha_X : C^\infty(M) \to C^\infty(M)$

$: C^\infty(M) \to C^\infty(M)$

derivation

$\Rightarrow [X,Y] \in \mathfrak{X}(M)$ (2)

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(5)

• For $\omega \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$ we can talk about $\omega(X) \in C^\infty(M)$

(the function $p \longmapsto \omega_p(X_p)$)

Hence, each $\omega \in \Omega^1(M)$ induces

$$\omega : \mathfrak{X}(M) \to C^\infty(M)$$

which is $C^\infty(M)$-linear:

$$\omega(X+Y) = \omega(X) + \omega(Y)$$

$$\omega(fX) = f\,\omega(X)$$

Thm (4.20): This gives: $\Omega^1(M) \xleftrightarrow{1-1} \left\{ \begin{array}{l} \xi : \mathfrak{X}(M) \to C^\infty(M) \\ \text{which are } C^\infty(M)\text{-linear} \end{array} \right\}$

Idea of the proof: Surjectivity: Start with ξ. Look for $\boxed{\omega}$ s.t.

$$\boxed{\xi(X)(p) = \omega_p(X_p) \quad (\forall)\, X \in \mathfrak{X}(M),\ p \in M}$$

Fix $p \in M$. Want to define $\omega_p \in T_p^*M$, i.e. $\omega_p : T_pM \to \mathbb{R}$.

It sends $v \in T_pM$ to what? $\boxed{\text{Choose } X \in \mathfrak{X}(M) \text{ s.t. } X_p = v}$ and set $\omega_p(v) := \xi(X)(p)$

Problem: dependence on choice of X? If $X' \in \mathfrak{X}(M)$ s.t. $X'_p = v$ is $\xi(X')(p) = \xi(X)(p)$?

Denoting $X' - X =: \boxed{Y \in \mathfrak{X}(M)}$. Know $Y_p = 0$. To prove: $\xi(Y)(p) = 0$.

Claim: $\boxed{Y = f_1 Y^1 + \dots + f_k Y^k}$ with $Y^i \in \mathfrak{X}(M)$, $f_i \in C^\infty(M)$, $f_i(p) = 0 \Rightarrow \xi(Y) = f_1\,\xi(Y^1) + \dots$

at p it is 0!

Differenti...

• V = vector s...

• $\{e_1, \dots, e_m\}$ ...

• M, $p \in M$ ...

• ex: for $f \in$ ...

• ex: (U, x)-...

the dual basi...

Hence $(d$...

2'

Reminder:

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$$\mathfrak{X}(M) \xleftrightarrow{1-1} \left\{ \begin{matrix} L : C^\infty(M) \to C^\infty(M) \\ \text{derivation} \end{matrix} \right\}$$

2

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5

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Claim: $\boxed{Y = f_1 Y^1 + \dots + f_k Y^k \text{ with } Y^i \in \mathfrak{X}(M), f_i \in C^\infty(M), f_i(p) = 0} \Rightarrow \xi(Y) = f_1 \xi(Y^1) + \dots$ at p it is 0!

3

- Operation $\boxed{d : C^\infty(M) \longrightarrow \Omega^1(M)}$, $f \longmapsto df$ $(p \longmapsto (df)_p)$

Rk: $\boxed{df(X) = \mathcal{L}_X(f)}$

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Ex: For $\omega = df$ with $f \in C^\infty(\mathbb{R}^L)$: $df = \sum_{i=1}^{L} \frac{\partial f}{\partial x_i} \cdot dx_i$

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$\omega|_{S^2} = x\,dx + y\,dy + z\,dz$ as a 1-form on S^2 is 0

Pf: $p = (x, y, z) \in S^2$, $v_p = a\left(\frac{\partial}{\partial x}\right)_p + b\left(\frac{\partial}{\partial y}\right)_p + c\left(\frac{\partial}{\partial z}\right)_p \in T_pS^2$ i.e. $ax + by + cz = 0$

$\omega_p(v_p) = x \cdot a + y \cdot b + z \cdot c = 0$

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- V = vector space $\rightsquigarrow$ its du…
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  Its elements: cotan…
  (they…
- ex: for $f \in C^\infty(M)$, $p \in M$: $(df)_…$
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  the dual basis to $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_…$

Hence $(dx_i)_p : T_pM \to \mathbb{R}$
$X_p = \sum_j \mathrm{coeff}_j(p) \left(\frac{\partial}{\partial x_j}\right)_p \longmapsto$

Restrictions: $M \subseteq N$, $p \in$ …

- for $W \subseteq V$ vector subspace $\Rightarrow$ …
- Use it for $T_pM \subseteq T_pN \Rightarrow$
- Apply it to all $p \in M \Rightarrow$

$(\forall)\, X\in\mathfrak{X}(M)$, $p\in M$

set $\omega_p(v) := \xi(X)(p)$

$(dx_1)_p, \dots, (dx_m)_p$ basis of T_pM

the dual basis to $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ of T_pM

p) 3

that

$+g\cdot df$

n be

form on M.

equal

forms on M !

$ax+by+cz=0$

Restrictions: $M\subseteq N$, $p\in M$ 6

- for $W\subseteq V$ vector subspace $\Rightarrow$ restriction $V^*\longrightarrow W^*$, $\xi\mapsto \xi|_W$
- Use it for $T_pM\subseteq T_pN \Rightarrow$ — " — $T_p^*N\longrightarrow T_p^*M$
- Apply it to all $p\in M \Rightarrow$ — " — $\Omega^1(N)\longrightarrow \Omega^1(M)$

$$\omega \longmapsto \omega|_M \quad \text{denoted}$$

$\left(\text{Hence } (\omega|_M)_p(X_p) := \omega_p(X_p).\right)$ with $p\in M$, $X_p \in T_pM\subseteq T_pN$

General pull-backs: any $F: M\longrightarrow N$ smooth induces a map

$$F^*: \Omega^1(N)\longrightarrow \Omega^1(M),\quad \omega\longmapsto F^*(\omega)$$

with $\omega\in\Omega^1(N)$, $F^*(\omega)\in\Omega^1(M)$, given by

$$\boxed{F^*(\omega)_p(v) = \omega_{F(p)}\big((dF)_p(v)\big)}$$

where $v\in T_pM$, $(dF)_p(v)\in T_{F(p)}N$, $(dF)_p: T_pM\to T_{F(p)}N$

Ex: $M\subseteq N$ and F is the inclusion $i: M\to N$: $i^*(\omega)=\omega|_M$

Ex: $\boxed{F^*\left(\sum f_i\cdot dg_i\right)=\sum (f_i\circ F)\cdot d(g_i\circ F)}$ (exercise) May be useful for homework

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- $\{e_1, \dots, e_n\}$ – basis of V $\rightsquigarrow$ the dual basis $\{e^1, \dots, e^n\}$ of V^*: $e^i(e_j) = \delta^i_j$.

k-Forms: ⑦

- V = vector space, $k \in \mathbb{Z}_{\geq 1}$. just for help

space of k-covectors: $T^k V^* := \{\xi : \underbrace{V \times \dots \times V}_{k} \to \mathbb{R} \,/\, \xi = \text{linear in each argument}\}$

space of k-forms on the vector space V: $\Lambda^k V^* := \{\xi \in T^k V^* \,/\, \xi$ is also skew-symmetric$\}$

$\xi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma)\, \xi(v_1, \dots, v_k)$ $(\forall)\, \sigma \in S_k$.

One has
$$\mathrm{Alt} : T^k V^* \to \Lambda^k V^*$$
$$\mathrm{Alt}(\xi)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \xi(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

$\omega \in \Lambda^k V^* \Leftrightarrow \mathrm{Alt}(\omega) = \omega$

$\boxed{\text{Basis for } \Lambda^k V^* \text{ ??}}$ ← Need wedge

- Wedge product: first define $T^k V^* \times T^\ell V^* \to T^{k+\ell} V^*$, $(\omega, \eta) \longmapsto \omega \cdot \eta$

where $(\omega \cdot \eta)(v_1, \dots, v_{k+\ell}) = \omega(v_1, \dots, v_k)\, \eta(v_{k+1}, \dots, v_{k+\ell})$

Def: For $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^\ell V^*$ define
$$\omega \wedge \eta := \frac{(k+\ell)!}{k!\,\ell!} \mathrm{Alt}(\omega \cdot \eta) \in \Lambda^{k+\ell} V^*$$

(Left board fragments:) $\omega \wedge \omega = 0$ ... $\omega \in \Lambda^1 V^*$) ; $, v_3)\, \eta(v_2)$; $(\eta \wedge \xi)$; η, ξ. ; $(v_j))_{1 \le i,j \le k}$; $\dots, e^n \in V^*$; $\wedge \dots \wedge e^{i_k} \in \Lambda^k V^*$; a basis of $\Lambda^k V^*$; $\mathbb{R}$.

pf: Linearly indep? Assume: $\sum_I c_I \cdot e^I = 0$ with ...

Ex: $\omega, \eta \in \Lambda^1 V^* = V^* \Rightarrow \omega \wedge \eta \in \Lambda^2 V^*$ given by (8)

$(\omega \wedge \eta)(v_1, v_2) = \omega(v_1)\eta(v_2) - \omega(v_2)\eta(v_1)$. (in particular: $\omega \wedge \omega = 0$ for $\omega \in \Lambda^1 V^*$)

Ex: $\omega \in \Lambda^2 V^*, \eta \in \Lambda^1 V^* = V^* \Rightarrow \omega \wedge \eta \in \Lambda^3 V^*$ given by

$(\omega \wedge \eta)(v_1, v_2, v_3) = \omega(v_1, v_2)\eta(v_3) + \omega(v_2, v_3)\eta(v_1) - \omega(v_1, v_3)\eta(v_2)$

In general: $\omega \in \Lambda^k V^*, \eta \in \Lambda^l V^* \Rightarrow \begin{cases} \eta \wedge \omega = (-1)^{kl} \omega \wedge \eta \\ (\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta) \end{cases}$ $(\forall\, \omega, \eta, \zeta)$.

In particular:

- $\omega \wedge \omega = 0$ whenever $\omega \in \Lambda^k V^*$ with k = odd
- For $\xi_1, \dots, \xi_k \in V^* (= \Lambda^1 V^*) \Rightarrow$ get $\boxed{\xi_1 \wedge \dots \wedge \xi_k \in \Lambda^k V^*}$

And: $\boxed{\xi_{\sigma(1)} \wedge \dots \wedge \xi_{\sigma(k)} = \mathrm{sgn}(\sigma)\, \xi_1 \wedge \dots \wedge \xi_k}$ $\left((\xi_1 \wedge \dots \wedge \xi_k)(v_1, \dots, v_k) = \det\left((\xi_i(v_j))_{1 \le i,j \le k} \right) \right)$

Lemma: If $\{e_1, \dots, e_m\}$ - basis of V, consider the dual basis $e^1, \dots, e^m \in V^*$ and for $\boxed{I = (i_1, \dots, i_k) \text{ with } 1 \le i_1 < i_2 < \dots < i_k \le m}$ define $e^I := e^{i_1} \wedge \dots \wedge e^{i_k} \in \Lambda^k V^*$.
$I \in \mathrm{Ord}_k(m)$. Then $\{e^I : I \in \mathrm{Ord}_k(m)\}$ - is a basis of $\Lambda^k V^*$.

k-Forms:

- V = vect...

space of ...

space of ... on the ...

One has ...

Basis for ...

- Wedge prod...

Def: For ω ... define

$\omega \wedge$ ...

Corollary: $\dim \Lambda^k V^* = \begin{cases} \binom{m}{k} = \frac{m!}{k! \cdot (m-k)!} & 1 \le k \le m \\ 0 & k > m \end{cases}$ (9)

In particular $\Lambda^k V^* = 0$ $(\forall)$ $k > m$.

Pf: Linearly indep? Assume: $\sum_I c_I \cdot e^I = 0$ with $c_I \in \mathbb{R}$.

For each $J = (j_1, \dots, j_k) \in Ord_k(m)$ apply this to $(e_{j_1}, \dots, e_{j_k})$

Notice: $e^I(e_{j_1}, \dots, e_{j_k}) = \delta^I_J \Rightarrow c_J = 0 \Rightarrow \blacksquare$

Next: apply this to $V = T_pM \Rightarrow \Lambda^k T_p^* M$

with bases $(dx_{i_1})_p \wedge \dots \wedge (dx_{i_k})_p = (dx_I)_p$ for $I \in Ord_k(m)$

Ex: $\omega, \eta \in \Lambda^1 V^* = V^* \Rightarrow \omega \wedge \eta \in \Lambda^2 V^*$ given by (8)