1

k-forms: V = vector space, $k \geq 1$ integer (for $k=1$: V^*, $k=0$: $\mathbb{R}$)

$$\Lambda^k V^* := \{ \xi : \underbrace{V \times \dots \times V}_{k} \to \mathbb{R} \,/\, \xi = \text{multi-linear \& skew symmetric} \}$$

$$\cap|$$

$$T^k V^* := \{ \text{———— '' ————} \}$$

Alt

Convention: $\Lambda^0 V^* = \mathbb{R}$

- skew-symmetrization of $\xi \in T^k V^*$: $\overbrace{Alt(\xi)}^{\in \Lambda^k V}(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma)\, \xi(v_{\sigma(1)}, \dots, v_{\sigma(k)})$

- wedge product of $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^*$: $\boxed{\omega \wedge \eta := \frac{(k+l)!}{k!\, l!} Alt(\omega \otimes \eta) \in \Lambda^{k+l} V^*}$

(where $\omega \otimes \eta \in T^{k+l} V^*$ is $(\omega \otimes \eta)(v_1, \dots, v_{k+l}) := \omega(v_1, \dots, v_k)\, \eta(v_{k+1}, \dots, v_{k+l})$)

- Explicitly:

$$(\omega \wedge \eta)(v_1, \dots, v_{k+l}) = \sum_{\sigma \in S_{k,l}} sgn(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

(k,l)-shuffles: $\sigma \in S_{k+l}$ s.t.
$\sigma(1) < \dots < \sigma(k)$, $\sigma(k+1) < \dots < \sigma(k+l)$.

Basic properties: $\eta \wedge \omega = (-1)^{kl}\, \omega \wedge \eta$, $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$. 2

nsequence: any $\xi^1, \dots, \xi^k \in V^*$ induce

Basic properties: $\eta\wedge\omega=(-1)^{kl}\eta\wedge\omega$, $(\omega\wedge\eta)\wedge\zeta=\omega\wedge(\eta\wedge\zeta)$. 2

Consequence: any $\xi^1,\dots,\xi^k\in V^*$ induce $\xi^1\wedge\dots\wedge\xi^k\in\Lambda^k V^*$ $\big((v_1,\dots,v_k)\longmapsto \det\big((\xi^i(v_j))_{i,j}\big)\big)$.
and the following "rules" apply:

$$\boxed{\begin{array}{l}\xi_{\sigma(1)}\wedge\dots\wedge\xi_{\sigma(k)}=\mathrm{Sgn}(\sigma)\,\xi_1\wedge\dots\wedge\xi_k\\ \xi_1\wedge\dots\wedge\xi_k=0 \text{ if } \exists\, i\neq j \text{ with } \xi_i=\xi_j\end{array}}$$

- **Bases**: Let $m=\dim(V)$, $\{e_1,\dots,e_m\}$ - a basis of V $\Rightarrow$ dual basis $e^1,\dots,e^m$ of V^*.

Consider sets $I\in \mathrm{Ord}_k(m)$, i.e. $I=\{i_1,\dots,i_k\}$ with $1\le i_1<\dots<i_k\le m$

Any $I \rightsquigarrow e^I := e^{i_1}\wedge\dots\wedge e^{i_k}\in\Lambda^k V^*$

Prop: These form a basis of $\Lambda^k V^*$. In particular, $\dim(\Lambda^k V^*)=\begin{cases}\frac{m!}{k!(m-k)!} & 1\le k\le m\\ 0 & \text{otherw}\end{cases}$

$\sigma(1)<\dots<\sigma(k)$, $\sigma(k+1)<\dots<\sigma(k+l)$.

2

Basic properties: $\eta\wedge\omega=(-1)^{kl}\omega\wedge\eta$, $(\omega\wedge\eta)\wedge\zeta=\omega\wedge(\eta\wedge\zeta)$ $\omega\wedge\eta\wedge\zeta$

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$e^i(e_j)=\delta^i_j$

- Bases: Let $\boxed{m=\dim(V)}$. $\{e_1,\dots,e_m\}$ – a basis of $V\Rightarrow$ dual basis $e^1,\dots,e^m$ of V^*.

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Apply this to $V=T_pM\Rightarrow$ spaces $\Lambda^k T_p^*M$

& $(\forall)(U,x)$ chart of $M\Rightarrow$ basis $(dx^I)_p:=(dx^{i_1})_p\wedge\dots\wedge(dx^{i_k})_p\in\Lambda^k T_p^*M$.

... $\to \mathbb{R}$) ... linear & skew symmetric $\}$ (1)

...symmetrization of $\xi \in T^kV^*$: $\underbrace{\mathrm{Alt}(\xi)}_{\in \Lambda^k V}(v_1,\dots,v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \xi(v_{\sigma(1)},\dots,v_{\sigma(k)})$

Convention: $\Lambda^0 V^* = \mathbb{R}$

- **Wedge product** of $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^\ell V^*$: $\boxed{\omega \wedge \eta := \frac{(k+\ell)!}{k!\,\ell!} \mathrm{Alt}(\omega \otimes \eta) \in \Lambda^{k+\ell} V^*}$

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Basic properties: $\eta \wedge \omega = (-1)^{k\ell}\, \omega \wedge \eta$, $(\omega\wedge\eta)\wedge\zeta = \omega\wedge(\eta\wedge\zeta) = \omega\wedge\eta\wedge\zeta$ (2)

Consequence: any $\xi^1,\dots,\xi^k \in V^*$ induce $\xi^1 \wedge \dots \wedge \xi^k \in \Lambda^k V^*$ $\left((v_1,\dots,v_k) \mapsto \det(\xi^i(v_j))_{i,j}\right)$
and the following "rules" apply:

$$\boxed{\begin{gathered} \xi_{\sigma(1)} \wedge \dots \wedge \xi_{\sigma(k)} = \mathrm{sgn}(\sigma)\, \xi_1 \wedge \dots \wedge \xi_k \\ \xi_1 \wedge \dots \wedge \xi_k = 0 \text{ if } \exists\, i \neq j \text{ with } \xi_i = \xi_j \end{gathered}}$$

- **Bases**: Let $\boxed{m = \dim(V)}$, $\{e_1,\dots,e_m\}$ – a basis of $V \Rightarrow$ dual basis $e^1,\dots,e^m$ of V^* $\quad e^i(e_j) = \delta^i_j$

Consider sets $I \in \mathrm{Ord}_k(m)$, i.e. $I = \{i_1,\dots,i_k\}$ with $1 \le i_1 < \dots < i_k \le m$

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Apply this to $V = T_pM \Rightarrow$ spaces $\Lambda^k T_p^*M$

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③

Reminder:

- for any M, $p \in M$: the tangent space $T_pM \rightsquigarrow$ the cotangent space T_p^*M
- any (U,x) around p gives: $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ - basis of $T_pM \rightsquigarrow$ basis $(dx_1)_p, \dots, (dx_m)_p$ of T_p^*M

• vector field on M: $M \ni p \longmapsto X_p \in T_pM$	1-form on M: $M \ni p \longmapsto \omega_p \in T_p^*M$
& smoothness: $(\forall)(U,x)$ $X_p = \sum \underbrace{\text{coeff}_i(p)}_{\text{smooth}} \left(\frac{\partial}{\partial x_i}\right)_p$	s.t. $(\forall)(U,x)$, writing $\omega_p = \sum_i \underbrace{\text{coeff}_i(p)}_{\text{smooth}} (dx_i)_p$
$\mathfrak{X}(M) =$ the collection of vector fields on M	$\Omega^1(M) =$ the collection of 1-forms on M
• $\mathfrak{X}(M) =$ a vector space: $(X+Y)_p := X_p + Y_p$	$\Omega^1(M) =$ a vector space $(\omega + \eta)_p := \omega_p + \eta_p$
• $\mathfrak{X}(M) = C^\infty(M)$-module: $(f \cdot X)_p := f(p) X_p$	$\Omega^1(M) = C^\infty(M)$-module: $(f \cdot \omega)_p := f(p)\omega_p$

• Globally: any $\left.\begin{array}{l}\omega \in \Omega^1(M) \\ X \in \mathfrak{X}(M)\end{array}\right\}$ pair to give ④ $\omega(X) \in C^\infty(M)$, $\omega(X) \dots$

$\Omega^1(M) \dots$

- $\mathfrak{X}(M) = C^\infty(M)$-module: $(f\cdot X)_p := f(p) X_p$

- Globally: any $\left.\begin{matrix}\omega \in \Omega^1(M) \\ X \in \mathfrak{X}(M)\end{matrix}\right\}$ pair to give $\boxed{\omega(X) \in C^\infty(M),\ \omega(X)(p) := \omega_p(X_p)}$ 4

$\Rightarrow\ \Omega^1(M) \xleftrightarrow{1-1} \left\{ \begin{matrix} \omega : \mathfrak{X}(M) \to C^\infty(M) \\ \text{s.t. } \omega = C^\infty(M)\text{-linear} \end{matrix} \right\}$

- $\underline{d}: C^\infty(M) \to \Omega^1(M)$, $\ df(X) = \mathcal{L}_X(f)$.

- in $\mathbb{R}^L$; $\omega \in \Omega^1(\mathbb{R}^L)$: $\quad \omega = \sum_{i=1}^{m} F^i \cdot dx_i$

These can be restricted to any embedded $M \subseteq \mathbb{R}^L$

- general pull-backs: any $F: M \to N$ induces

$$\boxed{F^*: \Omega^1(N) \to \Omega^1(M),\ (F^*\omega)(X_p) = \omega_{F(p)}\Big((dF)_p(X_p)\Big)}$$

Reminder:

3

- for any M, $p\in M$: the tangent space T_pM ⟿ the k-cotangent space $\Lambda^k T_p^*M$ ($V \longmapsto V^*$)
- any (U,x) around p gives: $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ - basis of T_pM ⟿ basis $\{(dx^I)_p : I\in \mathrm{Ord}_k(m)\}$ - basis of $\Lambda^k T_p^*M$
- Vector field on M: $M\ni p \longmapsto X_p\in T_pM$

& smoothness: $(\forall)(U,x)$ $X_p = \sum \underbrace{\mathrm{coeff}_i(p)}_{\text{smooth}} \left(\frac{\partial}{\partial x_i}\right)_p$

$\mathfrak{X}(M) \overset{\text{the}}{=}$ collection of vector fields on M

- $\mathfrak{X}(M)$ = a vector space: $(X+Y)_p := X_p + Y_p$
- $\mathfrak{X}(M) = C^\infty(M)$-module: $(f\cdot X)_p := f(p)X_p$

k-form on M: $M\ni p \longmapsto \omega_p \in \Lambda^k T_p^*M$ ⟿ $\omega_p(X_p^1,\dots,X_p^k)\in\mathbb{R}$

s.t. $(\forall)(U,x)$, writing $\omega_p = \sum_I \underbrace{\mathrm{coeff}_I(p)}_{\text{smooth}} (dx^I)_p$

$\Omega^k(M)$ = the collection of k-forms on M

$\Omega^k(M)$ = a vector space: $(\omega+\eta)_p = \omega_p+\eta_p$

$\Omega^k(M) = C^\infty(M)$-module: $(\underset{\Omega^0(M)}{f}\cdot \underset{\Omega^k(M)}{\omega})_p := f(p)\omega_p$

Ex: $k=1$: $\Omega^1(M)$

$k=0$: $\Omega^0(M)=C^\infty(M)$

More generally: wedge products:

$$\Omega^k(M)\times\Omega^l(M)\longrightarrow\Omega^{k+l}(M) \qquad \text{where } (\omega\wedge\eta)_p = \omega_p\wedge\eta_p$$
$$(\omega,\eta)\longmapsto\omega\wedge\eta$$

4

- Globally: any $\left.\begin{array}{l}\omega\in\Omega^k(M)\\ X^1,\dots,X^k\in\mathfrak{X}(M)\end{array}\right\}$ "pair" to give $\omega(X^1,\dots,X^k)\in C^\infty(M)$, $p\longmapsto\omega_p(X_p^1,\dots,X_p^k)$

$\omega: \mathfrak{X}(M)\times\dots\times\mathfrak{X}(M)\longrightarrow C^\infty(M)$

$C^\infty(M)$-linear in each argument & skew

$(\omega, \eta) \longmapsto \omega \wedge \eta$ where $(\omega \wedge \eta)|_p = \omega_p \wedge \eta_p$

- Globally: any $\omega \in \Omega^k(M)$, $X^1, \dots, X^k \in \mathfrak{X}(M)$ "pair" to give $\omega(X^1, \dots, X^k) \in C^\infty(M)$, $p \longmapsto \omega_p(X^1_p, \dots, X^k_p)$

$$\Longrightarrow \quad \Omega^k(M) \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \omega : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \longrightarrow C^\infty(M) \\ \text{s.t. } \omega = C^\infty(M)\text{-linear in each argument \& skew} \end{array} \right\}$$

- $\boxed{\underline{d} : \Omega^0(M) \longrightarrow \Omega^1(M),}$ $df(X) = \mathcal{L}_X(f) \dots$ to be seen

- in $\mathbb{R}^L$; $\omega \in \Omega^k(\mathbb{R}^L)$: $\omega = \sum_{I \in \mathrm{Ord}_k(m)} F_I \cdot dx^I$, $F_I \in C^\infty(\mathbb{R}^L)$

These can be restricted to any embedded $M \subseteq \mathbb{R}^L$

- general pull-backs: any $F : M \longrightarrow N$ induces $F^* : \Omega^k(N) \to \Omega^k(M)$, $(F^*\omega)(X_p) = \omega_{F(p)}\left((dF)_p(X_p)\right)$

$$(F^*\omega)\left(X^1_p, \dots, X^k_p\right) = \omega_{F(p)}\left((dF)_p(X^1_p), \dots, (dF)_p(X^k_p)\right)$$

For any vector field $V \in \mathfrak{X}(M)$:

De Rham differential $\boxed{d}$

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

$\to k=0$: $d : \Omega^0(M) \to \Omega^1(M)$, $f \mapsto df$

Hence $df(X) = \mathcal{L}_X(f)$

$\to k=1$: $d : \Omega^1(M) \to \Omega^2(M)$, $\omega \mapsto d\omega$

$d\omega(X,Y) := \mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X,Y])$

Check: – this is $C^\infty(M)$-linear in X, Y $\Rightarrow$ indeed $d\omega \in \Omega^2(M)$

– If ω is exact, i.e. if $\boxed{\omega = df}$ for some f $\Rightarrow$ $d\omega$ is closed, i.e. $\boxed{d\omega = 0}$

In general, it follows that for $\omega \in \Omega^k(M)$

$(d\omega)(X^0, X^1, \ldots, X^k) = \sum (-1)^i \mathcal{L}_{X_i}(\omega(X^0, \ldots, \widehat{X^i}, \ldots, X^k))$
$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X^0, \ldots, \widehat{X^i}, \ldots, \widehat{X^j}, \ldots, X^k)$

Good to know: $\boxed{d \circ d = 0}$.

Rk: The fact that $d(d(\eta)) = 0$ $\forall \eta$ means also

$\left\{\begin{array}{l} \text{if } \omega \in \Omega^k(M) \\ \text{is exact, i.e.} \\ \omega = d\eta \\ \text{for some } \eta \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \omega \text{ is closed} \\ \text{i.e.} \\ d\omega = 0 \end{array}\right.$

Conceptually:

(DR1) d is a degree 1 deriv, i.e. it increases the form degree by 1, is linear & Leibniz $\boxed{d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta}$

(DR2) $d \circ d = 0$

(DR0) d is linear and, on 0-forms: the usual df

Thm: On any M $\exists !$ d (in all degrees) satisfying (DR0), (DR1), (DR2)

The Lie derivatives $\boxed{\mathcal{L}_V}$

$\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega^k(M)$

$\mathcal{L}_V \circ d = d \circ \mathcal{L}_V$

Cartan's magic formula: $\mathcal{L}_V = d \circ i_V + i_V \circ d$

Explicitly: $\mathcal{L}_V(\omega) := \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega)$

OR $\mathcal{L}_V(\omega)(X_p^1, \ldots, X_p^k) = \frac{d}{dt}\Big|_{t=0} \omega_{\varphi_V^t(p)}((d\varphi_V^t)_p(X_p^1), \ldots)$ $(*)$

… can be further computed and we get

$\mathcal{L}_V(\omega)(X^1, \ldots, X^k) = \mathcal{L}_V(\omega(X^1, \ldots, X^k)) -$
$- \sum \mathcal{L}_{X^i}(\omega(X^1, \ldots, [V, X^i], \ldots, X^k))$

Conceptually:

- $\mathcal{L}_V$ is a $\boxed{\text{degree } 0 \text{ derivation}}$, i.e., it preserves the form degree k, linear & satisfies:
  $\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge \mathcal{L}_V(\eta)$ $\forall \omega, \eta$.
- this is the only one which is the usual one on 0-forms (= functions) and commutes with d

The interior products $\boxed{i_V}$

$i_V : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$

Explicitly: for $\omega \in \Omega^k(M)$, $i_V\omega \in \Omega^{k-1}(M)$ given by:

$\boxed{(i_V\omega)(X^1, \ldots, X^{k-1}) = \omega(V, X^1, \ldots, X^{k-1})}$

$(*)$ Remember $\frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(X) = [V, X]$

i.e. $\frac{d}{dt}\Big|_{t=0} (d\varphi_V^{-t})_{\varphi_V^t(p)}(X_{\varphi_V^t(p)}) = [V, X]_p$

$\Rightarrow$ expression $(d\varphi_V^{-t})_{\varphi_V^t(p)}(X_{\varphi_V^t(p)}) = X_p + t[V,X]_p + O(t^2)$

$\Rightarrow (d\varphi_V^t)_q(X_q) = X_{\varphi_V^t(p)} - t[V,X]_{\varphi_V^t(p)} + O(t^2)$

$\Rightarrow (*)$ is $\frac{d}{dt}\Big|_{t=0} \omega_{\varphi_V^t(p)}(X^1_{\varphi_V^t(p)} - t[V, X^1]_{\varphi_V^t(p)}, \ldots)$.

Conceptually:

- i_V is a degree -1 deriv, i.e., it lowers the form degree by k, linear & satisfies
  $i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)$
  $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$
- is the unique one which, on 1-forms: $i_V(\omega) = \omega(V)$

Prop: characterises i_V uniquely

Ex: In $M = \mathbb{R}^3$, $\omega \in \Omega^2(M)$

$\omega = (x^2 + yz)\, dx \wedge dy$

$d(\omega) = d(x^2 + yz) \wedge dx \wedge dy + (x^2 + yz)\, d(dx \wedge dy)$ (Leibniz)

$d(x^2 + yz) = dx^2 + d(yz) = 2x\,dx + y\,dz + z\,dy$

$d(dx \wedge dy) = \underbrace{d(dx)}_{0} \wedge dy - dx \wedge \underbrace{d(dy)}_{0} = 0$

$d\omega = (2x\,dx + y\,dz + z\,dy) \wedge dx \wedge dy =$
$= y\, dz \wedge dx \wedge dy = y\, dx \wedge dy \wedge dz$.

$\mathbb{R}^m$, $\omega = \sum_I f_I\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$

$d\omega = \sum_{i, I} \frac{\partial f_I}{\partial x^i}\, dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$

Ex: Compute $\mathcal{L}_V(\omega)$ for various V's
e.g. $V = x\frac{\partial}{\partial \ldots}$

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③ … spaces $\Lambda^k T_p^* M$
$\omega \wedge \eta$
$\Omega^k(M) \to \Omega^{k+l}(M)$
wedge product
$F^*(\omega)_p(X_p^1, \ldots, X_p^k) = \omega_{F(p)}((dF)_p(X_p^1), \ldots, (dF)_p(X_p^k))$
… to be seen
$\boxed{\Omega^k(M) = 0 \quad (\forall) k > \dim(M)}$

De Rham diff

$\boxed{d}$

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

$\mathcal{L}_V \circ d = d \circ \mathcal{L}_V$

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Hence $df(X) = \mathcal{L}_X(f)$

$\to k=1$: $d : \Omega^1(M) \longrightarrow \Omega^2(M)$, $\omega \mapsto d\omega$

$d\omega(X,Y) := \mathcal{L}_X(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X,Y])$

Check: - this is $C^\infty(M)$- linear in X, Y $\Rightarrow$ indeed $d\omega \in \Omega^2(M)$.

- If ω is exact, i.e. if $\boxed{\omega = df}$ $\boxed{\text{for some } f}$ $\Rightarrow$ ω is closed i.e. $\boxed{d\omega = 0}$

In general, it follows that: for $\omega \in \Omega^k(M)$

$$(d\omega)(X^0, X^1, \dots, X^k) = \sum_i (-1)^i \mathcal{L}_{X^i}(\omega(X^0, \dots, \hat{X^i}, \dots, X^k)) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X^0, \dots, \hat{X^i}, \dots, \hat{X^j}, \dots, X^k)$$

Explicitly:

OR: $\mathcal{L}_V(\omega)(X^1_p, \dots$

... can be f...

get $\boxed{\mathcal{L}_V(\omega)(X}$

Good to know: $\boxed{d \circ d = 0.}$

Rk: The fact that $d(d(\eta)) = 0$ $\forall \eta$ means also:

$\left\{\begin{array}{l}\text{if } \omega \in \Omega^k(M) \\ \text{is exact, i.e.} \\ \omega = d\eta \\ \text{for some } \eta\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\omega \text{ is closed} \\ \text{i.e.} \\ d\omega = 0\end{array}\right\}$

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(DR1) d is a degree 1 deriv, i.e.
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(DR2) $d \circ d = 0$

(DR0) d is linear and, on 0-forms: the usual df

Thm: On any M, $\exists!$ d (in all degrees) satisfying (DR0), (DR1), (DR2)

Conceptually:

- $\mathcal{L}_V$ is a de...
preserves th...
$\mathcal{L}$...
- this is th...
is the us...
and com...

...aces $\Lambda^k T^*_p M$

: $I \in \mathrm{Ord}_k(M)$ of

$T^*_p M$

$(dx^I)_p$

...s on M

$)_p := \omega_p + \eta_p$

$)_p := f(p)\omega_p$

...rally: wedge product

$\Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$

$(\omega, \eta) \longmapsto \omega \wedge \eta$

$X^1, \dots, X^k \in C^\infty(M)$, $p \mapsto \omega_p(X^1_p, \dots, X^k_p)$

$\times \mathfrak{X}(M) \to C^\infty(M)$

(M)-linear in each argument & skew

$X) = \mathcal{L}_X(f)$... to be seen

$\sum F_I \cdot dx^I$, $F_I \in C^\infty(\mathbb{R}^L)$, $I \in \mathrm{Ord}_k(M)$

...y embedded $M \subseteq \mathbb{R}^L$

$\boxed{\begin{array}{l}\Omega^k(M) = 0 \\ \forall k > \dim(M)\end{array}}$

$: M \to N$ induces

$F^* : \Omega^k(N) \to \Omega^k(M)$, $(F^*\omega)(X_p) = \omega_{F(p)}((dF)_p(X_p))$

$(F^*\omega)(X^1_p, \dots, X^k_p) = \omega_{F(p)}((dF)_p(X^1_p), \dots, (dF)_p(X^k_p))$

d...

dw...

For any vector fie…

…fferential

$\Omega^{k+1}(M)$

$, f \mapsto df$

$= \mathcal{L}_X(f)$

$(M),\ \omega \mapsto d\omega$

$(\omega(Y)) - \mathcal{L}_Y(\omega(X)) - \omega([X,Y])$

…ar in X, Y ⇒ indeed $d\omega \in \Omega^2(M)$.

if $\boxed{\omega = df}$ for some f ⇒ ω is closed i.e. $\boxed{d\omega = 0}$

…t: for $\omega \in \Omega^k(M)$

$= \sum_i (-1)^i \mathcal{L}_{X^i}(\omega(X^0,\dots,\widehat{X^i},\dots,X^k))$

$(-1)^{i+j}\omega([X_i,X_j],X^0,\dots,\widehat{X^i},\dots,\widehat{X^j},\dots,X^k)$

The Lie derivatives $\boxed{\mathcal{L}_V}$

$$\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega^k(M)$$

$$\mathcal{L}_V \circ d = d \circ \mathcal{L}_V$$

Cartan's magic formula: $\mathcal{L}_V = d \circ i_V$ …

Explicitly: $\mathcal{L}_V(\omega) := \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega)$

OR: $\mathcal{L}_V(\omega)(X_p^1,\dots,X_p^k) = \frac{d}{dt}\Big|_{t=0} \omega_{\varphi_V^t(p)}\big((d\varphi_V^t)_p(X_p^1),\dots\big)$ (*)

… can be further computed and we get

$$\boxed{\mathcal{L}_V(\omega)(X^1,\dots,X^k) = \mathcal{L}_V(\omega(X^1,\dots,X^k)) - \sum \mathcal{L}_{X^i}(\omega(X^1,\dots,[V,X^i],\dots,X^k))}$$

Explic…

i.e.: d…

⇒ expr…

⇒ $(d\varphi_V$…

⇒ (*) is…

know: $\boxed{d \circ d = 0.}$

…fact that $d(d(\eta)) = 0$ ∀η

…se:

…(M) i.e. ⇒ ω is closed i.e. $d\omega = 0$.

…ally:

…a degree 1 deriv, i.e.

…ases the form degree by 1

& Leibniz: $\boxed{d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta}$

…= 0

linear and, on 0-forms: the usual df

…m: On any M, ∃! d (in all degrees) satisfying (DR0), (DR1), (DR2)

Conceptually:

- $\mathcal{L}_V$ is a degree 0 derivation, i.e., it preserves the form degree k, linear & satisfies:

$$\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge \mathcal{L}_V(\eta) \quad \forall \omega, \eta.$$

- this is the only one which is the usual one on 0-forms (= functions) and commutes with d

Ex: In $M = \mathbb{R}^3$, $\omega = (x^2 + yz)\,dx \wedge dy$, $\omega \in \Omega^2(M)$

$d(\omega) \underset{\text{Leibniz}}{=} d(x^2+yz) \wedge dx \wedge dy + (x^2+yz)\, d(dx \wedge d$…

$d x^2 + d(yz) = 2x\,dx + y\,dz + z\,dy$

$d\omega = (2x\,dx + y\,dz + z\,dy$…

$= y\,dz$…

Concept…

- i_V … the…
- is t… whi…

→ Prop…

r field $V \in \mathfrak{X}(M)$:

The interior products $\boxed{i_V}$

$$i_V : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$$

mula: $\mathcal{L}_V = d \circ i_V + i_V \circ d$

(ω)

$(\varphi_V^t)_p (X_p^1)$

d we

$, X^k)) -$

$, [V, X^1], X^k)$

Explicitly: for $\omega \in \Omega^k(M)$, $i_V \omega \in \Omega^{k-1}(M)$ given by:

$$\boxed{(i_V \omega)(X^1, \dots, X^{k-1}) = \omega(V, X^1, \dots, X^{k-1})}$$

(*) Remember $\frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^* (X) = [V, X]$

i.e. $\frac{d}{dt}\Big|_{t=0} (d\varphi_V^{-t})_{\varphi_V^t(p)} (X_{\varphi_V^t(p)}) = [V, X]_p$

$\Rightarrow$ expression $(d\varphi_V^{-t})_{\varphi_V^t(p)} (X_{\varphi_V^t(p)}) = X_p + t[V, X]_p + O(t^2)$

$\Rightarrow (d\varphi_V^t)_q (X_q) = X_{\varphi_V^t(p)} - t\, [V, X]_{\varphi_V^t(p)} + O(t^2)$

$\Rightarrow$ (*) is $\frac{d}{dt}\Big|_{t=0} \omega_{\varphi_V^t(p)} \left(X^1_{\varphi_V^t(p)} - t [V, X^1]_{\varphi_V^t(p)}, \dots \right)$.

fies:

$, \eta$.

ous)

Conceptually:

- i_V is a degree -1 deriv, i.e., it lowers the form degree by 1, linear & satisfies
  $$i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)$$
  $\omega \in \Omega^k(M), \eta \in \Omega^\ell(M)$
- is the unique one which, on 1-forms: $i_V(\omega) = \omega(V)$

$\to$ Prop: characterises i_V uniquely

$^2(M)$

$) d(dx \wedge dy)$

$\underbrace{\underbrace{d(dx)}_{0} \wedge dy - dx \wedge \underbrace{d(dy)}_{0}}_{0}$

$\mathbb{R}^m$, $\omega = \sum_I f_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$d\omega = \sum_{i, I} \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Ex: Compute $\mathcal{L}_V(\omega)$ for various V's

e.g. $V = x \frac{\partial}{\partial x}$, $V = \frac{\partial}{\partial y} - z \frac{\partial}{\partial x}$

Try also to ...