Reminder:

- k-forms ω on M: $\underset{p}{\overset{M}{\cup}} \longmapsto \underset{\omega_p}{\overset{\Lambda^k T_p^* M}{\cup}}$ (1)

  i.e. $\omega_p : \underbrace{T_pM \times \dots \times T_pM}_{k} \to \mathbb{R}$

- globally:

$$\Omega^k(M) \overset{1-1}{\longleftrightarrow} \left\{ \omega : \overbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}^{k} \to C^\infty(M) \;\middle|\; C^\infty(M)\text{-multilinear, skew} \right\} \dots$$

- locally: $\omega \in \Omega^k(M)$ written, w.r.t. charts (U, x):

$$\omega_p = \sum_{1 \le i_1 < \dots < i_k \le m} \operatorname{coeff}^{i_1, \dots, i_k}(p)\, (dx_{i_1})_p \wedge \dots \wedge (dx_{i_k})_p$$

(OR: $\boxed{\omega = \sum_{I \in \mathrm{Ord}_k(m)} \operatorname{coeff}^I (dx_{i_1}) \wedge \dots \wedge (dx_{i_k})}$

- Cheap 1-forms (exact 1-forms): $df \in \Omega^1(M)$ for $f \in C^\infty(M)$ $(df(X) = L_X(f))$

- Wedge products

Recall
$\Omega^0(M) = C^\infty(M)$
$f \wedge \omega = f\omega$

$$\Omega^k(M) \times \Omega^\ell(M) \longrightarrow \Omega^{k+\ell}(M)$$

$$(\omega \wedge \eta)(X^1, \dots, X^{k+\ell}) = \sum_{\sigma \in S_{k,\ell}} \operatorname{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \dots)\, \eta(X_{\sigma(k+1)}, \dots)$$

Prop

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Also

Explic

$\Downarrow$

(2) even more cheap (k-)forms: $f\, dg_1 \wedge \dots \wedge dg_k \in \Omega^k(M)$ (and sums of such) $(f, g_1, \dots, g_k \in C^\infty(M))$

- In $M = \mathbb{R}^L$: any k-form looks like

$$\boxed{\omega = \sum_I F^I\, dx_{i_1} \wedge \dots \wedge dx_{i_k}}$$

(Ex: $\boxed{df = \sum_i \frac{\partial f}{\partial x_i} dx_i}$)

$d(fg) = f\,dg + g\,df$

$\eta \wedge \omega = (-1)^{k\ell} \omega \wedge \eta$

- For embedded $M \subseteq \mathbb{R}^L$

& more generally, for $M \subseteq N \Rightarrow$ Restrictions $\begin{cases} \Omega^k(N) \longrightarrow \Omega^k(M) \\ \omega \longmapsto \boxed{\omega|_M} \end{cases}$

where: $(\omega|_M)_p(X_p^1, \dots, X_p^k) := \dots$

Comp
$\omega =$
d
d
Rk

- Cheap 1-forms (exact 1-forms): $df \in \Omega^1(M)$ for $f \in C^\infty(M)$
  $(df(X) = L_X(f))$
- Wedge products

Recall: $\Omega^0(M) = C^\infty(M)$, $f \wedge \omega = f\omega$

$$\Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$$

$$(\omega \wedge \eta)(X^1, \ldots, X^{k+l}) = \sum_{\sigma \in S_{k,l}} \mathrm{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \ldots)\, \eta(X_{\sigma(k+1)}, \ldots)$$

$\Downarrow$ even more cheap (k-)forms $f\, dg_1 \wedge \ldots \wedge dg_k \in \Omega^k(M)$ (and sums of such) $(f, g_1, \ldots, g_k \in C^\infty(M))$

- In $M = \mathbb{R}^L$ any k-form looks like
$$\boxed{\omega = \sum_I F^I\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}}$$
$\left(\text{Ex: } \boxed{df = \sum_i \frac{\partial f}{\partial x_i} dx_i}\right)$

$\boxed{d(fg) = f\,dg + g\,df}$

$\eta \wedge \omega = (-1)^{kl}\, \omega \wedge \eta$

- For embedded $M \subseteq \mathbb{R}^L$ & more generally, for $M \subseteq N$ $\Rightarrow$ Restrictions
$$\begin{cases} \Omega^k(N) \longrightarrow \Omega^k(M) \\ \omega \longmapsto \boxed{\omega|M} \end{cases}$$
where $(\omega|_M)_p(X_p^1, \ldots, X_p^k) = \omega_p(X_p^1, \ldots)$ for $X_p^i \in T_pM \ldots \subseteq T_pN$

- General pull-backs for $F: M \to N$ $\Rightarrow$
$$\begin{cases} F^*: \Omega^k(N) \longrightarrow \Omega^k(M) \\ \omega \longmapsto \boxed{F^*(\omega)} \end{cases}$$
where $(F^*\omega)_p(X_p^1, \ldots, X_p^k) = \omega_{F(p)}((dF)_p(X_p^1), \ldots)$ for $X_p^i \in T_pM \left(\ldots \xrightarrow{(dF)_p} T_{F(p)}N\right)$ $\omega \wedge d\eta + d\omega \wedge \eta$

- Important: $\boxed{F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)}$
  $\boxed{F^*(d\omega) = d(F^*(\omega))}$

Compu…
$\omega =$
$d\omega =$
$d\omega = d$…
Rk: In $\mathbb{R}$…

Prop (Cor 4.47) On any M, $\exists !$ operators (one for each k) ③

$$\boxed{d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)}$$

d degree 1

st (DR0) d is $\mathbb{R}$-linear & on 0-forms: the usual $\boxed{df}$

(DR1) d satisfies graded Leibniz

$$\boxed{d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta}$$ $(\forall)\ \omega \in \Omega^k(M),\ \eta \in \Omega^\ell(M)$

(DR2) $\boxed{d \circ d = 0}$

Also important: $\boxed{F^* \circ d = d \circ F^*}$

Explicitly: … (Koszul formula)

Prop (Prop 4.55) For any $V \in \mathfrak{X}(M)$, $\exists !$ operators ④

$$\boxed{\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega^k(M)}$$

$\mathcal{L}_V$ degree 0

st (0) $\mathcal{L}_V$ is $\mathbb{R}$-linear & on 0-forms the usual $\boxed{\mathcal{L}_V(f)}$

(1) satisfies Leibniz

$$\boxed{\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge \mathcal{L}_V(\eta)}$$

(2) $\boxed{\mathcal{L}_V \circ d = d \circ \mathcal{L}_V}$

Explicitly $\mathcal{L}_V(\omega) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega) = \dots$

Prop (4.57) $(\forall)\ V$, $\exists !$ ⑤

$$\boxed{i_V : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)}$$

i_V degree -1

st (0) i_V = $\mathbb{R}$-linear & on 1-forms evaluation on V $\boxed{i_V(\omega) = \omega(V)}$

(1) graded Leibniz

$$\boxed{i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)}$$

Explicitly $i_V(\omega)(X^1, \dots, X^{k-1}) =$
$= \omega(V, X^1, \dots, X^{k-1})$

Cartan's magic $\boxed{\mathcal{L}_V = \underbrace{d \circ i_V + i_V \circ d}_{\mathcal{L}_V'}}$

Proof of Cartan:

(0) $\mathcal{L}_V'(f) = d(\underbrace{i_V(f)}_{0}) + i_V(df) = df(V) = L_V(f)$

(2) $\mathcal{L}_V' \circ d = d \circ i_V \circ d + i_V \circ \underbrace{d \circ d}_{0}$

$d \circ \mathcal{L}_V' = \underbrace{d \circ d}_{0} \circ i_V + d \circ i_V \circ d$

(1) $\mathcal{L}_V'(\omega \wedge \eta) = d(\overbrace{i_V(\omega \wedge \eta)}^{\text{Leib}}) + i_V(\overbrace{d(\omega \wedge \eta)}^{\text{Leib}})$
$= d(i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)) + \dots = \dots$

…tations ⑥

$x e^y\, dy \wedge dz \in \Omega^2(\mathbb{R}^3)$, Leibniz

$d(x e^y\, dy \wedge dz) = d(x e^y) \wedge dy \wedge dz + x e^y\, d(dy \wedge dz)$

⑦

$V = x \frac{\partial}{\partial y}$

$\mathcal{L}_V(\omega) = \mathcal{L}_{x\frac{\partial}{\partial y}}(x e^y\, dy \wedge dz) =$

⑧

$L_V(f)(p) = \partial_{v_p}(f) = \dots$

$i_V(\omega) = i_{x\frac{\partial}{\partial y}}(x e^y\, dy \wedge dz)$ $\quad v_p = \dots$

$L_V(f) = \frac{d}{dt}\Big|_{t=0} f(\varphi_V^t(p))$

$= i_{x\frac{\partial}{\partial y}}(x e^y\, dy) \wedge dz \dots$

Proof of Cartan:

(0) $\mathcal{L}'_V(f) = d(\underbrace{i_V(f)}_{0}) + i_V(df) = df(V) = L_V(f)$

(2) $\mathcal{L}'_V \circ d = d \circ i_V \circ d + i_V \circ \underbrace{d \circ d}_{0}$

$d \circ \mathcal{L}'_V = \underbrace{d \circ d \circ i_V}_{0} + d \circ i_V \circ d$

(1) $\mathcal{L}'_V(\omega \wedge \eta) = d(i_V(\omega \wedge \eta)) + i_V(d(\omega \wedge \eta))$

$= d(i_V(\omega) \wedge \eta + \ldots) + \ldots = \ldots$

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Computations

$\omega = x e^y \, dy \wedge dz \in \Omega^2(\mathbb{R}^3)$

$d\omega = d(\underbrace{x e^y}_{0} \underbrace{dy \wedge dz}_{2}) = \underbrace{d(x e^y)}_{e^y dx + x e^y dy} \wedge dy \wedge dz + x e^y \underbrace{d(dy \wedge dz)}_{= \underbrace{d(dy)}_{0} \wedge dz - dy \wedge \underbrace{d(dz)}_{0}}$

$= (e^y dx + x e^y dy) \wedge dy \wedge dz$

$= e^y dx \wedge dy \wedge dz + x e^y dy \wedge dy \wedge dz = \boxed{e^y \, dx \wedge dy \wedge dz}$

$d\omega = d(x e^y dy \wedge dz) = \ldots = \text{—} \;\; \text{—}$

Rk in $\mathbb{R}^k$: $d(\underbrace{F^I dx_{i_1} \wedge \ldots \wedge dx_{i_k}}) = \sum_i \frac{\partial F^I}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$

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$V = x \frac{\partial}{\partial y}$

$\mathcal{L}_V(\omega) = L_{x\frac{\partial}{\partial y}}(x e^y \, dy \wedge dz) =$

$= \underbrace{L_{x\frac{\partial}{\partial y}}(x e^y)}_{x^2 e^y} dy \wedge dz + \cancel{x e^y} \, \underbrace{L_{x\frac{\partial}{\partial y}}(dy \wedge dz)}$

$\underbrace{\underbrace{L_{x\frac{\partial}{\partial y}}(dy)}_{d(\underbrace{L_{x\frac{\partial}{\partial y}}(y)}_{x}) = dx} \wedge dz + dy \wedge L_{x\frac{\partial}{\partial y}}(dz)}_{dx \wedge dz}$

$= \underline{x^2 e^y dy \wedge dz + x e^y dx \wedge dz}$

OR: Cartan

$\mathcal{L}_V(\omega) = d(\underbrace{i_V(\omega)}_{x^2 e^y dz}) + i_V(\underbrace{d\omega}_{e^y dx \wedge dy \wedge dz}) =$

$= \underline{x e^y dx \wedge dz + x^2 e^y dy \wedge dz} + x e^y \cancel{dx} \wedge dz$

8

$i_V(\omega) = i_{x\frac{\partial}{\partial y}}(\underbrace{x e^y \wedge dy}_{1} \wedge \underbrace{dz}_{1})$

$= i_{x\frac{\partial}{\partial y}}(x e^y dy) \wedge dz - x e^y dy \wedge i_{x\frac{\partial}{\partial y}}(dz)$

$= \underline{x^2 e^y \wedge dz}$

$L_V(f)(p) = \partial_{v_p}(f) = f \circ \gamma(0)$

$v_p = \dot{\gamma}(0)$

$L_V(f) = \frac{d}{dt}\Big|_{t=0} f(\varphi_V^t(p))$

Prop (Cor 4.47) On any M, $\exists!$ operators (one for each k)

$$\boxed{d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)}$$

d degree 1

s.t. (DR0) d is $\mathbb{R}$-linear & on 0-forms the usual $\boxed{df}$

(DR1) d satisfies graded Leibniz

$$\boxed{d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k \omega\wedge d\eta} \quad (\forall\ \omega\in\Omega^k(M),\ \eta\in\Omega^\ell(M))$$

(DR2) $\boxed{d\circ d = 0}$

Also compatible: $\boxed{F^*\circ d = d\circ F^*}$

Explicitly … (Koszul formula)

Prop (Prop 4.55) For any $V\in\mathfrak{X}(M)$, $\exists!$ operators

$$\boxed{\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega^k(M)}$$

$\mathcal{L}_V$ degree 0

s.t. (0) $\mathcal{L}_V$ is $\mathbb{R}$-linear & on 0-forms the usual $\boxed{\mathcal{L}_V(f)}$

(1) satisfies Leibniz

$$\boxed{\mathcal{L}_V(\omega\wedge\eta) = \mathcal{L}_V(\omega)\wedge\eta + \omega\wedge\mathcal{L}_V(\eta)}$$

(2) $\boxed{\mathcal{L}_V\circ d = d\circ\mathcal{L}_V}$

Explicitly $\mathcal{L}_V(\omega) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega) = \dots$

Prop (4.57) $(\forall)\ V$, $\exists!$

$$\boxed{\iota_V : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)}$$

ι_V degree −1

s.t. (0) ι_V = $\mathbb{R}$-linear & on 1-forms evaluation on V $\boxed{\iota_V(\omega) = \omega(V)}$

(1) graded Leibniz

$$\boxed{\iota_V(\omega\wedge\eta) = \iota_V(\omega)\wedge\eta + (-1)^k\omega\wedge\iota_V(\eta)}$$

Explicitly $\iota_V(\omega)(X^1, \dots, X^{k-1}) = \omega(V, X^1, \dots, X^{k-1})$

Cartan's magic $\boxed{\mathcal{L}_V = \underbrace{d\circ\iota_V + \iota_V\circ d}_{\mathcal{L}_V'}}$

Proof of Cartan:

(0) $\mathcal{L}_V'(f) = d(\underbrace{\iota_V(f)}_{0}) + \iota_V(df) = df(V) = L_V(f)$

(2) $\mathcal{L}_V'\circ d = d\circ\iota_V\circ d + \iota_V\circ \underbrace{d\circ d}_{0}$

$d\circ\mathcal{L}_V' = \underbrace{d\circ d}_{0}\circ\iota_V + d\circ\iota_V\circ d$

(1) $\mathcal{L}_V'(\omega\wedge\eta) = d(\overbrace{\iota_V(\omega\wedge\eta)}^{\text{Leib}}) + \iota_V(\overbrace{d(\omega\wedge\eta)}^{\text{Leib}})$

$= d(\iota_V(\omega)\wedge\eta + (-1)^k\omega\wedge\iota_V(\eta)) + \dots = \dots$

Computations

$\omega = x e^y\, dy\wedge dz \in \Omega^2(\mathbb{R}^3)$

$d\omega = d(\underbrace{x e^y}_{0}\underbrace{dy\wedge dz}_{2}) \overset{\text{Leibniz}}{=} \underbrace{d(x e^y)}_{e^y dx + x e^y dy}\wedge dy\wedge dz + x e^y \underbrace{d(dy\wedge dz)}_{\underbrace{d(dy)}_{0}\wedge dz - dy\wedge \underbrace{d(dz)}_{0}}$

$= (e^y dx + x e^y dy)\wedge dy\wedge dz$

$= e^y dx\wedge dy\wedge dz + x e^y dy\wedge dy\wedge dz = e^y dx\wedge dy\wedge dz$

$d\omega = d(x e^y dy \wedge dz) = \dots = \text{—} \,\text{''}\, \text{—}$

In $\mathbb{R}^n$: $d(F^I dx_{i_1}\wedge\dots\wedge dx_{i_k}) = \sum_i \frac{\partial F^I}{\partial x_i} dx_i\wedge dx_{i_1}\wedge\dots\wedge dx_{i_k}$

$V = x\frac{\partial}{\partial y}$

$\mathcal{L}_V(\omega) = \mathcal{L}_{x\frac{\partial}{\partial y}}(x e^y\, dy\wedge dz) =$

$= \underbrace{L_{x\frac{\partial}{\partial y}}(x e^y)}_{x^2 e^y} dy\wedge dz + x e^y\, \mathcal{L}_{x\frac{\partial}{\partial y}}(dy\wedge dz)$

$= x^2 e^y dy\wedge dz + x e^y dx\wedge dz$

$\underbrace{\mathcal{L}_{x\frac{\partial}{\partial y}}(dy)}_{d(L_{x\frac{\partial}{\partial y}}(y)) = dx}\wedge dz + dy\wedge \mathcal{L}_{x\frac{\partial}{\partial y}}(dz)$

$dx\wedge dz$

OR: Cartan

$\mathcal{L}_V(\omega) = d(\underbrace{\iota_V(\omega)}_{x^2 e^y dz}) + \iota_V(\underbrace{d\omega}_{e^y dx\wedge dy\wedge dz}) =$

$= \cancel{x e^y dx\wedge dz} + x^2 e^y dy\wedge dz + x e^y dx\wedge dz$

$L_V(f)(p) = \partial_{V_p}(f) = \dot{f}\circ\gamma(0)$, $V_p = \dot\gamma(0)$

$L_V(f)(p) = \frac{d}{dt}\Big|_{t=0} f(\varphi_V^t(p))$

$\iota_V(\omega) = \iota_{x\frac{\partial}{\partial y}}(x e^y \wedge dy\wedge dz)$

$= \iota_{x\frac{\partial}{\partial y}}(x e^y dy)\wedge dz - x e^y dy\wedge \iota_{x\frac{\partial}{\partial y}}(dz)$

$= x^2 e^y \wedge dz$

Prop (Cor 4.47): On any M, $\exists!$ operators (one for each k) ③'

$$\boxed{d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)}$$

d degree 1

s.t. (DR0) d is $\mathbb{R}$-linear &
on 0-forms: the usual $\boxed{df}$

(DR1) d satisfies graded Leibniz

$$\boxed{d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k \omega\wedge d\eta} \quad (\forall)\ \omega\in\Omega^k(M),\ \eta\in\Omega^\ell(M)$$

(DR2) $\boxed{d\circ d = 0}$

Also important: $\boxed{F^*\circ d = d\circ F^*}$

Explicitly: ... (Koszul formula)

Some more proofs: (For $\mathcal{L}_V$ given by $(*)$: (0) ✓

② $\mathcal{L}_V(d(\omega)) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(d\omega)$

$= d\left(\frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*\omega\right)$

$= d(\mathcal{L}_V(\omega))$

Prop (Prop 4.55): For any $V\in\mathfrak{X}(M)$, $\exists!$ operators ④'

$$\boxed{\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega^k(M)}$$

$\mathcal{L}_V$ degree 0

s.t. (0) $\mathcal{L}_V$ is $\mathbb{R}$-linear &
on 0-forms the usual $\boxed{\mathcal{L}_V(f)}$

(1) satisfies Leibniz

$$\boxed{\mathcal{L}_V(\omega\wedge\eta) = \mathcal{L}_V(\omega)\wedge\eta + \omega\wedge\mathcal{L}_V(\eta)}$$

(2) $\boxed{\mathcal{L}_V\circ d = d\circ\mathcal{L}_V}$

Explicitly: $\boxed{\mathcal{L}_V(\omega) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega)} \overset{(*)}{=} \dots$

① $\mathcal{L}_V(\omega\wedge\eta) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega\wedge\eta) = \frac{d}{dt}\Big|_{t=0}\left((\varphi_V^t)^*(\omega)\wedge(\varphi_V^t)^*(\eta)\right)$

$= \lim_{t\to 0} \frac{(\varphi_V^t)^*(\omega)\wedge(\varphi_V^t)^*(\eta) - \omega\wedge\eta}{t}$

$= \lim_{t\to 0} \frac{(\varphi_V^t)^*(\omega) - \omega}{t}\wedge(\varphi_V^t)^*(\eta) + \lim_{t\to 0} \omega\wedge\frac{(\varphi_V^t)^*(\eta) - \eta}{t}$

$= \mathcal{L}_V(\omega)\wedge\eta + \omega\wedge\mathcal{L}_V(\eta)$.

Prop (4.57) $(\forall)\ V$ ...

$\boxed{i_V : \Omega^k \dots}$

s.t. (0) i_V = $\mathbb{R}$-linear & ...
on 1-forms: eval...

(1) graded Leibniz

$\boxed{i_V(\omega\wedge \dots}$

Explicitly: $i_V(\omega)(X^1, \dots$
$= \omega(V, X^1, \dots$

Cartan's magic: $\boxed{\mathcal{L}_V = \dots}$

Proof of Cartan:

(0) $\mathcal{L}'_V(f) = d(\underbrace{i_V(f)}_{0} \dots$

(2) $\mathcal{L}'_V\circ d = d\circ i_V \dots$

$d\circ\mathcal{L}'_V = \underbrace{d\circ d}_{0}\circ \dots$

(1) $\mathcal{L}'_V(\omega\wedge\eta) = d \dots$
$= d(\dots$

①

$\ldots \times T_pM \to \mathbb{R}$

...

ts (U, x^i):

$(X_k)_p$

for $f \in C^\infty(M)$

$f))$

$\eta(X_{\sigma(k+1)}, \ldots)$

Prop (Cor 4.47): On any M, $\exists!$ operators (one for each k) ③!

$$\boxed{d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)}$$

d degree 1

s.t. (DR0) d is $\mathbb{R}$-linear &

on 0-forms: the usual $\boxed{df}$

(DR1) d satisfies graded Leibniz

$$\boxed{d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta} \quad (\forall)\ \omega \in \Omega^k(M),\ \eta \in \Omega^\ell(M)$$

(DR2) $\boxed{d \circ d = 0}$

Also important: $\boxed{F^* \circ d = d \circ F^*}$

Explicitly: ... (Koszul formula)

Some more proofs: For $\mathcal{L}_V$ given by $(*)$: (0) ✓

② $\mathcal{L}_V(d(\omega)) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(d\omega)$

$= d\left(\frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*\omega\right)$

$= d(\mathcal{L}_V(\omega))$

Prop (Prop 4.55): For any $V \in \mathfrak{X}(M)$

$$\boxed{\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega}$$

s.t. (0) $\mathcal{L}_V$ is $\mathbb{R}$-linear &

on 0-forms: the usual $\boxed{\mathcal{L}_V(f)}$

(1) satisfies Leibniz:

$$\boxed{\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta +}$$

(2) $\boxed{\mathcal{L}_V \circ d = d \circ \mathcal{L}_V}$

Explicitly: $\boxed{\mathcal{L}_V(\omega) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega)}$ $(*)$

① $\mathcal{L}_V(\omega \wedge \eta) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega \wedge \eta) =$

$= \lim_{t \to 0} \frac{(\varphi_V^t)^*(\omega) \wedge (\varphi_V^t)^*(\eta) \overset{-\omega \wedge \omega}{}}{t}$

$= \lim_{t \to 0} \frac{(\varphi_V^t)^*(\omega) - \omega}{t} \wedge (\varphi$

$= \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge$

st. (DR0) d is $\mathbb{R}$-linear & on 0-forms: the usual $\boxed{df}$

(DR1) d satisfies graded Leibniz

$\boxed{d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k\omega\wedge d\eta}$ $(\forall)\ \omega\in\Omega^k(M),\ \eta\in\Omega^\ell(M)$

(0) $\mathcal{L}_V$ is $\mathbb{R}$-linear & on 0-forms the usual $\boxed{\mathcal{L}_V(f)}$

(1) satisfies Leibniz

$\boxed{\mathcal{L}_V(\omega\wedge\eta) = \mathcal{L}_V(\omega)\wedge\eta + \omega\wedge\mathcal{L}_V(\eta)}$

For De Rham d: (9)

Step 1: It is true on $\mathbb{R}^m$: $d(f\,dx_{i_1}\wedge\dots\wedge dx_{i_k}) = \sum_i \frac{\partial f}{\partial x_i} dx_i\wedge dx_{i_1}\wedge\dots\wedge dx_{i_k}$

Check the conditions. Notice $d\circ d=0$ comes out of $\frac{\partial^2 f}{\partial x_i\partial x_j} = \frac{\partial^2 f}{\partial x_j\partial x_i}$

Step 2: For general M, the prop is true for all $U\subseteq M$ domains of charts

pf: think for a minute.

Step 3: We are done, if we require d to be local.

$\boxed{\exists!\ d \text{ on } M, \text{ which is local } \& \ (\forall)\ U\subseteq M \text{ domain of a chart } d|_U \text{ is precisely the one from Step 2}}$

means what?

$\boxed{\text{"}d|_U\text{"}(\omega|_U) = (d\omega)|_U}$

Well defined? Yes if ω-local

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) \\ \downarrow \text{restr} & & \downarrow \text{restr} \\ \Omega^k(U) & \xrightarrow{\text{"}d|_U\text{"}} & \Omega^{k+1}(U) \end{array}$$

$(\forall)\ V\subseteq M$ open one has the implication:

$\underbrace{\omega\in\Omega^k(M),\ V\text{-supported}}_{\exists\ A\text{-closed},\ A\subseteq V \text{ st } \omega_p=0\ \forall p \text{ outside } A} \Rightarrow d\omega$ is V-supported

ω_p, $p\in M$

$\mu(V, \dots)$ S^n $\iota_V(\mu)(p)=0$

$\mu_{S^n} = \sum_i (-1)^i x_i\, dx_0\wedge\dots\wedge\widehat{dx_i}\wedge\dots\wedge dx_n$ $(x_0,x_1,\dots,x_n)\in S^n\subseteq\mathbb{R}^{n+1}$

(10) Step 4: (DR0) & (DR1) $\Rightarrow$ locality

$d(f\eta) = df\wedge\eta + f\,d\eta$

$\omega = f\eta$ f-vanishes as we want

$\hookrightarrow$ just have to handle functions

Last thing: $d\circ F^* = F^*\circ d$

Call $\omega\in\Omega^k(N)$ good if $d(F^*(\omega)) = F^*(d\omega)$

Notice: Sums of good are good

ω,η = good $\Rightarrow \omega\wedge\eta$ is good

$(\forall)\ f\in C^\infty(M)$ are good

ω = good $\Rightarrow d\omega$ = good

$\sum f\,dg_{i_1}\wedge\dots\wedge dg_{i_k}$ are good

$\Rightarrow$ all forms are good.

$dx\wedge dy\wedge dz$

Prop (4.5.7) ... $i_V : \Omega^k(M) \to \Omega^{k-1}(M)$

st. (0) $i_V = \mathbb{R}$-linear & on 1-forms: evaluation on V $\boxed{i_V(\omega) = \omega(V)}$

i_V degree -1

(1) graded Leibniz

$$\boxed{i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)}$$

11

Def: A volume form on a m-dimensional manifold M is any top-degree form

$$\mu \in \Omega^m(M)$$

such that $\mu_p \neq 0 \ \forall\, p \in M$.

$\ldots) = F^*(d(\ldots))$

... good

Ex In $\mathbb{R}^m$, μ must look

$$\mu = f \, dx_1 \wedge \ldots \wedge dx_m$$

$f \in C^\infty(\mathbb{R}^m)$ nowhere 0.

e.g. $\boxed{\mu_{can} = dx_1 \wedge \ldots \wedge dx_m}$ $\quad \sum x_i \frac{\partial}{\partial x_i}$

Rk: On any M, locally (U, χ)

$$\mu = f^\chi \, dx_1 \wedge \ldots \wedge dx_m \qquad f^\chi \in C^\infty(U)$$

Ex: $S^2 \subseteq \mathbb{R}^3$ $\qquad \cancel{dx \wedge dy \wedge dz}|_{S^2} = 0$

$$\mu_{S^2} = x\, dy \wedge dz + y\, dx \wedge dz + z\, dx \wedge dy$$

Volume form on S^2. $\mu \,|\, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \,|\, y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = \ldots$ $\quad (x,y,z) \in S^2$

$\mathcal{L}_V(\omega\wedge\eta) = \mathcal{L}_V(\omega)\wedge\eta + \omega\wedge\mathcal{L}_V(\eta)$

graded Leibniz

$\iota_V(\omega\wedge\eta) = \iota_V(\omega)\wedge\eta + (-1)^k\,\omega\wedge\iota_V(\eta)$

$\frac{\partial f}{\partial x_i}\, dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$

$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

$U \subseteq M$ domains

be local

$\iota_V(\mu)(p) = 0$

$(x_0, x_1, \ldots, x_n) \in S^n \subseteq \mathbb{R}^{n+1}$

$\sum f\, dg_{i_1} \wedge$

$\Rightarrow$ all forms are good.

$\boxed{\mu_{S^n} = \sum_i (-1)^i x_i\, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n}$

$dx \wedge dy \wedge dz$

11

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Rk: On any M, locally (U, X)

$$\mu = f^X\, dx_1 \wedge \ldots \wedge dx_m \qquad f^X \in C^\infty(U)$$

Ex: $S^2 \subseteq \mathbb{R}^3$ $\qquad \cancel{dx \wedge dy \wedge dz}\big|_{S^2} = 0$

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Volume form on S^2. $\mu \,\big|\, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\ y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \big| = \ldots$ $\quad (x, y, z) \in S^2$