Reminder: differential forms & operations (1)

$\omega \in \Omega^k(M)$ $\left(\begin{array}{l}\text{functions for } k=0\\ 0 \text{ for } k>\dim(M)\end{array}\right)$ | M = manifold $\Rightarrow$ ... $\Rightarrow$ $\Omega^0(M) \xrightarrow{d} \Omega^1(M)$ ...

- **wedges** $\Omega^k(M)\times\Omega^l(M) \longrightarrow \Omega^{k+l}(M)$, $(\omega,\eta)\longmapsto \omega\wedge\eta$
- **De Rham**: $d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, $d\circ d = 0$, Leibniz
- **pull-backs**: $F: M \longrightarrow N \rightsquigarrow F^*: \Omega^k(N) \to \Omega^k(M)$

  Compatible with everything $\begin{cases} F^*(d\omega) = d(F^*\omega) \\ (G\circ F)^* = F^*\circ G^* \end{cases}$
- **volume forms**: $\mu\in\Omega^m(M)$ s.t. $\mu_p \neq 0$ $\forall p\in M$
- **orientations** on M:
  - each volume form $\mu \rightsquigarrow$ orientation $\mathcal{O}_\mu$
  - for two of them: $\mathcal{O}_\mu = \mathcal{O}_{\mu'} \iff \mu' = f\mu$ with $f>0$.

$H^k(M) := \dfrac{\Omega^k_{cl}(M)}{\Omega^k_{ex}(M)}$

$\boxed{H^k(M) = \{[\omega] : d\omega ...}$

the k-th DeRham ...
When all of these are ...
Euler characteristic ...

(2)

Given $(M,\mathcal{O})$ oriented:

- oriented basis of T_pM: basis $\{v^1,\dots,v^m\}$ s.t. $\mu_p(v^1,\dots,v^m)>0$
- ... (U,χ): if $\mu_p\left(\left(\frac{\partial}{\partial x_1}\right)_p,\dots,\left(\frac{\partial}{\partial x_m}\right)_p\right)>0$ $\forall p\in U$

$f>0.$

Given $(M, \mathcal{O})$ oriented:

(2)

- $\pm$oriented basis of T_pM: basis $\{v^1, \dots, v^m\}$ s.t. $\mu_p(v^1, \dots, v^m) > 0$
- $\pm$oriented chart (U, x): if $\mu_p\left(\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p\right) > 0 \ \forall p \in U$
- Integration on $(M, \mathcal{O})$: $\int_M : \Omega_c^m(M) \longrightarrow \mathbb{R}$.

$\rightarrow$ for $\omega =$ small, i.e. supported in some $U \xrightarrow[\sim]{x} \Omega$ oriented

$\int_M \omega := \int_\Omega F_\omega^x \, dx$ where $(F_\omega^x$-coeff w.r.t. $x) \in C^\infty(\Omega)$

$\rightarrow$ for arbitrary $\omega \in \Omega_c^m(M)$, write $\omega = \sum_{i=1}^k \omega_i \leftarrow$ small ... part of 1

$\int_M \omega := \sum_i^k \int_M \omega_i$

- Given M, $\mu \in \Omega^m(M)$ vol form: $\boxed{Vol_\mu(M) := \int_M \mu}$ (using $\mathcal{O}_\mu$) >0.

Thm (Stokes): $(M, \mathcal{O})$-oriented $\Rightarrow \int_M \omega = 0 \quad \forall \omega$ of type $d\eta$ with $\eta \in \Omega_c^{m-1}(M)$.

Corollary: $M =$ cpct $\Rightarrow$ any volume form $\mu \in \Omega^m(M)$ is closed (i.e. $d\mu = 0$) but not exact (i.e. not a $d\eta$).

3

functions for $k=0$

0 for $k>\dim(M)$

$\to \omega\wedge\eta$

Leibniz

$^k(M)$

$\omega))$

M = manifold $\Rightarrow$

$\Rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to \cdots \to \Omega^m(M) \to 0$

$$\Omega^k(M) \supseteq \Omega^k_{cl}(M) = \{\omega\in\Omega^k(M) : d\omega=0\}$$

$d^2=0 \Rightarrow$

$$\Omega^k_{cl}(M) \supseteq \Omega^k_{ex}(M) = \{\omega=d\eta \mid \eta\in\Omega^{k-1}(M)\}$$

$$H^k(M) := \frac{\Omega^k_{cl}(M)}{\Omega^k_{ex}(M)}$$

$$\boxed{H^k(M) := \{[\omega] : d\omega=0\} \text{ where } [\omega]=[\omega'] \iff \omega-\omega'=d\eta \text{ for some } \eta}$$

the k-th DeRham cohomology space of M.

When all of these are finite dimensional one defines the Euler characteristic of M as

$$\boxed{\chi(M) := \sum_k (-1)^k \dim_{\mathbb{R}}(H^k(M))}$$

$\boxed{\mathrm{vol}_\mu(M) := \int_M \mu}$ (using $\mathcal{O}_\mu$) > 0.

(2')

Thm (Stokes) for $(M, \mathcal{O})$ = compact oriented $\Longrightarrow$

$\Rightarrow$ integration induces a map: $\int_M : H^m(M) \to \mathbb{R}$

$[\omega] \longmapsto \int_M \omega$.

and, for any $\mu \in \Omega^m(M)$ vol. form, $[\mu] \in H^m(M)$ is non-zero.

(functions for $k=0$
0 for $k>\dim(M)$)

$\to \omega\wedge\eta$

Leibniz

$\Omega^k(M)$

$(\omega))$

M.

O_m

M = manifold $\Rightarrow$ ③

$$\Rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to \cdots \to \Omega^m(M) \to 0$$

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the k-th DeRham cohomology space of M.

When all of these are finite dimensional one defines the Euler characteristic of M as

$$\boxed{\chi(M) := \sum_k (-1)^k \dim_{\mathbb{R}}(H^k(M))}$$

Thm (Poincaré Lemma): $H^k(\mathbb{R}^m) = \begin{cases} \mathbb{R} & k=0 \\ 0 & k\neq 0. \end{cases}$

Fun

Rk

$\to 0$

$\omega = 0\}$

$)\}$

or some η

<u>Functoriality</u>: Any $F: M \longrightarrow N$ induces

$$F^*: H^k(N) \longrightarrow H^k(M),\ [\omega] \longmapsto [F^*(\omega)]$$

$[\omega'] \quad \overset{?}{=} [F^* \omega']$

4

<u>Rk</u>: $(F \circ G)^* = G^* \circ F^*$, $Id^* = Id$ $\Rightarrow$ if F is a diffeom, then all $F^*: H^k(N) \to H^k(M)$ are isom.

<u>Thm</u>: (1) If $F_0, F_1: M \to N$ are homotopic then $\longrightarrow$ $\exists\ F: M \times [0,1] \to N$ s.t. $\begin{cases} F(x,0) = F_0(x) \\ F(x,1) = F_1(x) \end{cases}$ $F(x,t)$ $\boxed{\text{write } F_0 \sim F_1}$

$F_0^*, F_1^*: H^k(N) \to H^k(M)$ coincide

(2) If $F: M \to N$ is homotopy equivalence $\longrightarrow$ $\exists\ G: N \to M$ s.t. $\begin{cases} F \circ G \sim Id_N \\ G \circ F \sim Id_M \end{cases}$

$\Rightarrow$ $F^*: H^k(N) \to H^k(M)$ are isomorphisms.

<u>Ex 1</u>: $M = \mathbb{R}^n$, $N = \mathbb{R}^0 = \{0\}$ are h.e.

<u>Ex 2</u>: $M = \mathbb{R}^m \setminus \{0\}$, $N = S^{m-1}$ are h.e.

Corollary: $\begin{cases} i_0 : M \to M\times\mathbb{R},\ x \mapsto (x,0) \\ i_1 : M \to M\times\mathbb{R},\ x \mapsto (x,1) \end{cases}$

look in cohomology:

Ex: $m=1$: $0 \to \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \to 0$ $\quad f \mapsto f'\cdot dx$

$f=f(x) \in C^\infty(\mathbb{R})$ $\qquad \{g\cdot dx \mid g \in C^\infty(\mathbb{R})\}$

$H^0(\mathbb{R}) = \mathbb{R} \Leftarrow$ $\begin{cases} \Omega^0_{cl}(\mathbb{R}) = \mathbb{R} \\ \Omega^0_{ex}(\mathbb{R}) = 0 \end{cases}$ $\qquad \Omega^1_{cl}(\mathbb{R}) = \text{everything} -$

$\Omega^1_{ex}(\mathbb{R}) = \{f'dx \mid f \in C^\infty(\mathbb{R})\}$

$(\forall) g \in C^\infty(\mathbb{R})\ \exists f$ s.t. $g = f'$. Ok (e.g. $f(x) = \int_0^x g(y)dy$)

Homotopic to

For general m: proceed inductively & write $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$

In general, for any M look at $M\times\mathbb{R}$ and try to retain as much of possible from the argument:

k-form: look like $\omega = \alpha_t + \beta_t \wedge dt$

$\alpha_t \in \Omega^k(M)$

$\beta_t \in \Omega^{k-1}(M)$

$d_{M\times\mathbb{R}}(\omega) = d_M\alpha_t - \dot{\alpha}_t \wedge dt + d_M\beta_t \wedge dt$

Hence $\omega = \text{closed} \Rightarrow \dot{\alpha}_t = d\beta_t \Rightarrow \alpha_1 - \alpha_0 = d\left(\int_0^1 \beta_t\, dt\right)$

$\Rightarrow [\alpha_1] = [\alpha_0]$ in coh. of M

What does it mean to take "the α_0 of $\omega = \alpha_t + \beta_t \wedge dt$"?

$i_0 : M \hookrightarrow M\times\mathbb{R}$, $i_0(x) = (x,0)$ and $\alpha_0 = i_0^*(\omega)$

Ex 2

$H_0 =$

$H_1 =$

M) $\begin{pmatrix} \text{functions for } k=0 \\ 0 \text{ for } k>\dim(M) \end{pmatrix}$

$,\eta) \longmapsto \omega\wedge\eta$

$=0$, Leibniz

$M) \to \Omega^k(M)$

$) = d(F^*(\omega))$

$= F^* \circ G^*$

$\forall p\in M$

rientation O_μ

$\mu = f\mu$

with $f>0$.

M = manifold $\Rightarrow$

$\Rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to \cdots \to \Omega^m(M) \to 0$ (3)

$\Omega^k(M) \supseteq \Omega^k_{cl}(M) = \{\omega\in\Omega^k(M) : d\omega = 0\}$

$d^2=0 \Rightarrow \Omega^k_{cl}(M) \supseteq \Omega^k_{ex}(M) = \{\omega = d\eta \mid \eta\in\Omega^{k-1}(M)\}$

$$H^k(M) := \frac{\Omega^k_{cl}(M)}{\Omega^k_{ex}(M)}$$

$$\boxed{H^k(M) := \{[\omega] : d\omega = 0\}} \text{ where } [\omega]=[\omega'] \Leftrightarrow \omega-\omega' = d\eta \text{ for some } \eta$$

the k-th DeRham cohomology space of M.

When all of these are finite dimensional one defines the Euler characteristic of M as $\boxed{\chi(M) := \sum_k (-1)^k \dim(H^k(M))}$

Thm (Poincaré Lemma): $H^k(\mathbb{R}^m) = \begin{cases} \mathbb{R} & k=0 \\ 0 & k\neq 0. \end{cases}$

Corollary: $\begin{cases} i_0 : M \to M\times\mathbb{R}, & x\longmapsto (x,0) \\ i_1 : M \to M\times\mathbb{R}, & x\longmapsto (x,1) \end{cases}$

look in cohomology:

$i_0^*, i_1^* : H^k(M\times\mathbb{R}) \to H^k(M)$ are equal: $i_0^* = i_1^*$.

Functoria[l]

Rk: $(F\circ G$

Thm

(So

Proof

Ex: $m=1$: $0 \to \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \to 0$ (6) $f \longmapsto f'\cdot dx$

Ex 1

M) → 0

) : dω = 0 }

$^{k-1}(M)$ }

y for some y

Functoriality: Any $F: M \to N$ induces $[\omega'] \overset{?}{=} [F^*\omega']$

$$F^*: H^k(N) \to H^k(M),\ [\omega] \mapsto [F^*(\omega)]$$

Rk: $(F\circ G)^* = G^*\circ F^*$, $Id^* = Id$ ⇒ if F is a diffeom, then all $F^*: H^k(N) \to H^k(M)$ are isom.

Thm: (1) If $F_0, F_1: M \to N$ are homotopic then $\to \exists\ F: M\times[0,1] \to N$ $F(x,t)$

s.t. $\begin{cases} F(x,0) = F_0(x) \\ F(x,1) = F_1(x) \end{cases}$ $\boxed{\text{write } F_0 \sim F_1}$

$F_0^*, F_1^*: H^k(N) \to H^k(M)$ coincide

(2) If $F: M \to N$ is homotopy equivalence $\to \exists\ G: N \to M$ s.t.

⇒ $F^*: H^k(N) \to H^k(M)$ are isomorphisms.

s.t. $\begin{cases} F\circ G \sim Id_N \\ G\circ F \sim Id_M \end{cases}$

(Say M, N are $\underbrace{\text{homotopic equivalent}}_{\text{h.e.}}$ if ∃ $F: M \to N$ homotopy equivalence)

Proof: (1) $F(x,0) = F_0(x)$ means $F\circ i_0 = F_0$
$F(x,1) = F_1(x)$ — " — $F\circ i_1 = F_1$

$\left.\right\}$ $F_0^* = (F\circ i_0)^* = i_0^* \circ F^*$
$F_1^* = (F\circ i_1)^* = i_1^* \circ F^*$

(2) $\begin{cases} F^*\circ G^* = (G\circ F)^* = Id^* = Id \text{ in cohom. } H^k(M) \\ G^*\circ F^* = (F\circ G)^* = Id^* = Id \text{ in coh } H^k(N) \end{cases}$ ☑

Ex1: $M = \mathbb{R}^m$, $N = \mathbb{R}^0 = \{0\}$ are homotopic equivalent

$$\mathbb{R}^m \underset{G}{\overset{F}{\rightleftarrows}} \{0\} \qquad F(x) = 0$$

$F\circ G: \{0\} \to \{0\}$, $0 \mapsto 0$

finite dimensional one defines the Euler characteristic of M as $\boxed{\chi(M) := \sum_k (-1)^k \dim(H^k(M))}$

Thm (Poincaré Lemma) $H^k(\mathbb{R}^m) = \begin{cases} \mathbb{R} & k=0 \\ 0 & k \neq 0 \end{cases}$

(2) If $F: M \to N$ is homotopy equivalence $\Rightarrow$ $F^*: H^k(N) \to H^k(M)$ are isomorphisms

(Say M, N are homotopic equivalent (h.e.) if $\exists\, F: M \to$

Proof: (1) $F(x,0) = F_0(x)$ means $F \circ i_0 = F_0$) $F_0^* = ($

Thm: S^m has a nowhere vanishing vector field $\Longleftrightarrow m = \text{odd}$ (7)

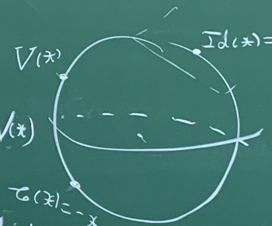
Pf: "$\Leftarrow$"

"$\Rightarrow$" Assume $\exists\, V: S^m \longrightarrow \mathbb{R}^{m+1} \setminus \{0\}$ s.t. $V(p) \perp p$ $(\forall)\, p \in S^m$

May assume $\|V(p)\| = 1$ $(\forall)\, p$ ($\|\cdot\|$ = Euclidean norm).

Claim: Id and $\underbrace{-\mathrm{Id}}_{\tau}: S^m \longrightarrow S^m$ are homotopic maps.

Pf: Take $\begin{cases} H: S^m \times \mathbb{R} \to S^m \\ H(x,t) = (\cos \pi t) \cdot x + (\sin \pi t) \cdot V(x) \end{cases}$



$\Downarrow$

$\tau^*: H^m(S^m) \longrightarrow H^m(S^m)$ is the identity

But $\begin{cases} \mu = \sum_{i=0}^{m} (-1)^i x_i \, dx_0 \ldots \widehat{dx_i} \ldots dx_m \\ \tau^*(\mu) = (-1)^{m+1} \mu \end{cases}$

$\left.\begin{array}{l} [\mu] = (-1)^{m+1} [\mu] \\ \text{But } [\mu] \neq 0 \end{array}\right\} \Rightarrow 1 = (-1)^{m+1} \Rightarrow m = \text{odd}$