(1)

Last time:

- smooth maps $f : \Omega \to \mathbb{R}^k$, with $\Omega \subset \mathbb{R}^n$ open, or, more generally, $f : M \to \mathbb{R}^k$, with $M \subset \mathbb{R}^L$
- differentials $(df)_p : \mathbb{R}^n \to \mathbb{R}^k$, $p \in \Omega$ — " —, $(df)_p : T_p^{geom} M \to \mathbb{R}^k$, $p \in M$

$$T_p^{geom} M = \left\{ \frac{d\gamma}{dt}(0) \;\middle/\; \gamma = \text{smooth curve in } M \text{ with } \gamma(0) = p \right\} \subset \mathbb{R}^L$$

- diffeomorphisms, immersions, submersions (on opens $\subseteq$ Euclidean spaces)

locally $(t_1, \dots, t_n) \mapsto (t_1, \dots, t_n, 0, \dots, 0)$ . . .

- (smooth) charts for $\mathbb{R}^n$: (U, x) with $U \subseteq \mathbb{R}^n$ open (domain of the chart)

$$x = (x_1, \dots, x_n) : U \longrightarrow x(U) \quad \text{diffeomorphism}$$

$p \in U$, $x(U) \subset \mathbb{R}^n$ open

... change coordinates ...

$\longrightarrow \exists$ po

$\longrightarrow \exists$ e

which is

$\longrightarrow \exists$ s

(U, x), $U \subset M$ open

equivalent

$m \in \mathbb{N}$

eomorphism

map

$\mathbb{R}^L$

immersion

$g(x) = 0\}$

d P i.e.

②

SORRY . . .

① Curves ...

② Surfaces — " —

$m = 2$

$\mathbb{R}^2 \{ z = 0.$

t, s

e.g. $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$

For general M ~~$\subseteq \mathbb{R}^L$~~, $p \in M$, the following (3) are equivalent

$\boxed{m \in \mathbb{N}}$

$\rightarrow \exists$ parametrization of M around p, i.e.

$$\text{par}: \Omega \longrightarrow U \ni p, \qquad \Omega \subseteq_{\text{open}} \mathbb{R}^m, \qquad U \subseteq_{\text{open}} M \ \cancel{\subseteq \mathbb{R}^L}$$

homeomorphism

as a map $\Omega \longrightarrow \mathbb{R}^L$ is an immersion

$\rightarrow \exists$ equation for M around p, i.e.

$$\text{eq}: \tilde{U} \longrightarrow \mathbb{R}^{L-m}, \qquad \tilde{U} \subseteq_{\text{open}} \mathbb{R}^m$$

which is submersion and s.t. $M \cap \tilde{U} = \{ x \in \tilde{U} \,/\, \text{eq}(x) = 0 \}$

$\rightarrow \exists$ smooth m-dimensional chart for M around p i.e.

(U, x), $U \subseteq_{\text{open}} M$

$$x = (x_1, \dots, x_m): U \longrightarrow x(U) = \Omega, \qquad U \subseteq_{\text{open}} M, \qquad \Omega \subseteq_{\text{open}} \mathbb{R}^m$$

which is a diffeomorphism.

Rk: Nicest "type of charts": when we have a chart in ④

$\tilde{x} : \tilde{U} \longrightarrow \tilde{\Omega}$ for the entire $\mathbb{R}^L$

$\tilde{U} \subseteq \mathbb{R}^L$, $\tilde{\Omega} \subseteq \mathbb{R}^L$

$(\tilde{U}, \tilde{x}) =$ = charts of $\mathbb{R}^L$ addapted to M

s.t., when restricted to M:

$$\underbrace{\tilde{U} \cap M}_{U} \xrightarrow{\underbrace{\tilde{x}|_U}_{x}} \underbrace{\tilde{\Omega} \cap (\mathbb{R}^m \times \{0\})}_{\Omega \subseteq_{open} \mathbb{R}^m}$$

$\Rightarrow (U, x)$ chart for M around p.

Def: A smooth m-dimensional submanifold of $\mathbb{R}^L$ is any subset $M \subseteq \mathbb{R}^L$ s.t., $(\forall)\, p \in M$, the previous conditions hold true.

4

...nture $\mathbb{R}^L$

...bset

... hold true.

$f:\mathbb{R}\to\mathbb{R}^2,\ f(x)=(x,0)$ 5 $f(x)=(x^2,0)$

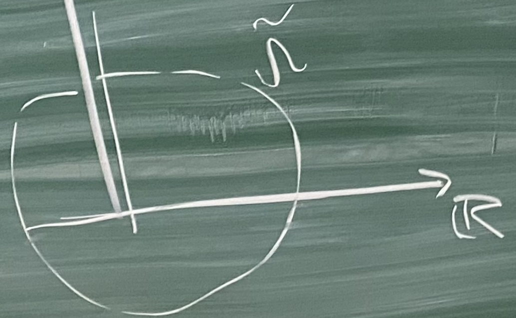
$$\begin{array}{ccc} (-\frac{\pi}{2},\frac{\pi}{2}) & \longrightarrow & \mathbb{R}^2, \quad t\longmapsto (tg(t),0) \\ \wr\big\downarrow tg & \nearrow f & \\ \mathbb{R} & & \end{array}$$

$(2x \quad 0)$

$f:\mathbb{R}\setminus\{0\}\to\mathbb{R}^2$

$L=2,\ m=1$

$$\mathbb{R}^m\times\underset{\mathbb{R}^{L-m}}{\{0\}}=\mathbb{R}^m\times\{\underbrace{(0,\dots,0)}_{L-m}\}$$



quivalent

$m \in \mathbb{N}$

orphism

map

$\mathbb{R}^L$

mmersion

$x) = 0$

p i.e.

$\boxed{m \in \mathbb{N}}$ ⑥ ABSTRACT MANIFOLDS (Chp II)

$\boxed{\text{Start with } M = \text{topological space}}$ (... set)

Q: what "extra-input" to given on M so that I can talk about "Smoothness in M" (e.g. $\gamma : (-\varepsilon, \varepsilon) \to M$ or $f : M \to \mathbb{R}$ being smooth)

... smooth structure

- an m-dimensional chart for M: a home~~omorphism~~ bijection

$$\chi = (\chi_1, \ldots, \chi_m) : \underset{\underset{M}{\cap \text{open}}}{U} \longrightarrow \underset{\underset{\mathbb{R}^m}{\cap \text{ open}}}{\chi(U)}$$

$\underset{U \text{ open}}{M}$

Call $(U, \chi) = (U, \chi_1, \ldots, \chi_m)$ a chart, U = domain of the chart

- given two charts (U, χ), (U', χ'), the change of coordinates (from χ' to χ) map is:

$$C_\chi^{\chi'} := \chi' \circ \chi^{-1} : \chi(U \cap U') \longrightarrow \chi'(U \cap U')$$

$\subseteq \mathbb{R}^m$ $\supset$ open

$$\left(C_\chi^{\chi'} = (C_1, \ldots, C_m), \text{ then } \chi'_i(p) = C_i(\chi_1(p), \ldots, \chi_m(p)) \right)$$

$x=(x_1,\dots,x_m): U \longrightarrow x(U)$ bijecti…

(7)

- say that (U,x) & (U',x') are <u>smoothly compatible</u> if $C_x^{x'}$ is a diffeomorphism.

- an m-dimensional <u>smooth atlas on M</u>: a collection $\mathcal{A}$ of m-dimensional charts of M s.t.

  1. each $p \in M$ is in the domain of at least one chart from $\mathcal{A}$;
  2. any two (U,x), (U',x') from $\mathcal{A}$ are smoothly compatible.

- in principle, a smooth structure on M is provided by a smooth atlas.

(4)

d tral.

a diffeom...

locally $(t_1,\dots,t_n) \mapsto (t_1,\dots,t_n,0,\dots,0)$

(8)

<u>Ex</u>: $M = \mathbb{R}^m$: some different atlases:

$\mathcal{A}_{\mathbb{R}^m} = \{ \mathrm{Id}_{\mathbb{R}^m} \}$ (only one chart: $\mathrm{Id}_{\mathbb{R}^m} : \underset{\mathbb{R}^m}{\overset{U}{\shortparallel}} \to \mathbb{R}^m$).

$\mathcal{A}'_{\mathbb{R}^m} = \{ \mathrm{Id}_\Omega \,/\, \Omega \subseteq \mathbb{R}^m \text{ open} \}$.

$\mathcal{A}^{max}_{\mathbb{R}^m}$ = all the smooth charts of $\mathbb{R}^m$ in the sense of Chp I.

<u>Ex</u>: $M \subseteq \mathbb{R}^L$ is a smooth submanifold:

$\mathcal{A}^{max}_M$ = the collection of all smooth charts of M —— " ——

$\mathcal{A}^{adapt}_M$ = —— " —— induced by an adapted chart of $\mathbb{R}^L$.

<u>Rk</u>: Once we have fixed a smooth atlas $\mathcal{A}$ on $M \Longrightarrow$ we can talk about smoothness (w.r.t. $\mathcal{A}$) of functions $f : M \to \mathbb{R}$:

Use $(\forall)(U,\chi) \in \mathcal{A}$ and require: $f_\chi := f \circ \chi^{-1} : \underset{\mathbb{R}^m}{\overset{\chi(U)}{\cap \text{open}}} \to \mathbb{R}$ to be smooth. its Ch I

$\Rightarrow \mathcal{C}^\infty(M,\mathcal{A}) = \{ f : M \to \mathbb{R} \,/\, f = \text{smooth w.r.t. } \mathcal{A} \}$.

(9)

$C^\infty(\mathbb{R}^m, \mathcal{A}^{\max}_{\mathbb{R}^m}) = C^\infty(\mathbb{R}^m)$ (usual sense)

Ex: $C^\infty(\mathbb{R}^m, \mathcal{A}_{\mathbb{R}^m}) = C^\infty(\mathbb{R}^m, \mathcal{A}'_{\mathbb{R}^m}) = C^\infty(\mathbb{R}^m, \mathcal{A}^{\max}_{\mathbb{R}^m}) = C^\infty(\mathbb{R}^m)$

$\mathcal{A}_{\mathbb{R}^m}, \mathcal{A}'_{\mathbb{R}^m}, \mathcal{A}^{\max}_{\mathbb{R}^m}$: provide the same "smooth structure".

Def: An m-dim. smooth structure on (a top. space) M is an $\mathcal{A}$ as above which is maximal, in the sense that there is no larger smooth atlas $\tilde{\mathcal{A}}$ containing $\mathcal{A}$.

Ex: $\boxed{\mathcal{A}^{\max}_{\mathbb{R}^m}}$, $\mathcal{A}^{\max}_M$.

$\mathcal{A}^{\max} :=$ the collection of all charts (U, x) of M, that are smoothly compatible with all charts in $\mathcal{A}$

Rk: $\mathcal{A}$ is maximal $\Leftrightarrow$ ($\mathcal{A}^{\max}$ ⊆ …) must be in $\mathcal{A}$.

We have:

- $\mathcal{A}$ is maximal $\Leftrightarrow \mathcal{A}^{\max} = \mathcal{A}$
- $\mathcal{A}^{\max}$ is indeed a smooth atlas, and is, indeed maximal. i.e. $\mathcal{A}^{\max}$ is a smooth structure on M.

$\Rightarrow$ each smooth atlas $\mathcal{A}$ provides / gives rise to / induces a smooth structure on