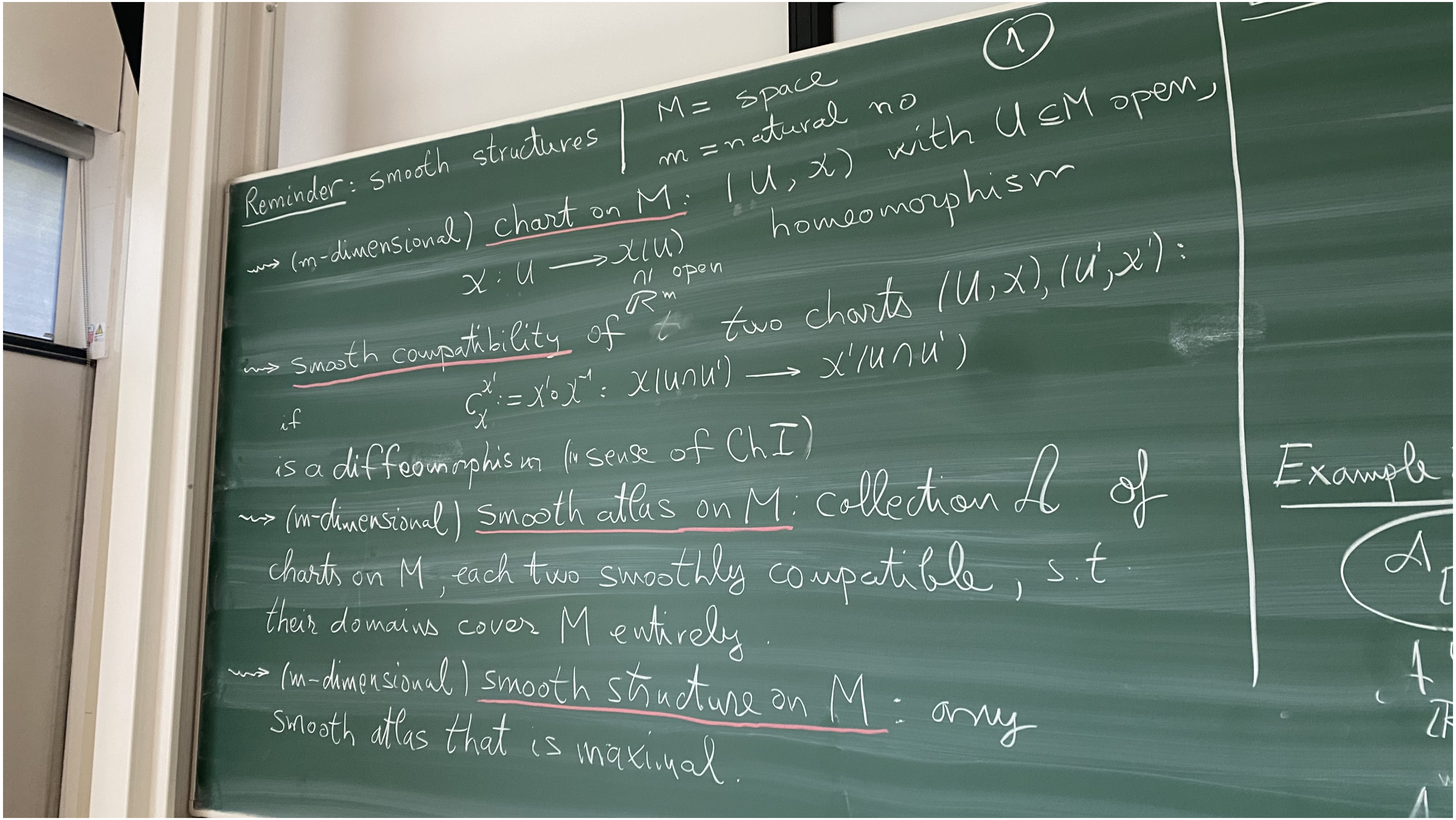


①

Reminder: smooth structures | M = space
m = natural no

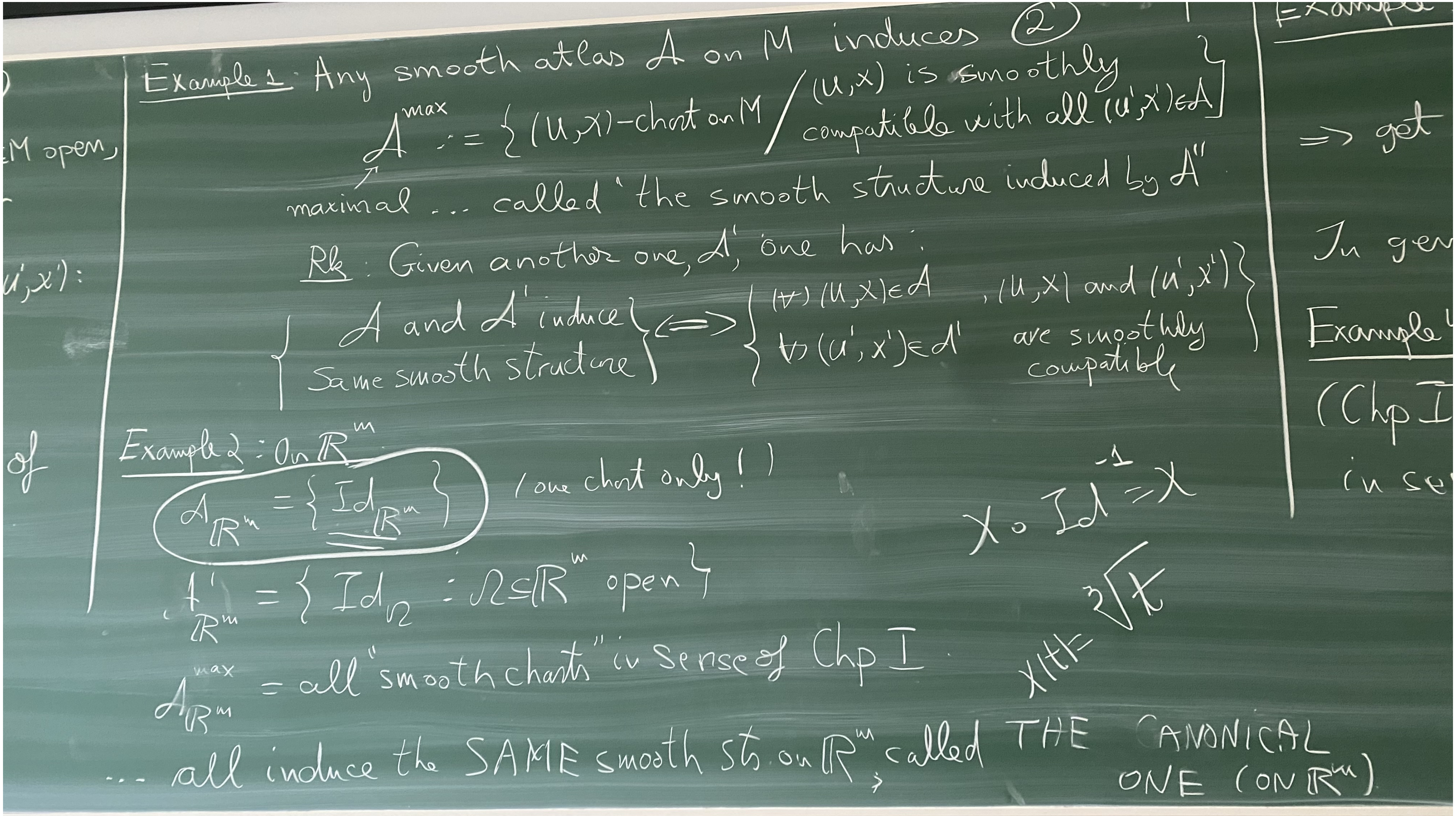
⇝ (m-dimensional) **chart on M**: (U, x) with $U \subseteq M$ open,
$x : U \longrightarrow x(U)$ homeomorphism
$x(U) \subseteq \mathbb{R}^m$ open

⇝ **smooth compatibility** of two charts $(U, x), (U', x')$:
if $c_x^{x'} := x' \circ x^{-1} : x(U \cap U') \longrightarrow x'(U \cap U')$
is a diffeomorphism (in sense of Ch I)

⇝ (m-dimensional) **smooth atlas on M**: collection $\mathcal{A}$ of charts on M, each two smoothly compatible, s.t. their domains cover M entirely.

⇝ (m-dimensional) **smooth structure on M**: any smooth atlas that is maximal.

Example



$\in M$ open,

(u',x'):

of

Example 1: Any smooth atlas $\mathcal{A}$ on M induces ②

$$\mathcal{A}^{max} := \left\{ (U,x)\text{-chart on } M \;/\; \begin{matrix}(U,x) \text{ is smoothly} \\ \text{compatible with all } (U',x')\in\mathcal{A}\end{matrix} \right\}$$

$\nearrow$ maximal ... called "the smooth structure induced by $\mathcal{A}$".

Rk: Given another one, $\mathcal{A}'$, one has:

$$\left\{\begin{matrix}\mathcal{A} \text{ and } \mathcal{A}' \text{ induce} \\ \text{same smooth structure}\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}(\forall)\,(U,x)\in\mathcal{A} & ,\,(U,x) \text{ and } (U',x') \\ \forall\,(U',x')\in\mathcal{A}' & \text{are smoothly compatible}\end{matrix}\right\}$$

Example 2: On $\mathbb{R}^m$

$$\boxed{\mathcal{A}_{\mathbb{R}^m} = \{ \underline{Id_{\mathbb{R}^m}} \}}$$ (one chart only!)

$x \circ Id^{-1} = x$

$$\mathcal{A}'_{\mathbb{R}^m} = \{ Id_\Omega : \Omega \subseteq \mathbb{R}^m \text{ open} \}$$

$x(t) = \sqrt[3]{t}$

$\mathcal{A}^{max}_{\mathbb{R}^m}$ = all "smooth charts" in sense of Chp I.

... all induce the SAME smooth str. on $\mathbb{R}^m$, called THE CANONICAL ONE (ON $\mathbb{R}^m$).

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<u>Example 3</u>: on $\mathbb{R}^m$, given ③ any homeomorphism

$$\chi : \mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^m \quad \underline{\text{possibly non-smooth}}$$

$\Rightarrow$ get another atlas (smooth!)

$$\boxed{A_\chi = \{\underline{\underline{\chi}}\}}$$ (one chart only!)

In general, this does not induce the canonical smooth str!

<u>Example 4</u>. For smooth submanifolds $M \subseteq \mathbb{R}^L$ (Chp I) the collection of all "smooth charts" in sense of Chp I $\Rightarrow$ maximal atlas (smooth str) on M.

↳ called THE CANONICAL SMOOTH STRUCTURE ON M.

$= \phi$

table.

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④

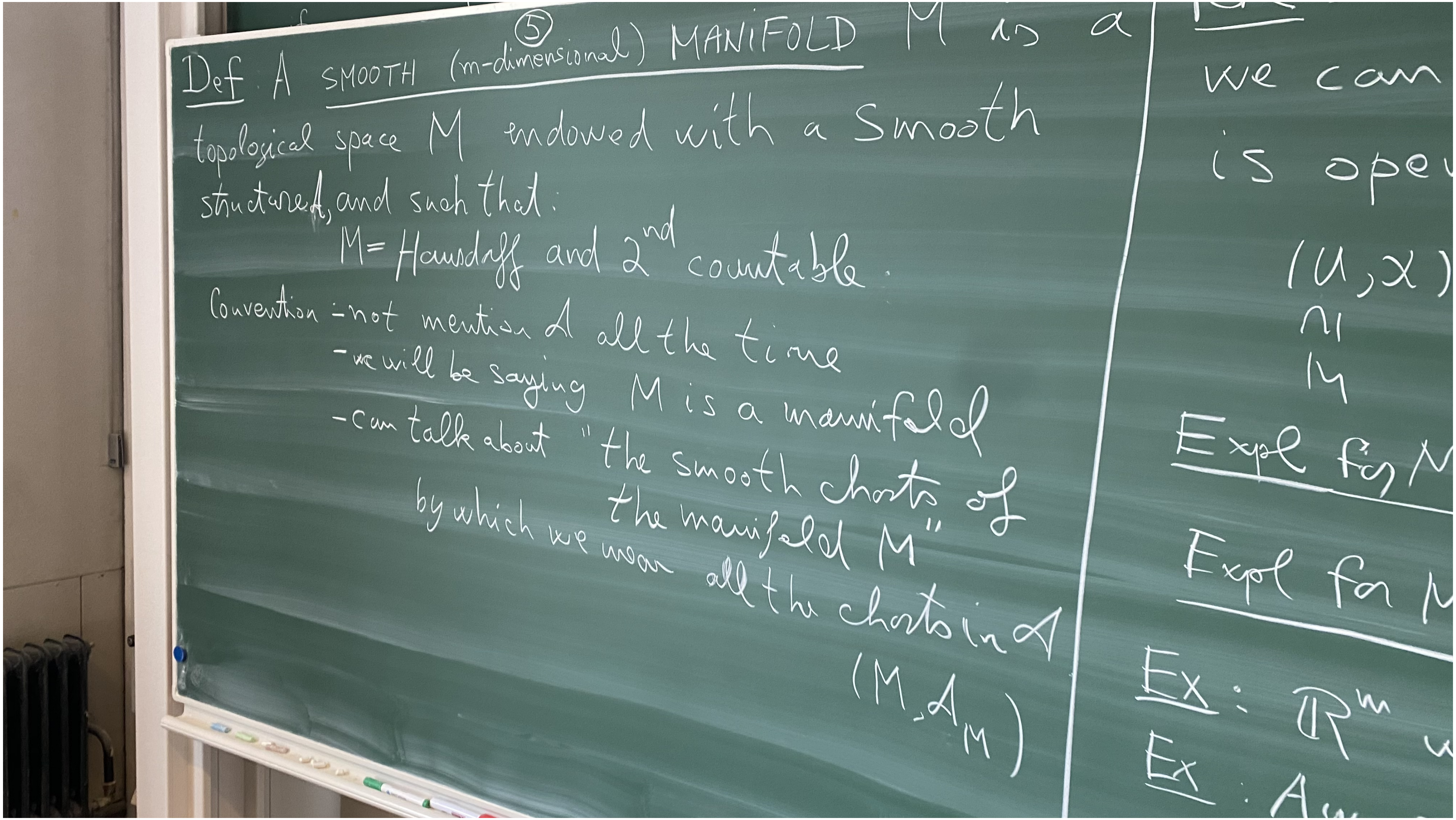
ANALOGY with: Topology $\mathcal{T}$ on M: collection of subsets of M . . . .

Topological space: $(\underset{\uparrow\ \text{set}}{M}, \underset{\uparrow\ \text{topology}}{\mathcal{T}})$; the members of $\mathcal{T}$ are called the opens of the space $(M, \mathcal{T})$

Convention is: never mention $\mathcal{T}$

- say $M =$ (topological) space

→ allowed to talk about: the opens of the space M



⑤

Def. A SMOOTH (m-dimensional) MANIFOLD M is a topological space M endowed with a smooth structure $\mathcal{A}$, and such that:

$M =$ Hausdorff and 2nd countable.

Convention: - not mention $\mathcal{A}$ all the time

- we will be saying M is a manifold
- can talk about "the smooth charts of the manifold M" by which we mean all the charts in $\mathcal{A}$

$(M, \mathcal{A}_M)$

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Ex: $\mathbb{R}^m$ w

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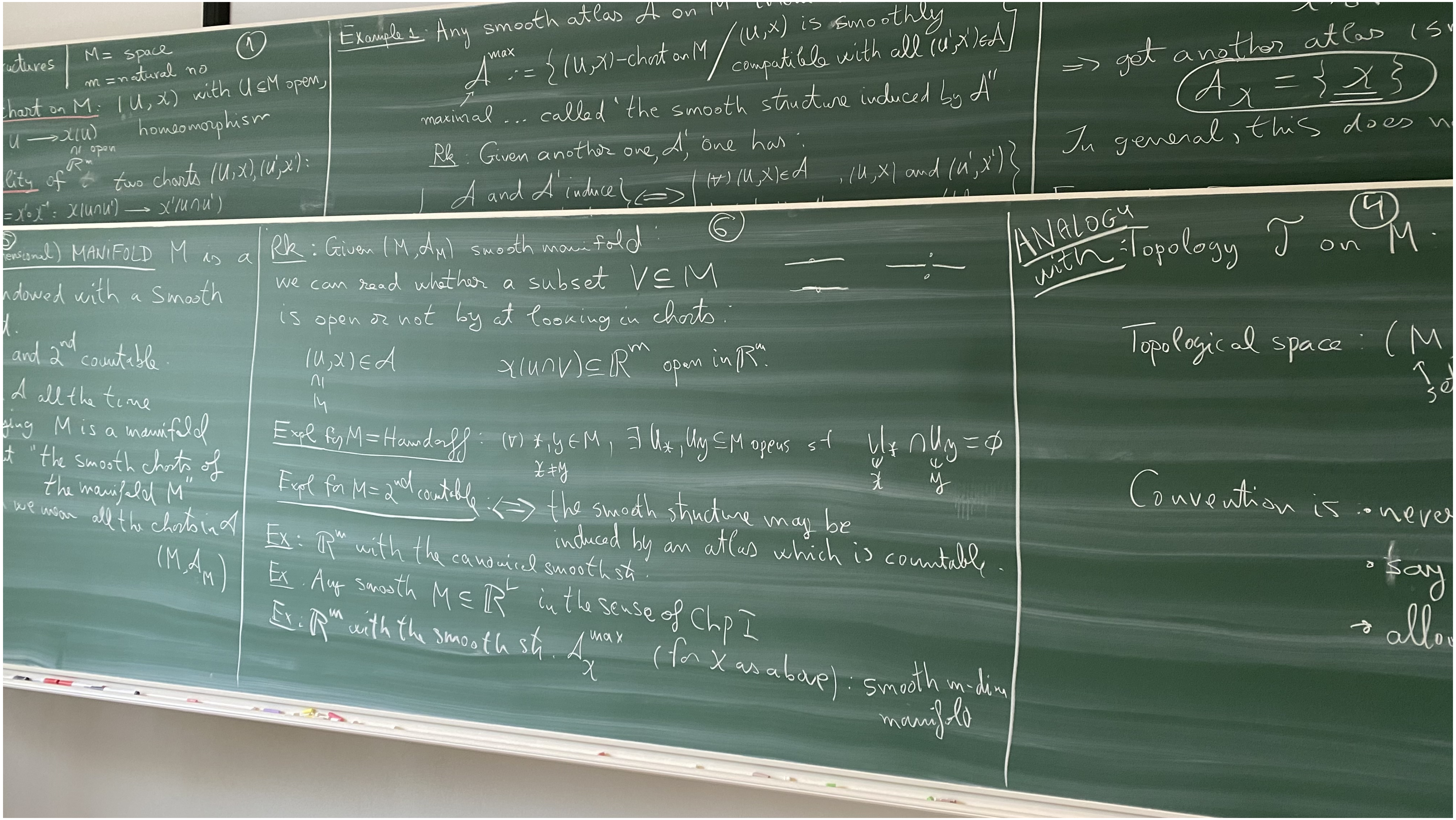

①

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m = natural no

hart on M: (U, x) with $U \subseteq M$ open,

$U \longrightarrow x(U)$ homeomorphism

$\underset{\mathbb{R}^m}{\cap}$ open

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$= x' \circ x^{-1} : x(U \cap U') \longrightarrow x'(U \cap U')$

Example 1: Any smooth atlas $\mathcal{A}$ on M

$\mathcal{A}^{max} := \{ (U,x)\text{-chart on } M \,/\, (U,x) \text{ is smoothly compatible with all } (U',x') \in \mathcal{A} \}$

↗ maximal ... called 'the smooth structure induced by $\mathcal{A}$"

Rk: Given another one, $\mathcal{A}'$, one has:

$\mathcal{A}$ and $\mathcal{A}'$ induce $\Longleftrightarrow$ $(\forall)\ (U,x) \in \mathcal{A}$, (U,x) and (U',x')

$\Longrightarrow$ get another atlas (sm

$$\boxed{\mathcal{A}_x = \{ \underline{\underline{x}} \}}$$

In general, this does n

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we mean all the charts in $\mathcal{A}$

$(M, \mathcal{A}_M)$

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Rk: Given $(M, \mathcal{A}_M)$ smooth manifold:
we can read whether a subset $V \subseteq M$
is open or not by at looking in charts:

$(U,x) \in \mathcal{A}$ $\quad$ $x(U \cap V) \subseteq \mathbb{R}^m$ open in $\mathbb{R}^m$.
$\underset{M}{\cap}$

Expl for M = Hausdorff: $(\forall)\ x, y \in M$, $x \neq y$, $\exists\ U_x, U_y \subseteq M$ opens s.t. $\underset{x}{U_x} \cap \underset{y}{U_y} = \emptyset$

Expl for M = 2^{nd} countable: $\Longleftrightarrow$ the smooth structure may be induced by an atlas which is countable.

Ex: $\mathbb{R}^m$ with the canonical smooth st.

Ex. Any smooth $M \subseteq \mathbb{R}^L$ in the sense of Chp I

Ex: $\mathbb{R}^m$ with the smooth st. $\mathcal{A}_x^{max}$ (for x as above): smooth m-dim manifold

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ANALOGY with: Topology $\mathcal{T}$ on M:

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⇝ (m-dimensional) smooth atlas on M.

Def A SMOOTH (m-dimensional) MANIFOLD ⑦ M is a topological space M endowed with a smooth structure $\mathcal{A}$, and such that:

$M =$ Hausdorff and 2^{nd} countable.

Ex: $\mathbb{R}^m$ together with the canonical smooth structure or, more generally, any $M \subseteq \mathbb{R}^L$ smooth submanifold in sense of Ch I

(i.e.: any "concrete manifold" is also an "abstract manifold").

Ex. For any $\mathcal{X}$ as above $\Rightarrow (\mathbb{R}^m, \mathcal{A}^{max}_{\mathcal{X}}) =$ manifold.

[Q: Given a top. space M, how many non-diffeomorphic smooth structures on M can one find? ? ?

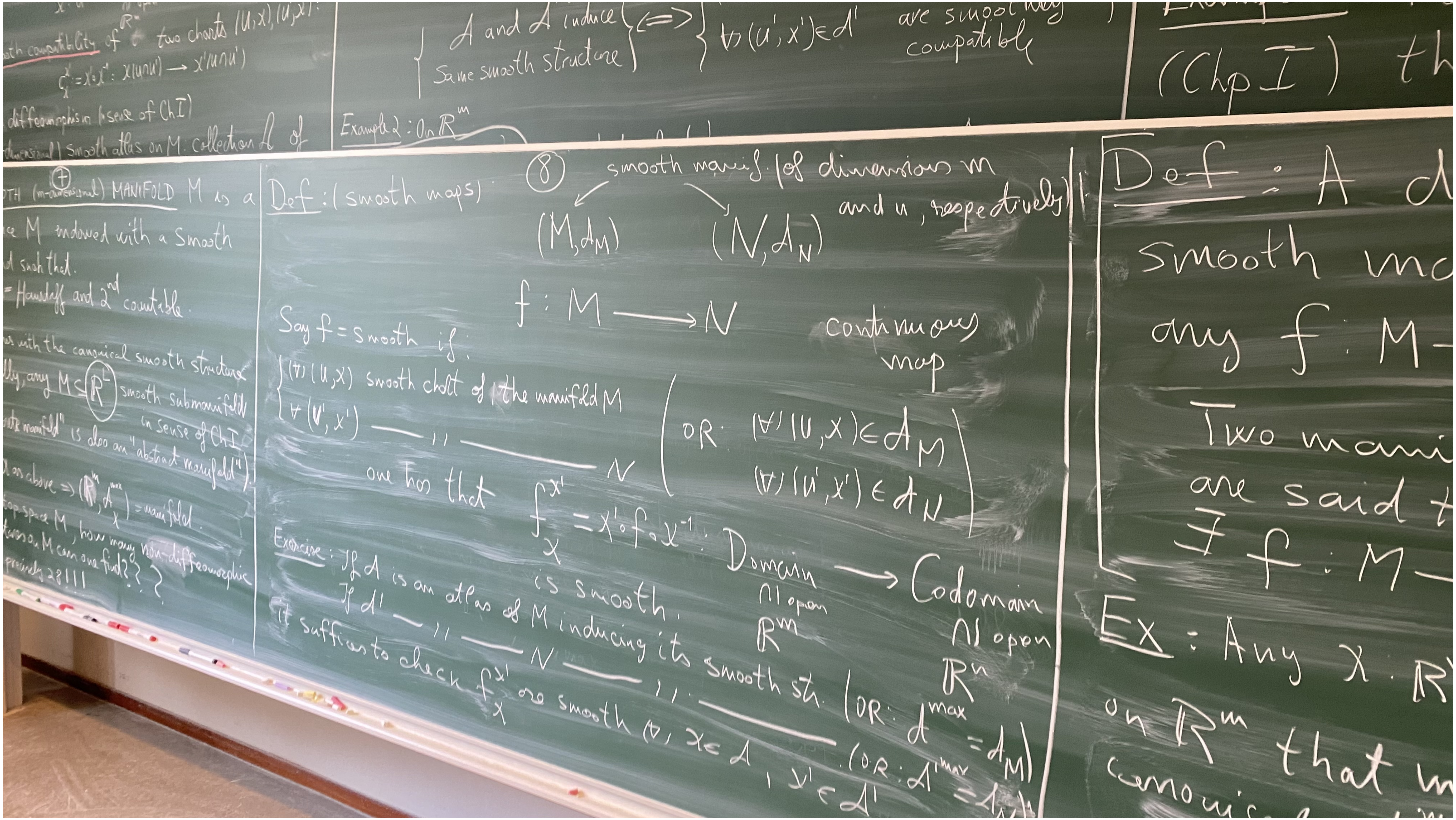
S^7: precisely 28!!!!

Def: (s

Say $f =$ s

$\begin{cases} (\forall) (U, x) \\ \forall (V', x') \end{cases}$

Exercise: $\exists \mathcal{L}$ o



oth compatibility of two charts $(U,x),(U',x')$:

$c^{x'}_{x} = x'\circ x^{-1} : x(U\cap U') \longrightarrow x'(U\cap U')$

diffeomorphism (in sense of Ch I)

dimensional) smooth atlas on M: collection $\mathcal{A}$ of

$\left\{ \begin{array}{l} \mathcal{A} \text{ and } \mathcal{A}' \text{ induce} \\ \text{same smooth structure} \end{array} \right\} \Longleftrightarrow \Big\{ \forall (U',x')\in \mathcal{A}'$ are smooth[ly]

compatible)

Example 2: On $\mathbb{R}^m$

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⑦

...TH (m-dimensional) MANIFOLD M is a

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= Hausdorff and 2^{nd} countable.

...s with the canonical smooth structure

...lly, any $M \subseteq \mathbb{R}^L$ smooth submanifold in sense of Ch I

...ate manifold" is also an "abstract manifold").

...l as above $\Rightarrow (\mathbb{R}^m, \mathcal{A}^{max}_{x})$ = manifold.

...top. space M, how many non-diffeomorphic

...tures on M can one find?? ?

...precisely 28!!!

Def: (smooth maps) ⑧ smooth manif. (of dimension m and n, respectively)

$(M, \mathcal{A}_M)$ $(N, \mathcal{A}_N)$

$f : M \longrightarrow N$ continuous map

Say f = smooth if:

$\left\{ \begin{array}{l} (\forall)\,(U,x) \text{ smooth chart of the manifold } M \\ \forall\,(U',x') \text{ —— '' —— } N \end{array} \right.$ $\left(\begin{array}{l} \text{OR: } (\forall)\,(U,x)\in \mathcal{A}_M \\ (\forall)\,(U',x')\in \mathcal{A}_N \end{array} \right)$

one has that $f^{x'}_{x} = x'\circ f \circ x^{-1}$: Domain $\longrightarrow$ Codomain

is smooth. $\underset{\text{open}}{\cap}$ $\mathbb{R}^m$ $\underset{\text{open}}{\cap}$ $\mathbb{R}^n$

Exercise: If $\mathcal{A}$ is an atlas of M inducing its smooth str. (OR: $\mathcal{A}^{max} = \mathcal{A}_M$)

If $\mathcal{A}'$ —— '' —— N —— '' —— (OR: $\mathcal{A}'^{max} = \mathcal{A}_N$)

it suffices to check $f^{x'}_{x}$ are smooth $(\forall)\ x\in\mathcal{A}$, $x'\in\mathcal{A}'$

Def: A d

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Ex: Any x: $\mathbb{R}$

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(Chp I) the collection of all (smooth) charts

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R: $A^{max} = A_N$)

Def: A diffeomorphism ⑨ between two smooth manifolds M and N is any $f : M \to N$, bijective, with f both f^{-1} smooth.

Two manifold M and N are said to be diffeomorphic if $\exists\, f : M \to N$ diffeomorphism.

Ex: Any $\chi : \mathbb{R}^m \to \mathbb{R}^m$ as above $\Rightarrow$ a smooth str A^{max}_{χ} on $\mathbb{R}^m$ that may be different than the canonical $A^{max}_{Id} = A^{max}_{\mathbb{R}^m}$. However one has a diffeom

$$\chi^{-1} = f : \left(\mathbb{R}^m, \underset{\{Id\}}{A^{max}_{\mathbb{R}^m}}\right) \xrightarrow[\chi \circ f = Id]{} \left(\mathbb{R}^m, \underset{\{\chi\}}{A^{max}_{\chi}}\right)$$

⇝ Smooth comp

⑩

Particular cases/classes of smooth maps:

- diffeomorphisms
- $M = I$ interval $\subseteq \mathbb{R}$ $\Rightarrow$ smooth maps $\gamma : I \longrightarrow N$ ← any manifold

  called "smooth curves" in N
- $N = \mathbb{R}$ $\Rightarrow$ smooth $\mathbb{R}$-valued maps $f : M \to \mathbb{R}$

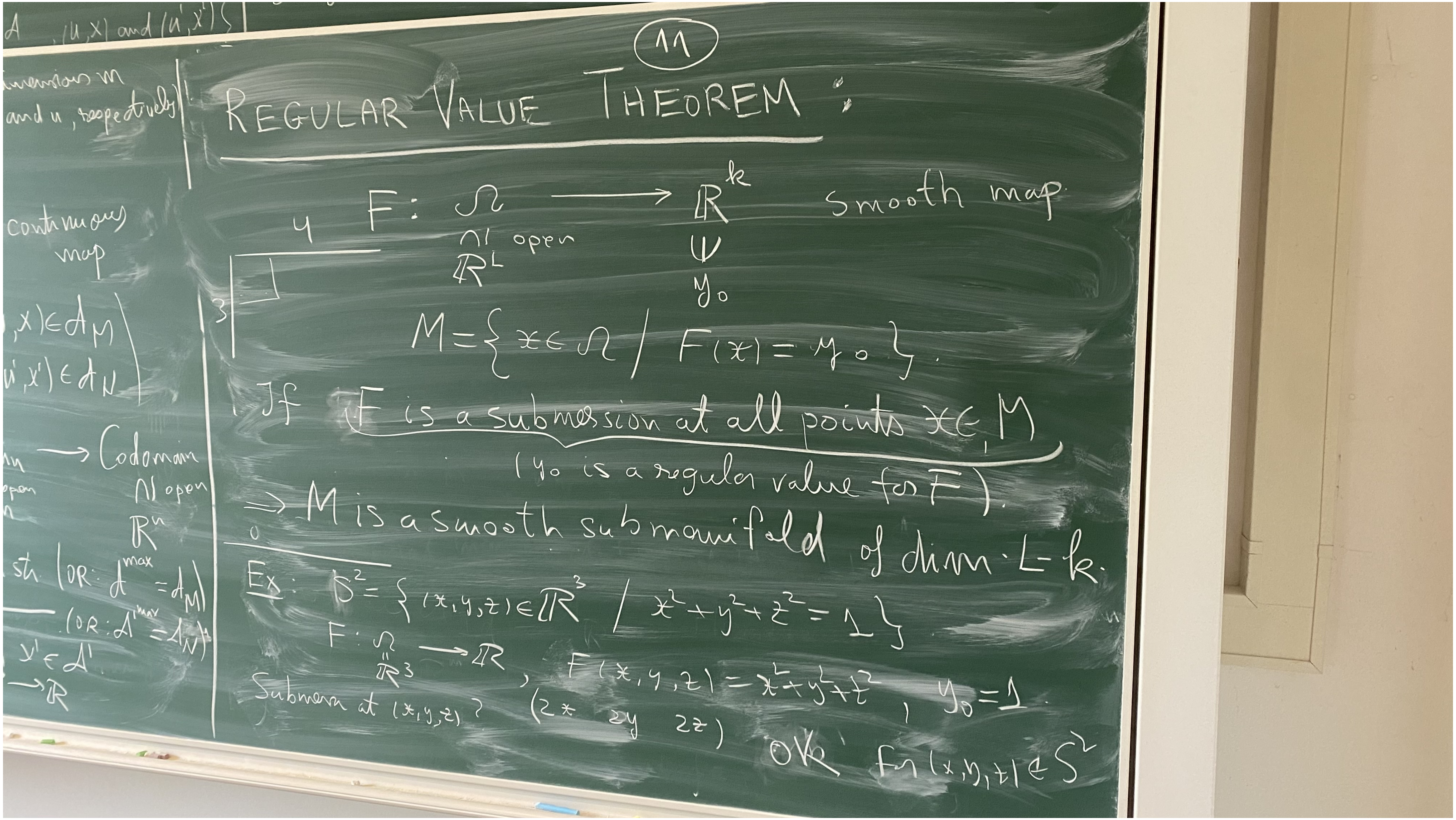
  ("smooth observable")
- $f : M \to N$ called immersion (or submersion) if all $f_x^{x'}$ (as in Def 1) are immersion (or submersion)).

Def I.

Say f

$\begin{cases} (\forall)\,(U, x \\ \forall\,(V', x \end{cases}$

Exercise:



A ,(u,x) and (u',x') {

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and n, respectively)

continuous
map

,x)∈ A_M
u',x') ∈ A_N

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open ∩| open
n $\mathbb{R}^n$

sth (OR: $A^{max}=A_M$)
(OR: $A'^{max}=A_N$)
$y'\in A'$
$\to\mathbb{R}$

(11)

REGULAR VALUE THEOREM:

$$F:\underset{\underset{\mathbb{R}^L}{\cap|\text{ open}}}{\Omega}\longrightarrow\underset{\underset{y_0}{\Psi}}{\mathbb{R}^k}\quad\text{smooth map}$$

$$M=\{x\in\Omega \,/\, F(x)=y_0\}.$$

If $\underbrace{F\text{ is a submersion at all points }x\in M}_{}$

(y_0 is a regular value for F).

$\Rightarrow$ M is a smooth submanifold of dim. $L-k$.

Ex: $S^2=\{(x,y,z)\in\mathbb{R}^3 \,/\, x^2+y^2+z^2=1\}$

$F:\underset{\underset{\mathbb{R}^3}{\parallel}}{\Omega}\longrightarrow\mathbb{R}$, $F(x,y,z)=x^2+y^2+z^2$, $y_0=1$.

Submers at (x,y,z)? $(2x\quad 2y\quad 2z)$ OK for $(x,y,z)\in S^2$