Reminder: smooth m-dimensional manifold M : ① the manifold $(M, \mathcal{A}_M)$ / M

a space M that is Hausdorff and 2^{nd} ~~countable~~

& an (m-dimensional) smooth structure $\mathcal{A}_M$ on M

i.e. smooth) maximal atlas

Members of $\mathcal{A}_M$: the "smooth charts" of the manifold $(M, \mathcal{A}_M)$ / M

Rk: In practice, to have a manifold one needs:

- a space (... or set) M
- exhibit a smooth atlas $\mathcal{A}$, hopefully finite, or countable

(... - make sure Hausdorfness).

$\Rightarrow$ the smooth structure $\mathcal{A}_M := \mathcal{A}^{max}$

(hence "smooth charts" of this manifold: all charts of M that are smoothly compatible with all charts from $\mathcal{A}$)

require $f = co$

Rk: If $\mathcal{A}_M = \mathcal{A}$

Example 0: $\mathbb{R}^m$ w

Same for $\Omega \subseteq \mathbb{R}$

Example 1: Smo

in

how?? ... Reg

① ... e manifold ... $(M, \mathcal{A}_M)$ ... $\mathcal{A}_M)$. ... from $\mathcal{A}$)

② **Smoothness of maps** $f:(M,\mathcal{A}_M)\longrightarrow(N,\mathcal{A}_N)$.

require f = continuous and

$$f_x^{x'} = x'\circ f\circ x^{-1} \text{ – smooth in sense of I} \quad \forall \begin{cases} x\in\mathcal{A}_M \\ x'\in\mathcal{A}_N \end{cases} \qquad (*)$$

Rk: If $\mathcal{A}_M=\mathcal{A}^{max}$, $\mathcal{A}_N=\mathcal{B}^{max}$: suffices for $\begin{cases} x\in\mathcal{A} \\ x'\in\mathcal{B} \end{cases}$.

Example 0: $\mathbb{R}^m$ with "usual smooth charts": can use $\mathcal{A}=\{Id_{\mathbb{R}^m}\}$

Same for $\Omega\subseteq\mathbb{R}^m$ open: use $\{Id_\Omega\}$

Example 1: Smooth submanifolds in sense of Chp I $\underline{\underline{M}}\subseteq\mathbb{R}^L$

how?? ... Regular value thm: write M by equations:

$$M=\{\underline{\underline{x}} \;/\; F(x)=y_0\} \text{ where } F: \underset{\substack{\cap \text{ open} \\ \mathbb{R}^L}}{\text{Domain}} \longrightarrow \mathbb{R}^k \text{ smooth}$$

s.t. F = a submersion at all $x\in M$.

③ **Example 2**:

One way:

At $x\in M$ ...

$\Rightarrow S^n$ ...

What are ...

A smooth ...

homeom...

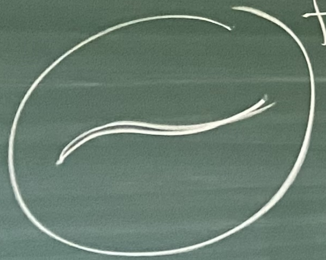
$\mathcal{A}_M$ (*)

$\mathcal{A}_N$

B.

...se $\mathcal{A} = \{ Id_{\mathbb{R}^m} \}$

$\hat{p}$



...ems:

Domain $\longrightarrow \mathbb{R}^k$ smooth

$\cap$| open $\mathbb{R}^l$

Example 2: $\underset{\mathbb{R}^{m+1}}{\overset{S^m}{\cap}} = \{ \underbrace{(x_0, x_1, \dots, x_m)}_{x} \in \mathbb{R}^{m+1} \,/\, (x_0)^2 + \dots + (x_m)^2 = 1 \}$

One way: $F : \mathbb{R}^{m+1} \to \mathbb{R}$, $F(x) = \dots$

$M = F^{-1}(1)$.

At $x \in M$: $\left(\frac{\partial F}{\partial x_0}(x) \; \dots \; \frac{\partial F}{\partial x_n}(x) \right) = \left(2x_0 \; \dots \; 2x_m \right)$

rank (maximal) 1 because $x \neq 0$ for $x \in M$.

$\Rightarrow$ S^m carries the "canonical smooth str"

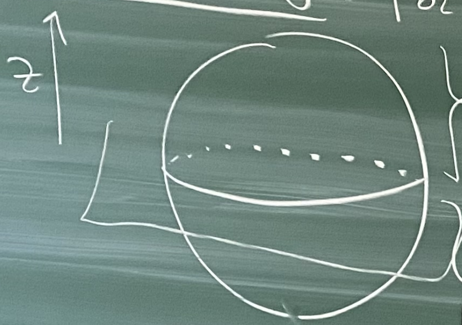
What are the smooth charts for this manifold?

A smooth chart for S^m is any homeomorphism $f : \underset{\cap \text{ open } S^m}{U} \xrightarrow{\sim} \underset{\cap \text{ open } \mathbb{R}^m}{x(U)}$ which is smooth in sense of chp I, i.e. $(\forall)\, x \in U$, $\exists\, \tilde{U}_x \subseteq \mathbb{R}^{m+1}$ neigh, $\exists\, \tilde{f}_x : \tilde{U}_x \to \mathbb{R}^m$ smooth, s.t. $\tilde{f}_x|_{\tilde{U}_x \cap S^m} = f|_{\tilde{U}_x \cap S^m}$

x^{-1} = smooth

4

Another way: for $m=2$.



$U = \{(x,y,z) \in S^m : z>0\}$

$X_U : U \longrightarrow \mathbb{R}^2$, $X_U(x,y,z) = (x,y)$

$V = \{z<0\}$ $\quad (U, X_U)$

X_V $\quad (V, X_V)$

Using y instead of z $\quad {>0 \atop <0}$ $\Rightarrow$ 2 more charts.

or x —"— —"— $\Rightarrow$ —"— —"—

6 charts $\Downarrow$ smooth atlas $\Downarrow$ smooth str.

Better another way: can enlarge a bit U and V to U', V' (and figure out what $X_{U'}$, $X_{V'}$ to take)

Best way: $U = S^2 \setminus \{\underbrace{(0,0,-1)}_{P_S}\} \xrightarrow{X_U} \mathbb{R}^2$ $\Rightarrow$ 2 charts: the stereographic projection

$V = S^2 \setminus \{\underbrace{(0,0,1)}_{P_N}\} \xrightarrow{X_V} \mathbb{R}^2$

Members of ...

Rk: In practice, to have a manifold one needs: ...

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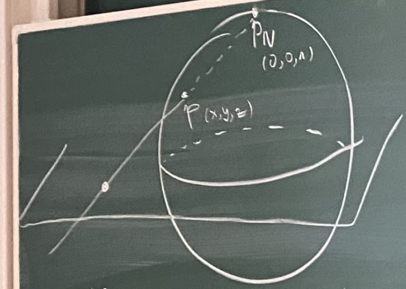


$x_u : S^2 \setminus \{p_N\} \longrightarrow \mathbb{R}^2$

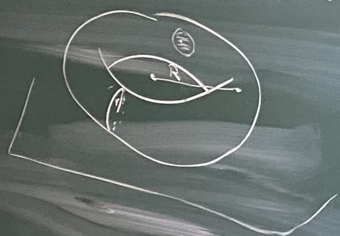
$x_u(p)$ = the intersection of $p_N p$ with the $z=0$ plane

$$x(x,y,z) = \left(\frac{x}{1-z} , \frac{y}{1-z} \right)$$

⇒ atlas made of two charts ⇒ smooth atlas ⇒ smooth str. on S^2.

The same smooth structure!

Example 3: Surfaces, e.g. the 2-torus



use intuition

. . . . . .

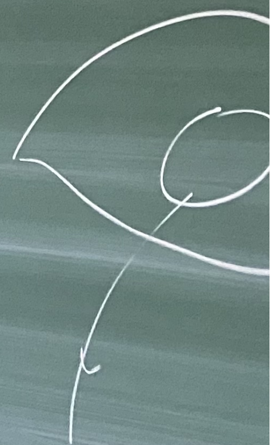
or use regular value thm.

equation for torus ⇒ OK.

same ... work ...

Example 4

$\mathbb{P}^n =$



Example 4

$\dots + \sin(y) + e^{\dots} \to 12x^2 - 42\cos(xy)\}$

$(0,\infty)$

$\{f(x)>0\}$

...al

... line $\mathbb{R}^{n^2}$

$\dots, a_{n1}, \dots, a_{nn})$ ∈

...ly just $\mathbb{R}^{n^2}$)

The (real) projective space (6.1)

$\mathbb{P}^n =$ collection of all lines in $\mathbb{R}^{n+1}$ going through the origin

$$= \{ \ell \subseteq \mathbb{R}^{n+1} \,/\, \ell = 1\text{-dimensional vector subspace} \}$$

For $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$: NOTATION: $\ell_x = \mathbb{R} \cdot x = [x_0 : x_1 : \dots : x_n]$

$$= \{ \lambda \cdot x \,/\, \lambda \in \mathbb{R} \} \subseteq \mathbb{R}^{n+1}$$

$\mathbb{P}^n = \{ \ell_x : x \in \mathbb{R}^{n+1} \setminus \{0\} \}$ where $\ell_x = \ell_y \iff y = \lambda x$ for some $\lambda \in \mathbb{R}^*$

$= \{ \ell_u : u \in S^n \}$ where $\ell_u = \ell_v \iff v = u$ or $v = -u$.

With $[:]$ notation:

$[x_0 : x_1 : \dots : x_n] = [y_0 : y_1 : \dots : y_n] \iff \exists \lambda \in \mathbb{R}^*$ s.t.

$y_0 = \lambda x_0$

$y_1 = \lambda x_1$

Therefore: [...

Moreover: [...

$\iff$

i.e. we ca...

i.e. $\mathbb{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})$ ...

i.e. $\mathbb{P}^n = S^n / \mathbb{Z}_2$

$\implies$ a c...

$_{n\times n} \mid A^T A = Id\}$. (TBC)

6.1

$\mathbb{R}^{m+1}$

...imensional} ...space

$[x_0 : x_1 : \dots : x_m]$

$\subseteq \mathbb{R}^{m+1}$

$= \lambda x$ for some $\lambda \in \mathbb{R}^*$

$v = u$ or $v = -u$

$\Rightarrow \exists \lambda \in \mathbb{R}^*$ s.t.

$y_0 = \lambda x_0$

$y_1 = \lambda x_1$

...

(7)

Therefore:

$$[x_0 : x_1 : \dots : x_m] = \left[1 : \frac{x_1}{x_0} : \dots : \frac{x_m}{x_0}\right]$$

Moreover:

$$[1 : y_1 : \dots : y_m] = [1 : z_1 : \dots : z_m]$$

$$\Leftrightarrow y_1 = z_1, \dots, y_m = z_m$$

I.e. we can "parametrize" P^m by coordinates $(y_1, \dots, y_m)$.

$$\boxed{\begin{array}{l} \text{i.e. } \mathbb{P}^m = (\mathbb{R}^{m+1} \setminus \{0\}) / \mathbb{R}^* \\ \text{i.e. } P^m = S^m / \mathbb{Z}_2 \end{array}}$$

Careful with $x_0 = 0$!

Fixing the mistake:

$$U_0 = \{[x_0 : x_1 : \dots : x_m] \in \mathbb{P}^m \,/\, x_0 \neq 0\}$$

$$\chi : U_0 \to \mathbb{R}^m, \quad [x_0 : x_1 : \dots : x_m] \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_m}{x_0}\right)$$

$\Rightarrow$ a chart (U_0, χ_0) for P^m

Similarly $(U_1, \chi_1), \dots, (U_m, \chi_m)$ charts for P^m) smooth atlas !

$\{U_0, U_1, \dots, U_m\}$ — do cover P^m (Rk 2.6.3)

$\Downarrow$

$\mathbb{P}^m$ becomes a smooth man...

are smoothly compatible with all charts from $\mathcal{A}$)

Example 5: The (complex) projective space

$\mathbb{CP}^m$ = complex lines in $\mathbb{C}^{m+1}$

$= \{ [z_0 : z_1 : \dots : z_m]_{\mathbb{C}} : z \in \mathbb{C}^{m+1} \setminus \{0\} \}$

Since $\mathbb{C}^m \cong \mathbb{R}^{2m} \Rightarrow \dots \Rightarrow$ a smooth $2m$-dimensional structure on $\mathbb{CP}^m$.

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$\{ (x,y) \in \mathbb{R}^2 \,/\, e^x + \sin(y^2) + e^{y e^x} > 12x^2 - 42\cos(xy) \}$

$(0, \infty)$

$\{ f(x) > 0 \}$

Example 6: Have on $\mathbb{R}^m$ twice: $\begin{cases} \mathbb{R}^m \\ \mathbb{C}^k \end{cases}$ $m = 2k$.

$M = M_{n\times n}(\mathbb{R}) = \{ A = (a_{ij})_{1 \le i,j \le n} \,/\, a_{ij} \in \mathbb{R} \}$. looks like $\mathbb{R}^{n^2}$

$x: M \longrightarrow \mathbb{R}^{n^2}$, $x(A) = (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn})$

Atlas $\{x\} \Rightarrow$ smooth structure on $M_{n\times n}(\mathbb{R})$ (really just $\mathbb{R}^{n^2}$)

Example 4

$\mathbb{P}^m =$

$=$

For $x = (x_0, \dots, x_m) \in \mathbb{R}^{m}$

$\mathbb{P}^m = \{ \ell_x : x \in \mathbb{R}$

$= \{ \ell_u : u$

With

$\ldots, A_N\}$. (2)

$x \in A_M$ (*)

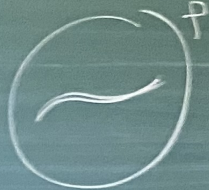
$x' \in A_N$

$x \in A$

$x' \in B$.

use $A = \{ Id_{\mathbb{R}^m} \}$

$\hat{P}$



ations.

. Domain $\longrightarrow \mathbb{R}^k$ smooth
$\cap$ open
$\mathbb{R}^L$

(9)

Example 7:

$$M = Gl_n(\mathbb{R}) = \{ A \in M_{n\times n}(\mathbb{R}) \;/\; \underbrace{A = \text{invertible}}_{\det(A) \neq 0} \}$$

$$\det : M_{n,n}(\mathbb{R}) \longrightarrow \mathbb{R} \quad \text{continuous}$$

$\Rightarrow Gl_n(\mathbb{R})$ is an open in $M_{n\times n}(\mathbb{R})$

$\Rightarrow Gl_n(\mathbb{R})$ carries a natural smooth str.

$\left(\text{use one single chart } \; x|_{Gl_n(\mathbb{R})} : Gl_n(\mathbb{R}) \longrightarrow \text{open} \subset \mathbb{R}^{n^2} \right)$

Example 8: $O(n) = \{ A \in M_{n\times n}(\mathbb{R}) : A \cdot A^T = Id \}$

$= \{ \text{linear maps } A : \mathbb{R}^n \to \mathbb{R}^n \;/\; \|A(v)\| = \|v\| \; \forall v \in \mathbb{R}^n \}$

$= \text{—— '' ——} \;/\; \langle A(u), A(v) \rangle = \langle u, v \rangle \}$

$= \{ A \in M_{n\times n} \;/\; A \cdot A^T = Id \}$.

TBC