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Reminder: ... atlases, smooth structures, manifolds, smooth maps

Exhibiting examples of manifolds M:

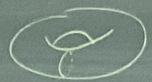
- "abstract": exhibit a smooth atlas $\mathcal{A}$ on M
- "concrete": embedded submanifolds $M \subseteq \mathbb{R}^L$

e.g. $M=\mathbb{R}^L$, or $M=\Omega \underset{\text{open}}{\subseteq} \mathbb{R}^L$ ... or use the RVT!

Examples:

① S^n (both approaches !!)

② torus



etc

③ $\mathbb{P}^n$

④ $\mathbb{CP}^n$

②

i.e.: $(\forall)\, p\in M$, $(\exists)$ chart $\tilde{x}:\tilde{U}\longrightarrow\underset{\mathbb{R}^L}{\overset{\cap}{\tilde{\Omega}}}$ for $\mathbb{R}^L$ around p

such that: $M\cap\tilde U = \{p\in\tilde U \,/\,$ last $L-m$ coord. of $\tilde x(p)$ vanish$\}$

$\Leftrightarrow$ $\tilde x(\underbrace{\tilde U\cap M}_{U}) = \underbrace{\tilde\Omega \cap \mathbb{R}^m\times\{0\}}_{\Omega}$, $\quad 0=\underbrace{(0,\dots,0)}_{L-m}$

$\Rightarrow$ $\tilde x|_U : U \longrightarrow \Omega$ serves as chart for M

$\Rightarrow$ a smooth structure on M (called "the induced one")

Smoothness: GOOD NEWS (Prop 2.34): for "concrete" $M\subseteq\mathbb{R}^L$, $N\subseteq\mathbb{R}^{L'}$;

a map $f:M\to N$ is smooth (in the abstract sense) IFF it is smooth as in Chp I.

IN PARTICULAR: To check that $f: M\to N$ is smooth, it suffices to show that f admits a smooth extension $\tilde f:\mathbb{R}^L\longrightarrow\mathbb{R}^{L'}$.

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⑤ $\mathcal{M}_{n\times n}(\mathbb{R})$ (... a copy of $\mathbb{R}^{n^2}$)

⑥ $GL_n(\mathbb{R})$ (... an open $\subseteq$)

⑦ $O(n)=\{A\in\mathcal{M}_{n\times n}(\mathbb{R}): A\cdot A^T=Id\}$

$\begin{cases} F: \mathcal{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathcal{M}^{symm}_{n\times n}(\mathbb{R}) \\ F(A)=A\cdot A^T \end{cases}$ $\quad$ $M=F^{-1}(Id)$.

a copy of $\mathbb{R}^{n(n+1)/2}$

⑤

⑩ Lie groups:

Def 2.112: A Lie group G: G has two structures:

- a group structure (with associated operations $m: G\times G\to G$, $i: G\to G$)
- a manifold structure

s.t. they are compatible in the sense that: m, i are smooth maps

1

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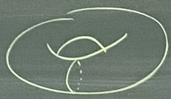
② torus  etc

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$$\begin{cases} F : \mathcal{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathcal{M}^{symm}_{n\times n}(\mathbb{R}) \\ F(A) = A\cdot A^T \end{cases}$$

a copy of $\mathbb{R}^{\frac{n(n+1)}{2}}$

$M = F^{-1}(Id)$.

IS: F a submersion at all $A\in O(n)$? Do it for $A = Id$.

To check: $(dF)_{Id} : \mathcal{M}_{n\times n}(\mathbb{R}) \longrightarrow \mathcal{M}^{sym}_{n\times n}(\mathbb{R})$ is surjective (?)

$$(dF)_{Id}(X) = \lim_{t\to 0} \frac{F(Id+tX)-F(Id)}{t}$$

$$= \lim_{t\to 0} \frac{(Id+tX)\cdot(Id+tX)^T - Id\cdot Id^T}{t}$$

$\Rightarrow (dF)_{Id}$ sends X to $X+X^T$

$Y = X + X^T$

$X = \frac{1}{2}Y$

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Examples:

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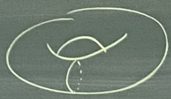
2. torus  etc
3. $\mathbb{P}^n$
4. $\mathbb{CP}^n$
5. $\mathcal{M}_{n\times n}(\mathbb{R})$ (... a copy of $\mathbb{R}^{n^2}$) ③
6. $GL_n(\mathbb{R})$ (... an open $\subseteq$).
7. $O(n) = \{ A \in \mathcal{M}_{n\times n}(\mathbb{R}) : A\cdot A^T = Id \}$

$$\begin{cases} F : \mathcal{M}_{n\times n}(\mathbb{R}) \longrightarrow \underbrace{\mathcal{M}^{symm}_{n\times n}(\mathbb{R})}_{\text{a copy of } \mathbb{R}^{\frac{n(n+1)}{2}}} \\ F(A) = A\cdot A^T \end{cases} \qquad M = F^{-1}(Id).$$

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nd p

... of $\tilde{x}(p)$ vanish}

$(0,\dots,0)$ ($L-m$)

chart for M

d

induced one)

$N \subseteq \mathbb{R}^{L'}$;

t sense)

More examples / constructions: (4) $\boxed{M\times N}$ set

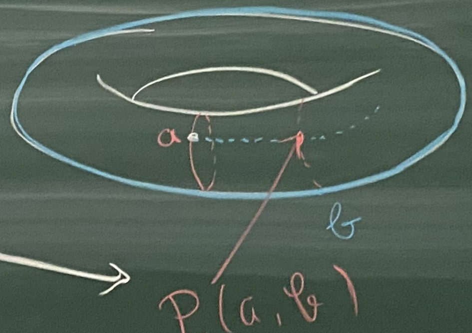
(9) Products of manifolds $(M, \mathcal{A}_M), (N, \mathcal{A}_N)$

$$\left.\begin{array}{ll} x: \underset{M}{\underset{\cap}{U}} \longrightarrow \underset{\mathbb{R}^m}{\underset{\cap}{x(U)}} & \text{chart for } M \\ x': \underset{N}{\underset{\cap}{U'}} \longrightarrow \underset{\mathbb{R}^n}{\underset{\cap}{x'(U')}} & \text{chart for } N \end{array}\right\} \rightsquigarrow x\times x' : \underset{M\times N}{\underset{\cap}{U\times U'}} \to \underset{\mathbb{R}^{m+n}}{\underset{\cap \text{ open}}{x(U)\times x'(U')}}$$

All such chart on $M\times N$ $\Rightarrow$ a smooth atlas $\Rightarrow$

$\Rightarrow$ a $\underbrace{\text{smooth str.}}_{\text{"the product smooth str."}}$ on $M\times N$ $\Rightarrow$ $M\times N$ becomes an $(m+n)$-dimensional manifold;

Ex: $S^1 \times S^1$ diffeomorphic to



$(a,b) \longmapsto P(a,b)$

Particular cases:

$\mathbb{R}\times M,\ (0,1)\times M,\ S^1\times M$

IN PARTICULAR : To check ...

5

(10) Lie groups :

Def 2.112 : A Lie group G : G has two structures :
- a group structure (with associated operations $m : G \times G \to G$, $i : G \to G$)
- a manifold structure

s.t. they are compatible in the sense that : m, i are smooth maps

Ex : $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ A Lie Group.

$\boxed{|z_1 z_2| = |z_1| \cdot |z_2|} \Rightarrow (S^1, \cdot)$ a group.

$\underset{(x,y)}{(x+iy)} \cdot \underset{(x',y')}{(x'+iy')} = \underset{(xx'-yy',\ xy'+x'y)}{xx'-yy'+i(xy'+x'y)}$

$(x^2+y^2)(a^2+b^2) = (xa+yb)^2 + (xb-ya)^2$

Ex : $S^3 = \{ \xi \in \mathbb{H} \mid |\xi| = 1 \} \Rightarrow$ A Lie group.

$\mathbb{H} = \{ \underbrace{x+iy+jz+kt}_{\xi} \mid x,y,z,t \in \mathbb{R} \}$

$|\xi_1 \xi_2| = |\xi_1| \cdot |\xi_2|$!

$|\xi| = \sqrt{x^2+y^2+z^2+t^2}$

NUMBER THEORY

$(x^2+y^2+z^2+t^2)(a^2+b^2+c^2+d^2) = (\quad)^2 + (\quad)$

$d)$.

$= Id$.

s surjective (?)

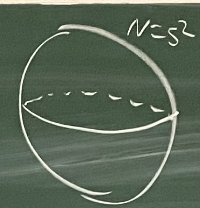
$- Id \cdot Id^T$

$Y = X + X^T$

$X = \frac{1}{2} Y$

Exhibiting examples of manifolds M.

(5) $\mathcal{M}_{n\times n}(\mathbb{R})$ (... a copy of $\mathbb{R}^{n^2}$) (6)



(6) $GL_n(\mathbb{R})$ (... an open $\cup$).

(7) $O(n) = \{ A \in \mathcal{M}_{n\times n}(\mathbb{R}) : A \cdot A^T = Id \}$

(11) The Lie group $O(n)$. Similarly: $SL_n(\mathbb{R}) = \{ A \,/\, \det A = 1 \}$

$SO(n) = O(n) \cap SL_n(\mathbb{R})$

(12) Replace $\mathbb{R}$ by $\mathbb{C} \Rightarrow \mathcal{M}_{n\times n}(\mathbb{C})$ ... a copy $\mathbb{R}^{(2n)\cdot(2n)} = \mathbb{R}^{4n^2}$

$\cup$

$GL_n(\mathbb{C})$

$U(n) = \{ A \in \mathcal{M}_{n\times n}(\mathbb{C}) \,/\, A \cdot A^* = Id \}$

$SU(n) = \{ A \in U(n) \,/\, \det A = 1 \}$.

all Lie Groups

(13) Examples of examples:

$U(1) = \{ z \in \mathbb{C} \,/\, z \cdot \bar{z} = 1 \} = S^1$

$SO(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{l} a^2+b^2=1 \\ c^2+d^2=1 \\ ac+bd=0 \end{array}, ad-bc=1 \right\} = \left\{ \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\} \Rightarrow SO(2) \cong S^1$

$SU(2) \cong S^3$

$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \xleftarrow{\;\cong_{\text{diffeom}}\;} S^1 \ni (x,y)$

(14) Emb...

Def 2...

Say ...

Ex: $N = S^2$, ...

Recall: t...

any contin...

f...

s.t., as a ...

$\{A=1\}$

$SL_n(\mathbb{R})$

$4n^2$

ups

$2) \cong S^1$

(14) Embedded submanifolds:

(7)

Def 2.74. N manifold, $M \subseteq N$ subset ($\dim n$ for N)

Say that M is an embedded submanif of N if

$(\forall) p \in M$, $(\exists)$ chart $\tilde{x} : \tilde{U} \longrightarrow \underset{\substack{\cap \text{ open} \\ \mathbb{R}^n}}{\tilde{\Omega}}$ for N around p

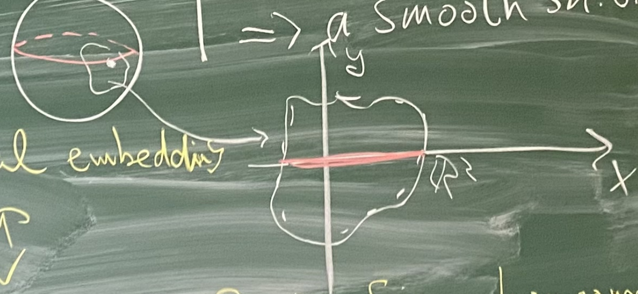
Such that $M \cap \tilde{U} = \{ p \in \tilde{U} \ / \ \text{last } n-m \text{ coord of } \tilde{x}(p) \text{ vanish} \}$

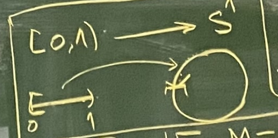
$\Longleftrightarrow \tilde{x}(\tilde{U} \cap M) = \dots$

$\dots\dots$

$\Rightarrow$ a smooth str. on M called the induced smooth str. from N

Ex: $N = S^2$, $M = S^1$





NICE: IF M = COMPACT and N = Hausdorff, then any continuous injection is an embedding

Recall: topological embedding = any continuous map $f : M \to N$ s.t., as a map from $\underbrace{M}_{\text{the original space}}$ to $\underbrace{f(M)}_{\text{endowed with the top. induced from } N}$, f is a homeomorphism.

Smooth embed...

Def: A ...

1.

2.

Ex: (1) $f : ($...

(2) N = toru...

$f, g, h : \underset{\substack{\| \\ S^1 \\ \mathbb{R}}}{M} \longrightarrow \underset{\substack{\| \\ \text{toru...}}}{N}$

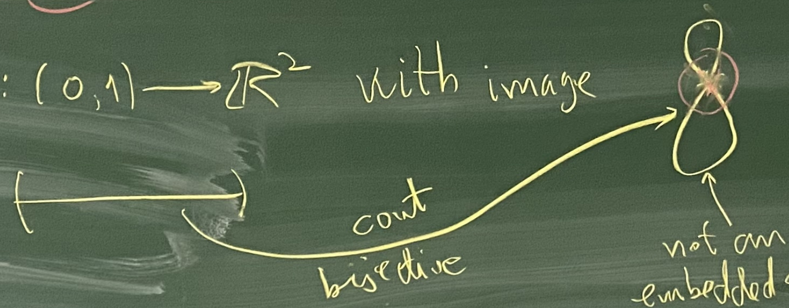
Smooth embeddings: 8

Def: A smooth map $f: M \longrightarrow N$ is smooth embedding if

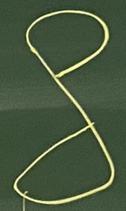
manifolds

1. $f(M) \subseteq N$ is an embedded submanifold
2. as a map from $\underbrace{M}_{\text{the original manifold}}$ to $\underbrace{f(M)}_{\text{endowed with the induced smooth str.}}$, f is a diffeomorphism.

Ex: ① $f: (0,1) \longrightarrow \mathbb{R}^2$ with image



or



② $N = \text{torus}$, $M = \mathbb{R}$

$f, g, h : M \longrightarrow N$

$M = S^1$, $\mathbb{R}$

$N = \text{torus}$