Reminder: • Problem: given $\begin{cases} N = \text{manifold} \\ M \subseteq N \text{ subset} \end{cases}$ ①

is there an "induced" smooth str. on M?

• $\Rightarrow$ the notion of embedded submanifold (of N):

$(\forall)\, p \in M$ $\left[\begin{array}{l} \exists\, \tilde{x} : \tilde{U} \to \tilde{\Omega} \text{ smooth chart of } N,\ p \in \tilde{U} \\ \text{s.t. } \tilde{U} \cap M = \{ q \in \tilde{U} \,/\, \text{last } n-m \text{ coord are } 0 \} \end{array}\right.$

s.c. of N adapted to M

$\Leftrightarrow \tilde{x}(\underbrace{\tilde{U} \cap M}_{U}) = \underbrace{\tilde{\Omega} \cap \mathbb{R}^m \times \{0\}}_{\Omega}$

$\Rightarrow \tilde{x}|_U : \underset{\cap\, M}{U} \to \underset{\cap\, \mathbb{R}^m}{\Omega} \Rightarrow$ smooth str. on M!

• smooth embedding of a manifold M_0 into another manifold N:

smooth map $f : M_0 \to N$ s.t.

1. $f(M_0) \subseteq N$ embedded submanifold
2. $f : M_0 \to f(M_0)$ is a diffeomorphism

continuous

homeomorphism

$\boxed{X}$

$\underline{\text{Rk 0}}$: $\underset{\text{embedded}}{M \subseteq N} \Rightarrow \begin{array}{l} i : M \to N \\ i(x) = x \end{array}$ is an smooth embedding ③

$\underline{\text{Rk 1}}$: For general manifold M_0

$\begin{pmatrix} \text{an embedding} \\ \text{of } M_0 \text{ in } N \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{a diffeomorphism of } M_0 \\ \text{with an embedded submanifold of } N \end{pmatrix}$

• Immersions: $f : M_0 \to N$ immersion $\Leftrightarrow$ ②

$\Leftrightarrow$ all $f_x^{x'}$ are immersions $\Leftrightarrow$

$\Leftrightarrow (\forall)\, p \in M_0\ \exists \begin{cases} (U, x) \text{ around } p \\ (U', x') \ldots \end{cases}$ s.t. $f_x^{x'}(x) = (x, 0)$

aware of difference between ... M and $f(M_0) \subseteq N$

Reminder: • Problem: given $\begin{cases} N = \text{manifold} \\ M \subseteq N \text{ subset} \end{cases}$ is there an "induced" smooth str. on M?

• $\Longrightarrow$ the notion of embedded submanifold (of N):

$(\forall)\, p \in M$ $\exists\, \tilde{x} : \tilde{U} \to \tilde{\Omega}$ smooth chart of N, $p \in \tilde{U}$

s.t. $\tilde{U} \cap M = \{ q \in \tilde{U} \,/\, \text{last } n-m \text{ coordinates } 0 \}$

s.c. of N adapted to M

$(\Leftrightarrow)\ \tilde{x}(\underbrace{\tilde{U} \cap M}_{U}) = \underbrace{\tilde{\Omega} \cap \mathbb{R}^m \times \{0\}}_{\Omega}$

$\Rightarrow \tilde{x}|_U : \underset{M}{\overset{\cap}{U}} \to \underset{\mathbb{R}^m}{\overset{\cap}{\Omega}} \Longrightarrow$ smooth str. on M!

• Smooth embedding of a manifold M_0 into another manifold N:

Smooth map $f : M_0 \to N$ s.t.

1. $f(M_0) \subseteq N$ embedded submanifold
2. $f : M_0 \to f(M_0)$ is a diffeomorphism

$\boxed{\text{continuous}}$ $\boxed{\text{homeomorphism}}$ $\boxed{X}$

②

• Immersions: $f : M_0 \to N$ immersion $\Longleftrightarrow$

$\Longleftrightarrow$ all $f_x^{x'}$ are immersions $\Longleftrightarrow$

$\Longleftrightarrow (\forall)\, p \in M_0\ \exists \begin{cases} (U, x) \text{ around } p \\ (U', x') \text{ around } f(p) \end{cases}$ s.t. $f_x^{x'}(*) = (*, 0)$

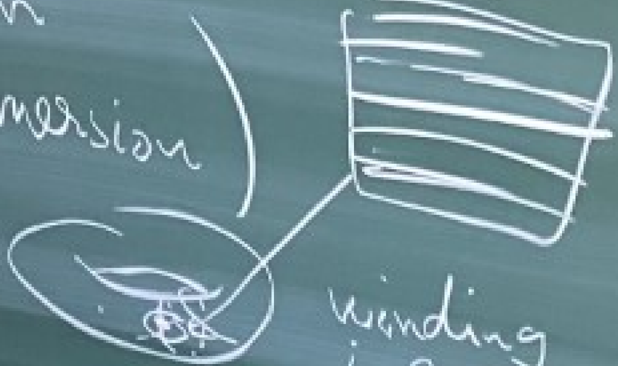
Be aware of the difference between $f : M_0 \to N$ and $f(M_0) \subseteq N$

Intuitively: "f makes M_0 sit inside N in a very very nice smooth way"

"parametrizes $f(M_0) \subseteq N$ in the smoothest possible way"

Rk2 $\left(\begin{array}{l} f : M_0 \to N \text{ a} \\ \text{smooth embedding} \end{array} \right) \Longrightarrow \left(\begin{array}{l} f \text{ is also an} \\ \text{injective immersion} \end{array} \right)$

... but not conversely: e.g. $f : \mathbb{R} \to T^2$



winding infinitely

OR $f : (0,1) \to \mathbb{R}^2$ by a nice formula but with image

Ex: $N = \mathbb{R}^2$

$f, g : \underset{M_0}{\overset{}{\mathbb{R}}} \to \underset{N}{\mathbb{R}^2}$

$f(t) = (t, t)$

$g(t) = (t^3, t^3)$

Reminder: • Problem: given $\begin{cases} N = \text{manifold} \\ M \subset N \text{ subset} \end{cases}$ ①

Rk ①: $M \subseteq N$ embedded $\Rightarrow$ $i: M \to N$, $i(x) = x$ is an smooth embedding ③

Relaxed

• Immersions: $f: M_0 \to N$ immersion $\Leftrightarrow$ ④

Be aware of the difference between $f: M_0 \to M$ and $f(M_0) \subseteq N$

$\Leftrightarrow$ all $f_x^{x'}$ are immersions $\Leftrightarrow$

$\Leftrightarrow$ $(\forall)\, p \in M_0$ $\exists \begin{cases} (U, x) \text{ around } p \\ (U', x') \text{ around } f(p) \end{cases}$ s.t. $f_x^{x'}(x) = (x, 0)$

Intuitively: "f makes M_0 sit inside N in a very very ~~smooth~~ nice smooth way"

"parametrizes $f(M_0) \subseteq N$ in the smoothest possible way"

Rk 2: $\begin{pmatrix} f: M_0 \to N \text{ a} \\ \text{smooth embedding} \end{pmatrix} \Rightarrow \begin{pmatrix} f \text{ is also an} \\ \text{injective immersion} \end{pmatrix}$

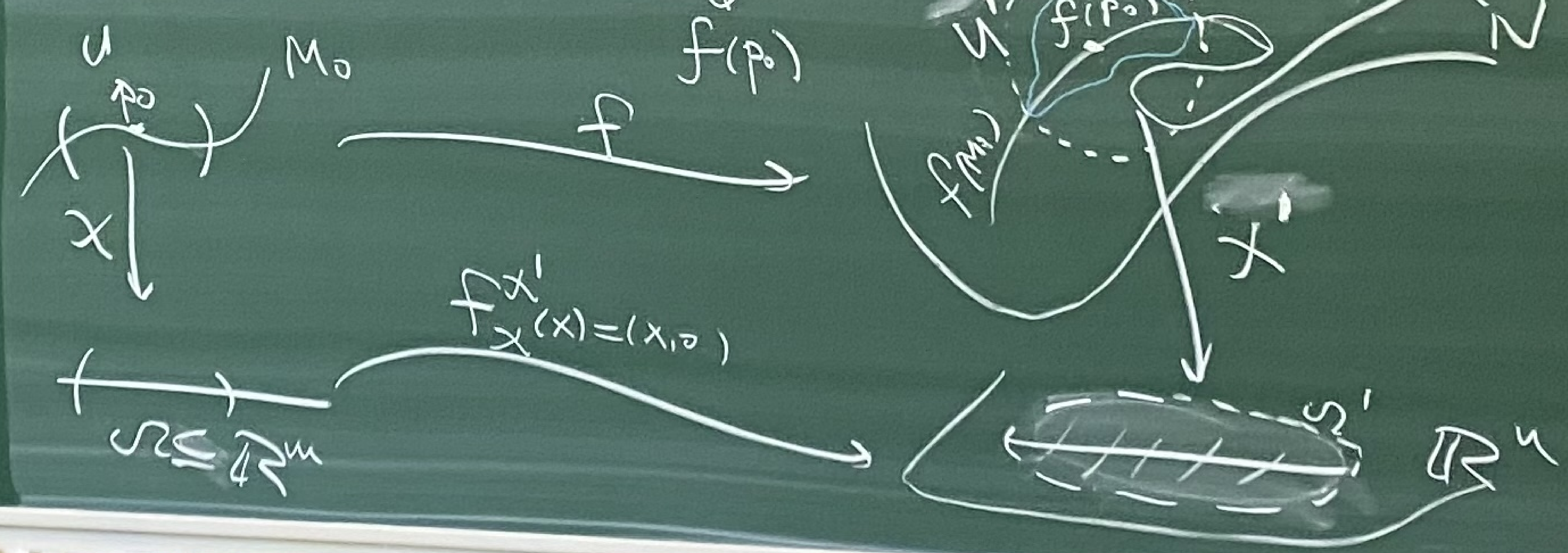
Thm 2.79: $f: M_0 \to N$ smooth.

$\begin{pmatrix} f: M_0 \to N \\ \text{smooth embedding} \end{pmatrix} \Leftrightarrow \begin{pmatrix} f = \text{immersion} \\ f = \text{topological embedding} \end{pmatrix}$

$\Updownarrow$ if $M_0 = \text{compact}$

$(f = \text{injective immersion})$

"sketch proof $\Leftarrow$": $\overbrace{f(M_0)}^{M} \subseteq N$ embedded submanifold?



$(U', x') = \text{adapted chart?}$

$f(U)$ - open in $f(M_0)$

$\| $

$f(M_0) \cap W$ $\quad$ W - open in N

$\Rightarrow$ $(W, x'|_W) = \text{adapted chart!}$

of N

Relaxed problem: given $\begin{cases} N = \text{manifold} \\ M \subseteq N \text{ subset} \end{cases}$ (5)

can one find a smooth structure on M such that the inclusion $i: M \to N$ is an immersion?

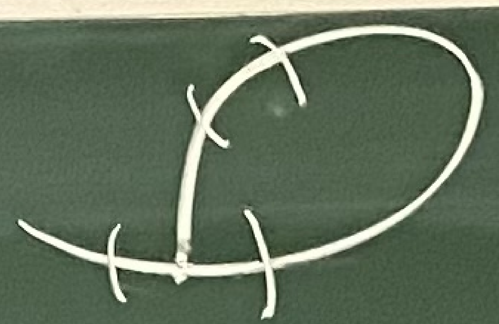
If yes, is it unique?

if yes & yes, we say that the subset $M \subseteq N$ has the unique smooth structure property

Def 2.88: An immersed submanifold of N: ~~any subset~~ any subset $M \subseteq N$ & a smooth structure on M (together with !!!!!) such that $i: M \to N$ is an immersion.

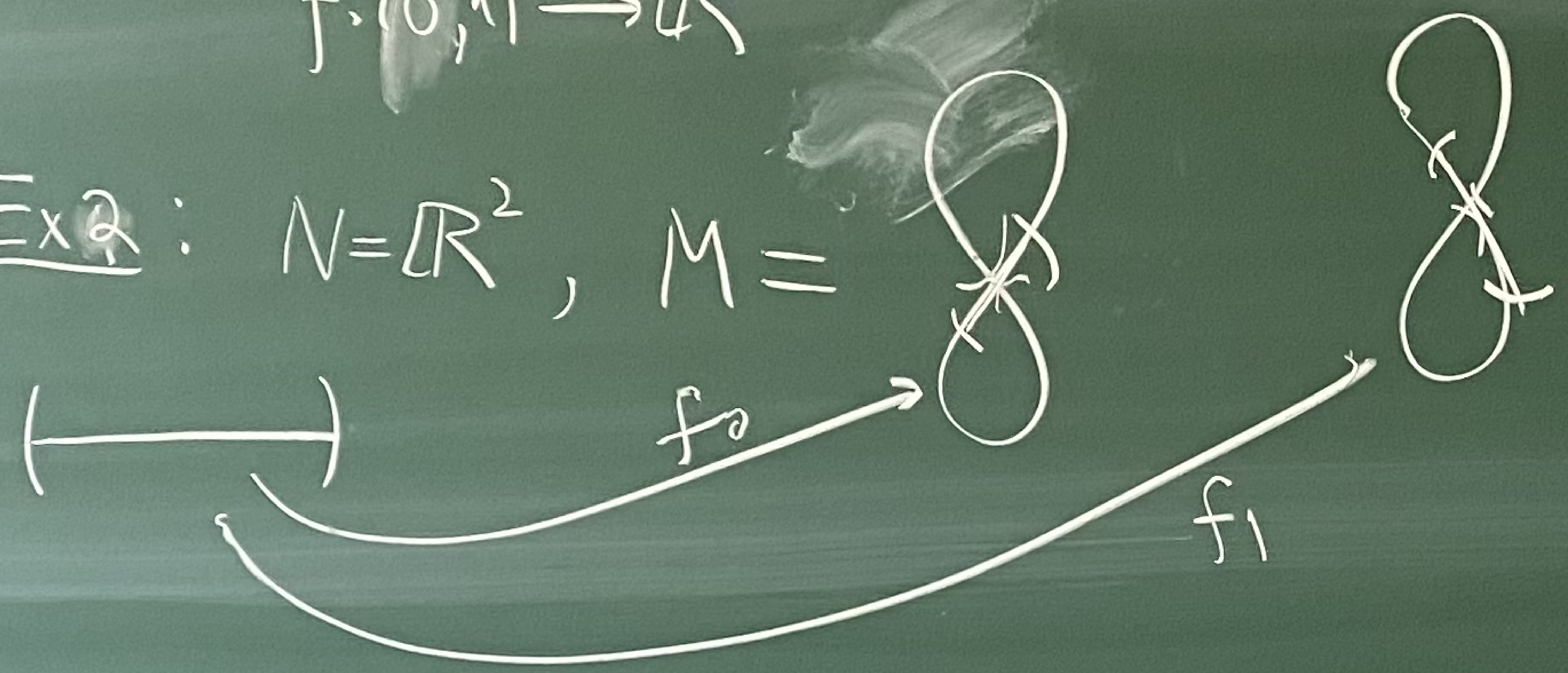
WARNING: UNLIKE THE CASE OF EMBEDED SBMN, IT IS NOT ENOUGH TO LOOK AT THE PICTURE ☹

Def 2.88: An immersed submanifold of N:

Ex1: $N=\mathbb{R}^2$, $M=$  ⑥ can be made into an immersed submanifold

$f:(0,1)\to\mathbb{R}^2$

Ex2: $N=\mathbb{R}^2$, $M=$



two ways to make M into an immersed submanifold.

Ex3:  no f_0, f_1, but one single f! Actually it has the unique SSP.

Thm: (Prop 2.9.3):

$M\subseteq N$ embedded $\Rightarrow$ M has the unique SSP

In particular: if

$\left.\begin{array}{l} f: M_0\to N \text{ inj. immersion} \\ f(M_0)\subseteq N \text{ embedded}\end{array}\right\} \Rightarrow$ f is a smooth embedding

$, x')$ = adapted chart?

(U) - open in $f(M_0)$

$(M_0)\cap W$ $\quad W$ - open in N

$, x'|_W)$ = adapted chart!

Reminder: Chp I : ① For $M \subseteq \mathbb{R}^n$, $p \in M$: (7)

$$T_p^{geom} M = \left\{ \frac{d\gamma}{dt}(0) \;/\; \gamma \in \underbrace{Curves_p(M)}_{\substack{\text{i.e. } \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n \text{ smooth,} \\ \text{landing in } M,\ \gamma(0) = p}} \right\} :$$

$T_p^{geom} M \subseteq \mathbb{R}^n$

When $\underline{M = \text{embeded submanifold}} \Rightarrow \boxed{T_p^{geom} M} \subseteq \mathbb{R}^n$ vector (sub) space :

② Any $F : M \to N$ smooth

induces $(dF)_p : T_p^{geom} M \longrightarrow T_{F(p)}^{geom} N$
$p \in M$

$$\frac{d\gamma}{dt}(0) \longmapsto \frac{d\, F \circ \gamma}{dt}(0)$$

Want :

TW1 : For

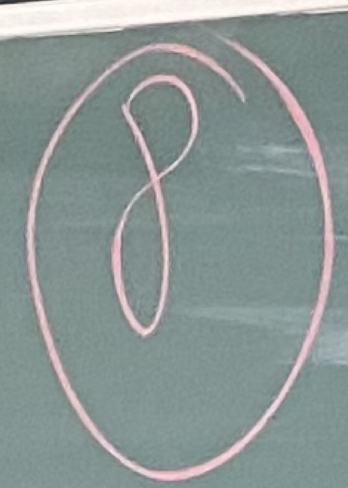
TW2 : For

lin

TW3 : Not

TW4 : le

In practice: M given by equations $F(x)=0$



$F: \mathbb{R}^n \to \mathbb{R}^k$ nice

$\gamma: (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ in M: $\quad F(\gamma(t)) = 0 \xrightarrow{\left.\frac{d}{dt}\right|_{t=0}} (dF)_p\left(\frac{d\gamma}{dt}(0)\right) = 0.$

Hence a vector $v \in \mathbb{R}^n$ is in $T_p^{geom} M \Rightarrow (dF)_p(v) = 0$

$\left(\Leftrightarrow \frac{\partial F}{\partial x_i}(p)\, v^i = 0\right)$

Ex. $M = S^2$. $\quad \underbrace{x^2+y^2+z^2-1}_{F} = 0$

$(dF)_p : \mathbb{R}^3 \to \mathbb{R}$, $\quad p = (x,y,z)$, $\quad (u,v,w) \mapsto 2(\underbrace{xu+yv+zw}_{=0})$

OR directly: $\quad T_p^{geom} S^2 = \{(u,v,w) : xu+yv+zw = 0\}$

$\gamma(t) = (x(t), y(t), z(t))$ in S^2

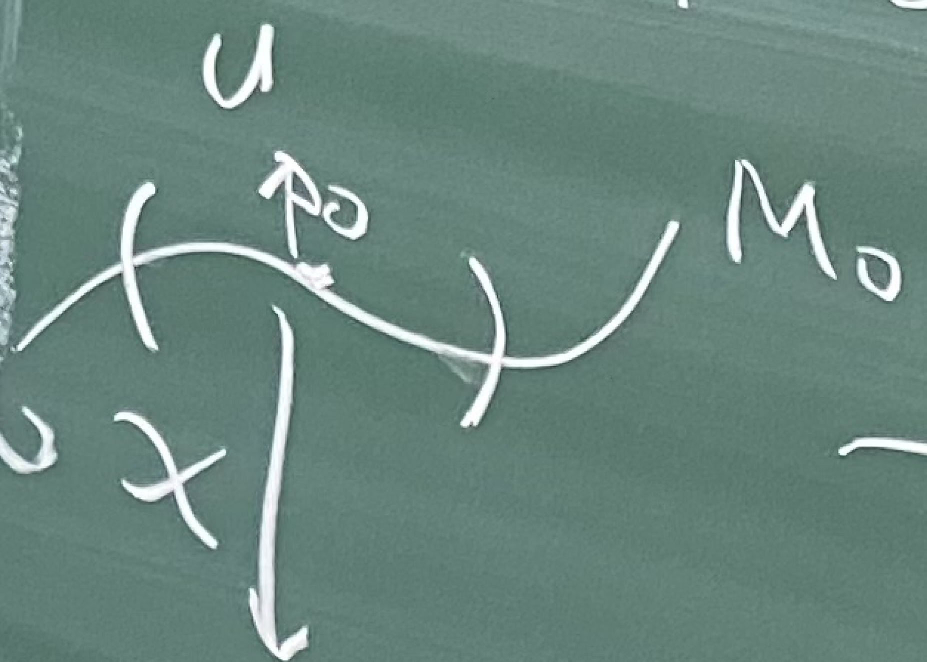
$x(t)^2 + y(t)^2 + z(t)^2 = 1$

$x(0)\underbrace{\frac{dx}{dt}(0)}_{u} + y(0)\underbrace{\frac{dy}{dt}(0)}_{v} + z(0)\underbrace{\frac{dz}{dt}(0)}_{w} = 0$

$\mathbb{P}^2 \to \mathbb{R}^4$ embedding

$\mathbb{P}^2 \to \mathbb{R}^3$

"sketch proof $\Leftarrow$"



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space :

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(0)

TANGENT SPACES (9)

Want:

TW1: For each manifold M, $p \in M$: a vector space T_pM

TW2: For smooth maps $F: M \to N$, $p \in M$:
linear map $(dF)_p : T_pM \to T_{F(p)}N$ satisfying
⊛ for $F = \mathrm{Id}_M : M \to M$, $(dF)_p = \mathrm{Id}$ of T_pM
⊛ chain rule $(d\,G\circ F)_p = (dG)_{F(p)} (dF)_p$.

TW3: Nothing new in $\mathbb{R}^m$ (and for embedded, as in Chp I)

TW4: leading slogan:

$$T_pM = \left\{ \text{"}\frac{d\gamma}{dt}(0)\text{"} \;\middle|\; \gamma \in \mathrm{Curves}_p(M) \right\}$$

$\gamma : (-\varepsilon, \varepsilon) \to M$ smooth
$\gamma(0) = p$

Relaxed proble…

if yes & yes, …
say that the …
subset $M \subseteq$ …
has the unique …
structure

Def 2.88: …

WARNING