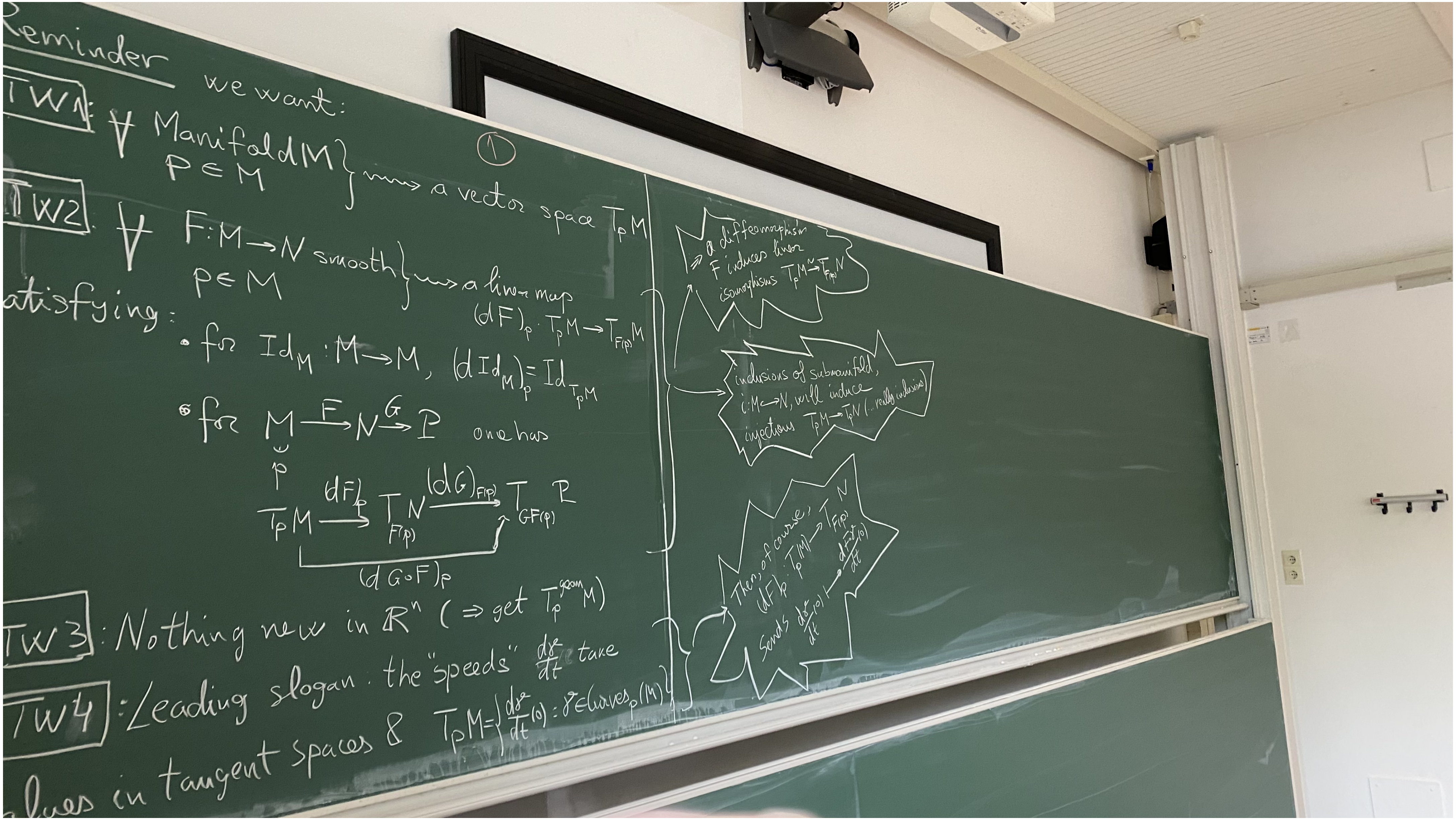


Reminder

we want: ①

$\boxed{TW1}$: $\forall$ $\left.\begin{array}{l}\text{Manifold } M\\ p\in M\end{array}\right\}$ $\rightsquigarrow$ a vector space T_pM

$\boxed{TW2}$: $\forall$ $\left.\begin{array}{l}F: M\to N \text{ smooth}\\ p\in M\end{array}\right\}$ $\rightsquigarrow$ a linear map $(dF)_p : T_pM \to T_{F(p)}M$

atisfying:

- for $Id_M : M\to M$, $(d\,Id_M)_p = Id_{T_pM}$
- for $\underset{\overset{\cup}{p}}{M} \xrightarrow{F} N \xrightarrow{G} P$ one has

$$\begin{tikzcd} T_pM \arrow[r, "(dF)_p"] \arrow[rr, bend right, "(dG\circ F)_p"'] & T_{F(p)}N \arrow[r, "(dG)_{F(p)}"] & T_{GF(p)}P \end{tikzcd}$$

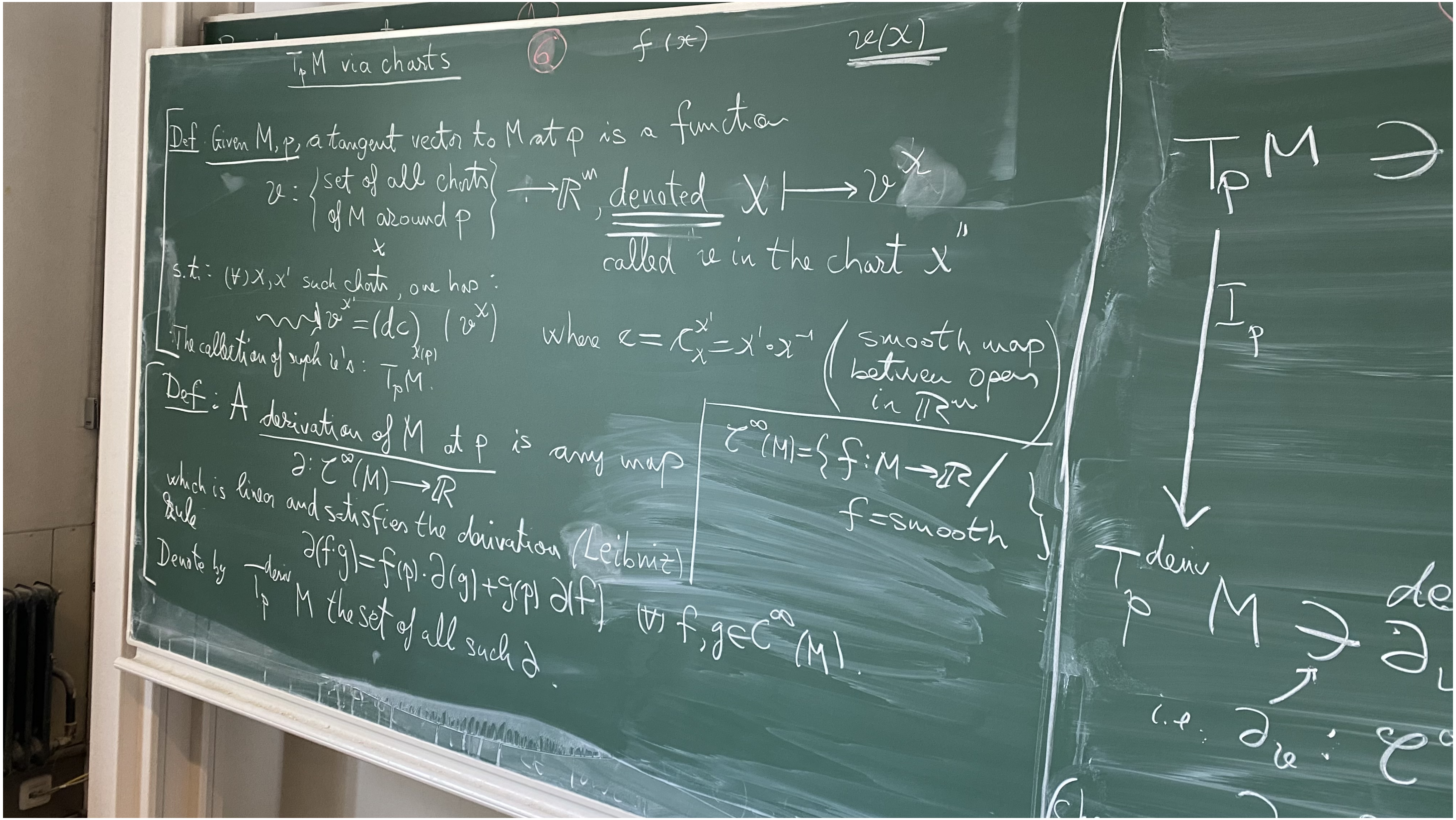
$\boxed{TW3}$: Nothing new in $\mathbb{R}^n$ ($\Rightarrow$ get $T_p^{geom}M$)

$\boxed{TW4}$: Leading slogan: the "speeds" $\frac{d\gamma}{dt}$ take alues in tangent spaces & $T_pM = \left\{\frac{d\gamma}{dt}(0) : \gamma \in Curves_p(M)\right\}$

$\Rightarrow$ a diffeomorphism F induces linear isomorphisms $T_pM \overset{\sim}{\to} T_{F(p)}N$

inclusions of submanifold, $i: M \hookrightarrow N$, will induce injections $T_pM \to T_pN$ (... really inclusions)

Then, of course, $(dF)_p : T_p(M) \to T_{F(p)}N$ sends $\frac{d\gamma}{dt}(0) \mapsto \frac{dF\circ\gamma}{dt}(0)$



T_pM via charts

⑥ $f(x)$ $\underline{v(X)}$

Def Given M, p, a tangent vector to M at p is a function

$$v : \left\{ \begin{matrix} \text{set of all charts} \\ \text{of } M \text{ around } p \end{matrix} \right\} \longrightarrow \mathbb{R}^n, \text{ denoted } X \longmapsto v^X$$

called "v in the chart X"

s.t.: $(\forall) X, X'$ such charts, one has:

$$v^{X'} = (dc)_{X(p)} (v^X)$$

where $c = c_X^{X'} = X' \circ X^{-1}$ (smooth map between open in $\mathbb{R}^n$)

The collection of such v's: T_pM.

Def: A derivation of M at p is any map

$$\partial : C^\infty(M) \longrightarrow \mathbb{R}$$

which is linear and satisfies the derivation (Leibniz) rule

$$\partial(f \cdot g) = f(p) \cdot \partial(g) + g(p)\, \partial(f) \quad (\forall)\, f, g \in C^\infty(M).$$

$C^\infty(M) = \{ f : M \to \mathbb{R} \,/\, f = \text{smooth} \}$

Denote by $T_p^{\text{deriv}} M$ the set of all such ∂.

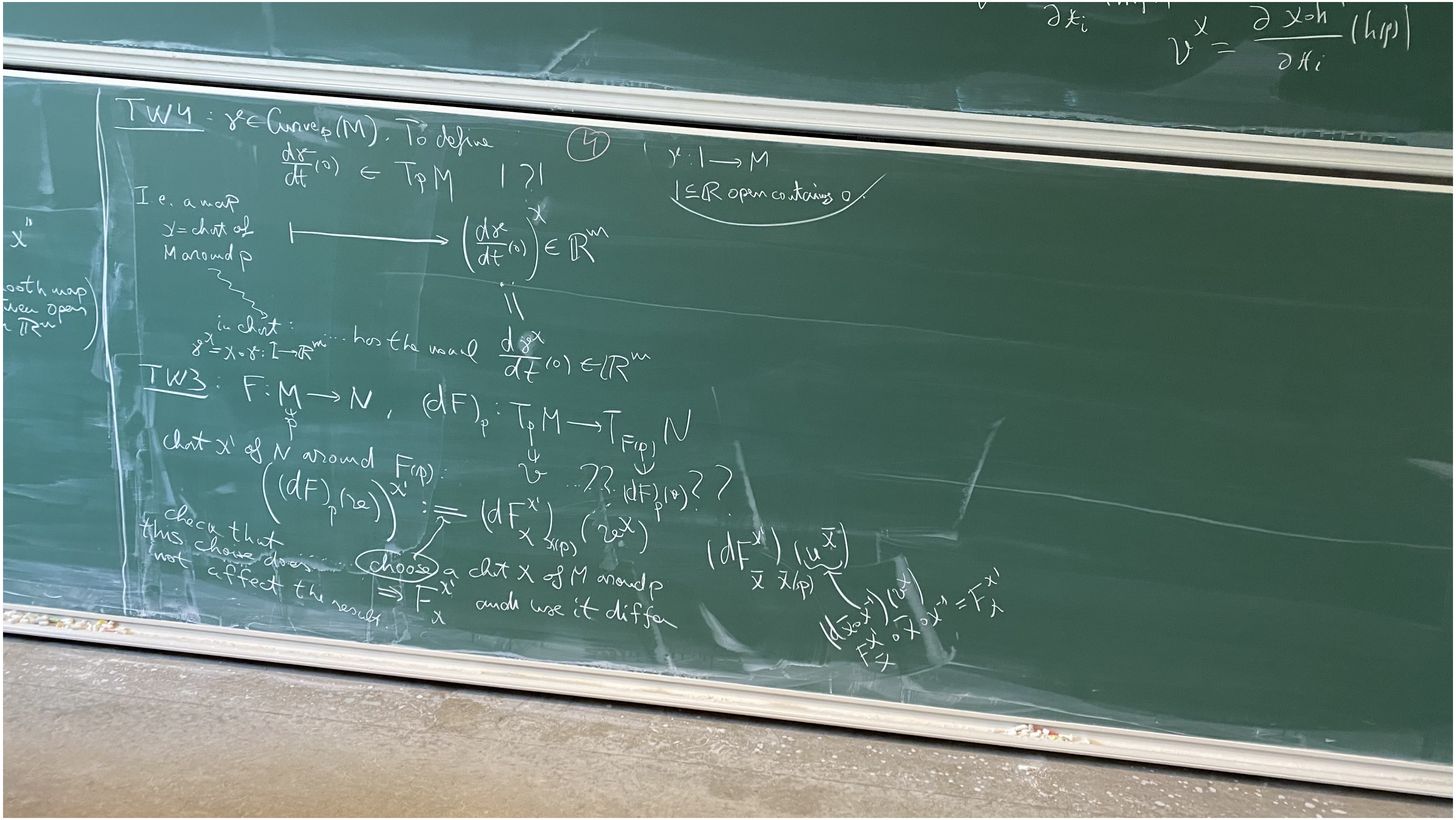
$$T_pM \ni$$

$$\Big\downarrow I_p$$

$$T_p^{\text{deriv}} M \ni \partial_v$$

de

i.e.: $\partial_v : C^\infty$



∂x_i

$$v^{x} = \frac{\partial\, x\circ h}{\partial x_i}(h(p))$$

x''

ooth map
tween open
n $\mathbb{R}^n$

TW4: $\gamma \in \mathrm{Curve}_p(M)$. To define $\frac{d\gamma}{dt}(0) \in T_pM$ | ?|

④ $\gamma : I \longrightarrow M$
$I \subseteq \mathbb{R}$ open containing 0.

I.e. a map
x = chart of
M around p $\longmapsto \left(\frac{d\gamma}{dt}(0)\right)^{x} \in \mathbb{R}^m$

$\|$

in chart:
$\gamma^{x} = x\circ\gamma : I \to \mathbb{R}^m$ ... has the usual $\frac{d\gamma^{x}}{dt}(0) \in \mathbb{R}^m$

TW3: $F : \underset{p}{\overset{\Psi}{M}} \longrightarrow N$, $(dF)_p : T_pM \longrightarrow T_{F(p)}N$

v ... ?? ... $(dF)_p(v)$? ?

chart x' of N around $F(p)$:

$$\left((dF)_p(v)\right)^{x'} := \left(dF^{x'}_{x}\right)_{x(p)}\left(v^{x}\right)$$

check that this choice does not affect the result.

choose a chart x of M around p $\Rightarrow F^{x'}_{x}$ and use it differ

$$\left(dF^{x'}_{\bar{x}}\right)_{\bar{x}(p)}\left(\underbrace{u^{\bar{x}}}\right)$$

$$\left(d\,\bar{x}\circ x^{-1}\right)\left(u^{x}\right)$$

$$F^{x'}_{\bar{x}} \circ \bar{x} \circ x^{-1} = F^{x'}_{x}$$

T_pM

TW1: T_pM = vector space $\ni v, w, \lambda \in \mathbb{R}$ ③

$v + w \in T_pM$ defined by $(v+w)^\chi := v^\chi + w^\chi$ ← addition is in $\mathbb{R}^m$

$\lambda \cdot v \in T_pM$ —"— $(\lambda \cdot v)^\chi := \lambda v^\chi$

Rk To know an element $v \in T_pM \Longleftrightarrow$ to know $v^\chi \in \mathbb{R}^m$ for a single chart χ around p.

Hence, any h FIX h!! induces an isomorphism $T_pM \xrightarrow{\sim} \mathbb{R}^m$, $v \longmapsto v^h$ $\ni e_1, \dots, e_m$

$\Rightarrow$ an induced (by h!) basis of T_pM. Denoted

$$\left(\frac{\partial}{\partial h_1}\right)_p, \dots, \left(\frac{\partial}{\partial h_m}\right)_p \longleftarrow \left(\left(\frac{\partial}{\partial h_i}\right)_p\right)^h = e_i \Rightarrow \text{gen formula for all } \chi'$$

Ex: For $M = \mathbb{R}^m$ (or for opens $\subseteq \mathbb{R}^m$): use $\chi = \text{Id} \Rightarrow$

$\Rightarrow$ $T_p\mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^m$ linear isomorphism ← the canonical isomorphism/identification

$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ – basis of $T_p\mathbb{R}^m$.

$$v = \left(\frac{\partial}{\partial h_i}\right)_p$$

$v^h = e_i$ by def.

On Arbitrary χ

$$v^\chi = (dc)(v^h) \qquad c = \chi \circ h^{-1}$$

e_i

$$v^\chi = \frac{\partial c}{\partial x_i}(h(p))$$

$$v^\chi = \frac{\partial\, \chi \circ h^{-1}}{\partial x_i}(h(p))$$

TW4: $\gamma \in \text{Curve}_p(M)$. To define ④ $\gamma : I \longrightarrow M$

$\frac{d\gamma}{dt}(0) \in T_pM$!?! $I \subseteq \mathbb{R}$ open containing 0.

$\underline{\underline{v(x)}}$

a

$x \longmapsto v^x$

the chart x''

$' \circ x^{-1}$ (smooth map between open in $\mathbb{R}^n$)

④

<u>TW4</u>: $\gamma \in \mathrm{Curve}_p(M)$. To define

$$\frac{d\gamma}{dt}(0) \in T_pM$$

I.e. a map
x = chart of
M around p

$$\longmapsto \left(\frac{d\gamma}{dt}(0)\right)^x \in \mathbb{R}^m$$

$$\parallel$$

in chart: $\gamma^x = x \circ \gamma : I \to \mathbb{R}^m$ ... has the usual $\frac{d\gamma^x}{dt}(0) \in \mathbb{R}^m$

<u>TW2</u>: $F : \underset{p}{\overset{\Psi}{M}} \to N$, $(dF)_p : T_pM \longrightarrow T_{F(p)}N$

$$\begin{tikzcd} T_pM \arrow[d, phantom, "\ni"] \arrow[r] & T_{F(p)}N \arrow[d, phantom, "\ni"] \\ v \arrow[r, dotted, "??"] & (dF)_p(v)\ ?? \end{tikzcd}$$

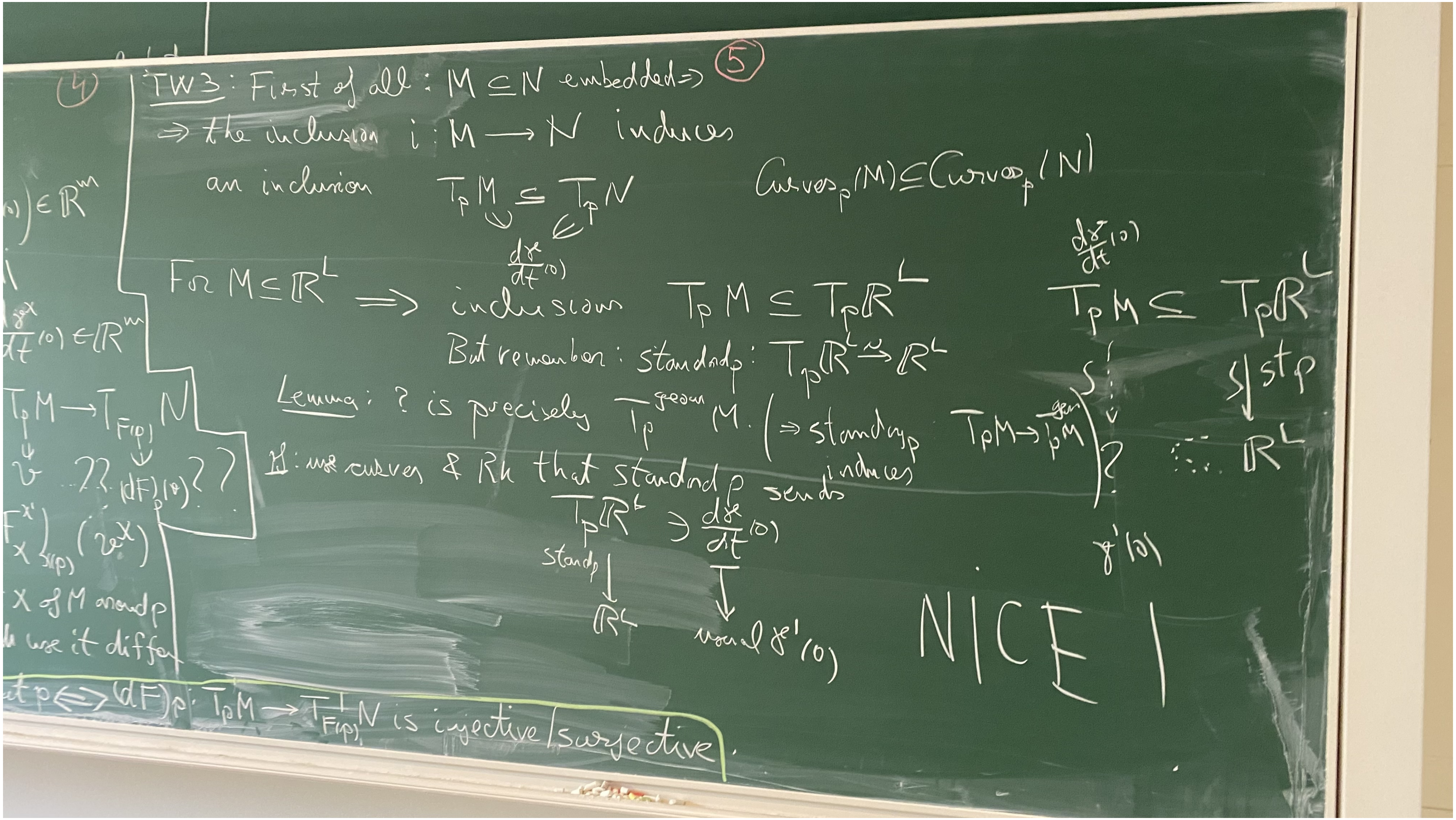
chart x' of N around $F(p)$:

$$\left((dF)_p(v)\right)^{x'} := \left(dF_x^{x'}\right)_{x(p)}\left(v^x\right)$$

check that this choice does not affect the result

...... choose a chart x of M around p $\Rightarrow F_x^{x'}$ and use it differ

<u>Consequence</u>: F = immersion/submersion at $p \Longleftrightarrow (dF)_p : T_pM \to T_{F(p)}N$ is injective/surjective.



④

$(0)) \in \mathbb{R}^m$

$\frac{d\gamma^x}{dt}(0) \in \mathbb{R}^m$

$$\begin{tikzcd} T_pM \arrow[r] & T_{F(p)}N \\ v \arrow[r, dotted, "??"] & (dF)_p(v)?? \end{tikzcd}$$

$F^{x'}_x)_{F(p)}(v^x)$

X of M around p

use it differ

at $p \iff (dF)_p : T_pM \to T_{F(p)}N$ is injective/surjective.

TW3: First of all: $M \subseteq N$ embedded ⇒ ⑤

⇒ the inclusion $i : M \longrightarrow N$ induces

an inclusion $T_pM \subseteq T_pN$ $\qquad$ $\mathrm{Curves}_p(M) \subseteq \mathrm{Curves}_p(N)$

$\frac{d\gamma}{dt}(0)$

For $M \subseteq \mathbb{R}^L \Longrightarrow$ inclusion $T_pM \subseteq T_p\mathbb{R}^L$ $\quad \frac{d\gamma}{dt}(0)$

But remember: $\mathrm{standard}_p : T_p\mathbb{R}^L \xrightarrow{\sim} \mathbb{R}^L$

$$\begin{tikzcd} T_pM \arrow[r, phantom, "\subseteq"] \arrow[d, dashed, "\wr"'] & T_p\mathbb{R}^L \arrow[d, "\wr"', "st_p"] \\ ? & \mathbb{R}^L \arrow[l, dotted] \end{tikzcd}$$

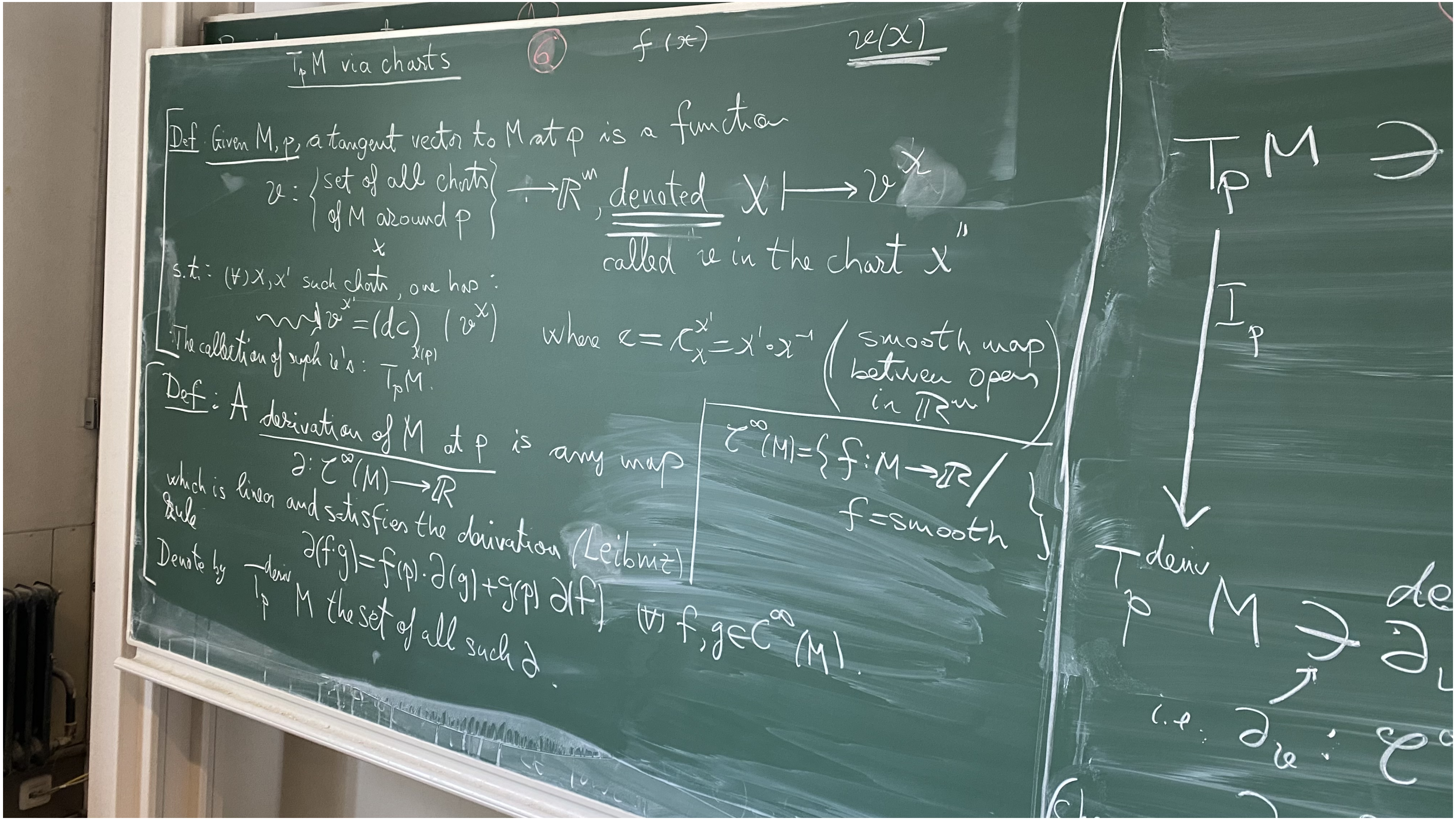
$\gamma'(0)$

Lemma: ? is precisely $T_p^{geom}M$. ($\Rightarrow \mathrm{standard}_p$ induces $T_pM \to T_p^{gen}M$)

Pf: use curves & Rk that $\mathrm{standard}_p$ sends

$$\begin{tikzcd} T_p\mathbb{R}^L \arrow[d, "\mathrm{stand}_p"'] & \ni & \frac{d\gamma}{dt}(0) \arrow[d, mapsto] \\ \mathbb{R}^L & & \text{usual } \gamma'(0) \end{tikzcd}$$

NICE!



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$$\overset{x}{\quad}$$

"called v in the chart x"

s.t.: $(\forall)\, x, x'$ such charts, one has:

$$v^{x'} = (dc)_{x(p)}(v^x) \quad \text{where } c = c_x^{x'} = x' \circ x^{-1}$$

(smooth map between open in $\mathbb{R}^n$)

The collection of such v's: T_pM.

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Denote by $T_p^{\text{deriv}}M$ the set of all such ∂.

$$\mathcal{C}^\infty(M) = \{ f : M \to \mathbb{R} \,/\, f = \text{smooth} \}$$

$$T_pM \ni$$

$$\Big\downarrow I_p$$

$$T_p^{\text{deriv}}M \ni \partial_v$$

de

i.e.: $\partial_v : \mathcal{C}$

⑦

TW1: Vector space: $(\partial + \partial')(f) := \partial(f) + \partial'(f)$

$(\lambda \cdot \partial) = \lambda \cdot \partial(f)$

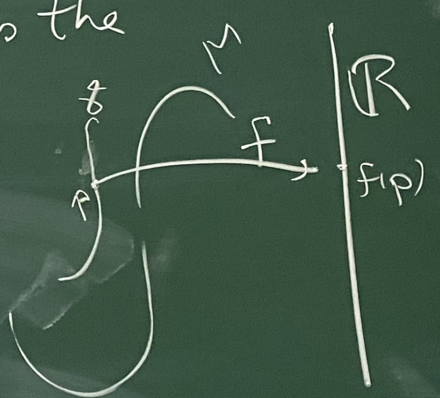
Ex: When $M = \Omega \subseteq \mathbb{R}^n$ open $\Rightarrow$ we have the usual

$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p : \mathcal{C}^\infty(\Omega) \longrightarrow \mathbb{R}$, hence $\in T_p^{\text{deriv}}\Omega$.

Prop: These form a basis of $T_p^{\text{deriv}}\Omega$ (to prove)

TW4: For $\gamma \in \text{Curves}_p(M)$ define $\dot{\gamma}(0) \in T_p^{\text{deriv}}M$ as the derivation $C^\infty(M) \longrightarrow \underset{\nearrow f}{\mathbb{R}}$, $f \longmapsto \boxed{(f \circ \gamma)'}(0)$

$\uparrow$ usual derivative of a function $\begin{array}{c} I \\ \cap\!| \\ \mathbb{R} \end{array} \longrightarrow \mathbb{R}$



TW2: For $F : M \longrightarrow N$; $p \in M$

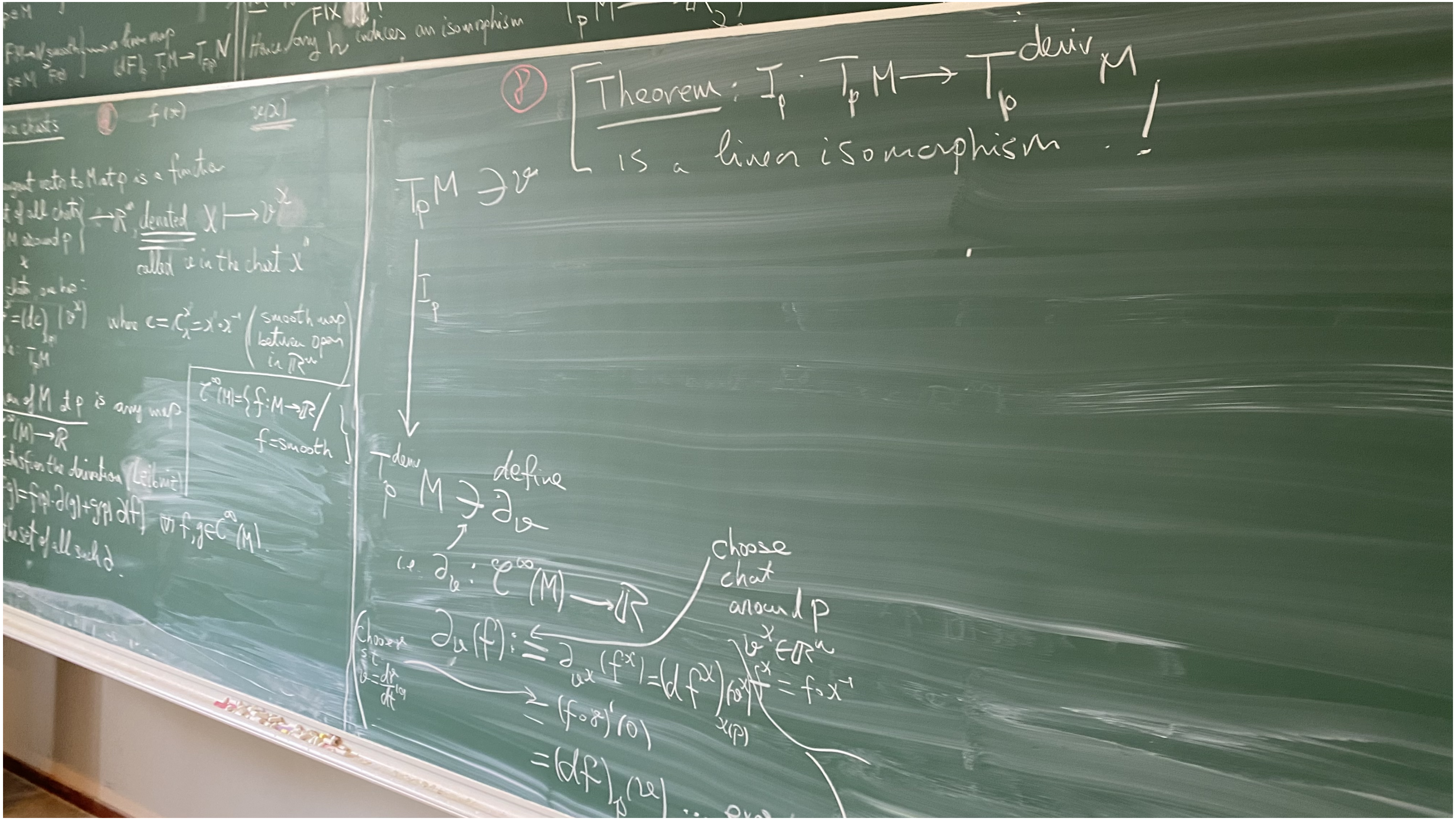
To define: $(dF)_p : T_p^{\text{deriv}}M \longrightarrow T_{F(p)}^{\text{deriv}}N$

let $\partial \longmapsto ?$

$(dF)_p(\partial) := \partial \circ F^*$

$$\begin{tikzcd}
f \in C^\infty(N) \arrow[r, "?"] \arrow[d, "F^*"'] & \mathbb{R} \\
f \circ F \in C^\infty(M) \arrow[ur, "\partial"'] &
\end{tikzcd}$$

$F^*(f) = f \circ F$



$p \in M$

$F: M \to N$ smooth function a linear map $(dF)_p: T_pM \to T_{F(p)}N$

$p \in M$ $F(p)$

FIX

Hence any h induces an isomorphism

T_pM

charts ⑦ $f(x)$ $x(x)$

tangent vector to M at p is a function

set of all charts of M around p $\} \to \mathbb{R}^n$, denoted $x \longmapsto v^x$

called v in the chart x

charts, one has:

$= (dc)_{x(p)} (v^x)$ where $c = C^x_{x} = x' \circ x^{-1}$ (smooth map between open in $\mathbb{R}^n$)

i.e. T_pM

$\mathcal{C}^\infty(M) = \{ f: M \to \mathbb{R} /$ f = smooth $\}$

of M at p is any map

$(M) \to \mathbb{R}$

satisfies the derivation (Leibniz)

$g) = f(p) \cdot \partial(g) + g(p)\, \partial(f)$ $\forall f, g \in C^\infty(M)$.

the set of all such ∂.

⑧ [**Theorem**: $I_p : T_pM \longrightarrow T_p^{deriv} M$ is a linear isomorphism !

$$\begin{tikzcd} T_pM \ni v \arrow[d, "I_p"] \\ T_p^{deriv} M \ni \partial_v \end{tikzcd}$$

define

i.e. $\partial_v : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}$

$\partial_v(f) :\stackrel{\leftarrow}{=} \partial_{v^x}(f^x) = (df^x)_{x(p)}(v^x)$ $f^x = f \circ x^{-1}$

choose chart around p $v^x \in \mathbb{R}^n$

Choose γ s.t. $v = \frac{d\gamma}{dt}(0)$

$= (f \circ \gamma)'(0)$

$= (df)_p(v)$ ... exa

map
pen

th

Theorem: $I_p : T_pM \longrightarrow T_p^{\text{deriv}}M$

is a linear isomorphism !

$$\begin{tikzcd} T_pM \ni v \arrow[d, "I_p"] \\ T_p^{\text{deriv}}M \ni \overset{\text{define}}{\partial_v} \end{tikzcd}$$

i.e. $\partial_v : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}$

$\partial_v(f) :\overset{!}{=} \partial_{v^x}(f^x) = (df^x)_{x(p)}\cancel{(f\circ x)}$

choose chat around p

$v^x \in \mathbb{R}^n$

$f^x = f\circ x^{-1}$

Choose γ s.t. $v = \frac{d\gamma}{dt}(0)$

$\underline{=} (f\circ\gamma)'(0)$

$= (df)_p(v)$ ... exercise