

$M, p, m = \dim(M)$.

a tangent vector to M at p : a function ζ 1

$\zeta: \left. \begin{array}{l} \text{charts } \chi \text{ of} \\ M \text{ around } p \end{array} \right\} \rightarrow \mathbb{R}^m, \text{ denoted } \zeta \mapsto \zeta^\chi$

Such that, $(\forall) \chi, \chi': \zeta^{\chi'} = (d\chi)_{\chi^{-1}(p)}(\zeta^\chi)$, where $c = \chi' \circ \chi^{-1}$.

$T_p M$: $T_p M$ vector space

\mathbb{R}^k : any χ around p induces $T_p M \xrightarrow{\sim} \mathbb{R}^m, \zeta \mapsto \zeta^\chi$
Therefore, also basis, denoted $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ for $T_p M$.

In particular, for $M = \Omega \subset \mathbb{R}^m$ open: can use $\chi = \text{Id} \Rightarrow$

\Rightarrow Standard $p: T_p \Omega \xrightarrow{\sim} \mathbb{R}^m, \zeta \mapsto \zeta^{\text{Id}}$
 $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ - basis of $T_p \Omega$

App's
 $T_p M$

deriv
 $T_p M$

a derivation of M at p : deriv_p
 $\partial: C^\infty(M) \rightarrow \mathbb{R}$
linear & deriv rule $\partial(fg) = f(p)\partial g + g(p)\partial f$

* $T_p M$: same ∂ $(\partial + \partial')(f) = \partial f + \partial' f$

\mathbb{R}^k : For $M = \Omega \subset \mathbb{R}^n$ open, usual ∂ -derivatives
 $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p \in T_p M$

Prop: This is a basis of $T_p M$
More generally, $(\forall) M, (\forall) \text{ chart } \chi \text{ around } p$

... the derivations
 $f \mapsto \frac{\partial f}{\partial x_i}(x(p))$
 \uparrow
 $C^\infty(M) \rightarrow \mathbb{R}$

T_p sends $v \in T_p M$
to $\partial_v \in T_p^{\text{deriv}} M$
which is
 $\partial_v: C^\infty(M) \rightarrow \mathbb{R}, f \mapsto \partial_v(f) = (df)_{x(p)}(v)$

choose χ
 $(df)_{x(p)}(v) = (df^\chi)_{x(p)}\left(\frac{dx^\chi}{dt}(t)\right) = \frac{d(f^\chi)}{dt}(t)$

TW 4: $\gamma \in \text{Curves}_p(M)$ has $\frac{d\gamma}{dt}(0) \in T_p M$... defined by a
 $\left(\frac{d\gamma}{dt}(0)\right)^x := \frac{d\gamma^x}{dt}(0) \in \mathbb{R}^m$ usual deriv. in \mathbb{R}^m

$\gamma: (-\varepsilon, \varepsilon) \rightarrow M$
 $\gamma(0) = p$

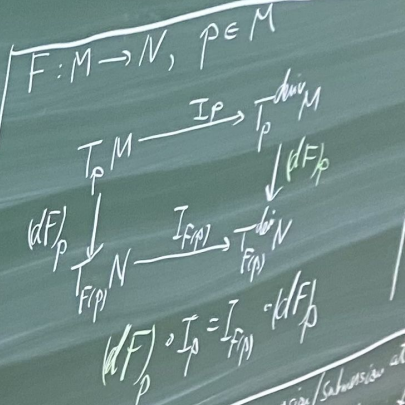
TW 2: Smooth maps $F: M \rightarrow N$ induce linear maps
 $(dF)_p: T_p M \rightarrow T_{F(p)} N$ & usual properties

Applied to inclusions $i: M \hookrightarrow N \Rightarrow$ get inclusions $\Leftrightarrow p \in M$
 $T_p M \subseteq T_p N$

TW 3: nothing new in \mathbb{R}^m for $M \subseteq \mathbb{R}^L$ embedded: $(p \in M)$
 $T_p M \xrightarrow{\text{standard } p} T_p \mathbb{R}^L$
 \downarrow
 $T_p^{\text{geom}} M \xrightarrow{\text{standard } p} \mathbb{R}^L$

TW 4: $\gamma \in \text{Curves}_p(M)$ has $\frac{d\gamma}{dt}(0) \in T_p M$... defined by $f \circ \gamma$
 $f: M \rightarrow N$

TW 2: c



CONSEQUENCE: $F: M \rightarrow N$ is immersion/submersion $\Leftrightarrow (dF)_p: T_p M \rightarrow T_{F(p)} N$ is injective/surjective

a function $\gamma: \mathbb{R} \rightarrow \mathbb{R}^m$, denoted $\gamma(t) \rightarrow \gamma^x$, where $\gamma = \gamma \circ \gamma^{-1}$.

$\gamma: \mathbb{R} \rightarrow \mathbb{R}^m$, $\gamma(t) \rightarrow \gamma^x$
 $\left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_m} \right)_p$ for $T_p M$.

open can use $\gamma = \text{Id} \Rightarrow$

$\gamma: \mathbb{R} \rightarrow \mathbb{R}^m$, $\gamma(t) \rightarrow \gamma^x$
 basis of $T_p \mathbb{R}^m$

I_p sends $v \in T_p M$ to $\partial_v \in T_p^{\text{deriv}} M$ which is $\partial_v: C^\infty(M) \rightarrow \mathbb{R}$,
 $f \mapsto \partial_v(f) = \left(\frac{df}{dt} \right)_{\gamma^{-1}(\gamma(p))} \circ \gamma^x$

a derivation of M at p : any $\partial: C^\infty(M) \rightarrow \mathbb{R}$
 linear & deriv. rule $\partial(fg) = f(p)\partial(g) + g(p)\partial(f)$
 $(\partial + \partial')(f) = \partial(f) + \partial'(f)$

Thm 1: same \leftarrow
 For $M = \Omega \subset \mathbb{R}^m$ open, usual ∂ -derivatives at p :
 $\left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_m} \right)_p \in T_p^{\text{deriv}} \Omega$

Prop: This is a basis of $T_p^{\text{deriv}} \Omega$

More generally, $(\nabla) M, (\nabla)$ chart χ around p : $\left(\frac{\partial}{\partial x_1} \right)_p, \dots, \left(\frac{\partial}{\partial x_m} \right)_p \in T_p^{\text{deriv}} M$

... the derivations

$$f \mapsto \frac{\partial f}{\partial x_i}(\chi(p))$$

$$\uparrow$$

$$C^\infty(M) \quad \uparrow \quad \mathbb{R}$$

$$(f \circ \chi^{-1}) \circ \chi \circ \gamma = f \circ \gamma$$

$$\left(\frac{df}{dt} \right)_{\chi^{-1}(\gamma(p))} \circ \gamma^x = \left(\frac{df}{dt} \right)_{\gamma^{-1}(\gamma(p))} \circ \gamma^x$$

by (2)

For $\gamma \in \text{Curve}_{\text{loop}}(M)$
 $\frac{d\gamma}{dt}(0) \mapsto \dots$
 In \mathbb{R}^m : $\left(\frac{\partial}{\partial x_i} \right)_p \mapsto \dots$
 Compatible with taking

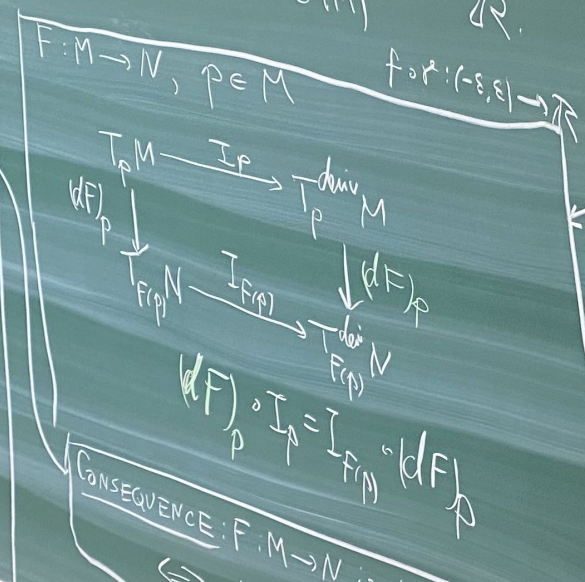
THEOREM: $(\nabla) M, (\nabla), \rho$
 $I_p: T_p M \rightarrow \dots$
 IS A LINEAR

IS A LINEAR ISOM

has $\frac{dx}{dt} \in T_p M \dots$ defined by $\textcircled{3}$
 $T_p M := \frac{dx}{dt} \Big|_p \left(\mathbb{R}^n \right)$ usual deriv in \mathbb{R}^n
 maps $F: M \rightarrow N$ induce linear maps
 $F_p: T_p M \rightarrow T_p N$ & usual properties

inclusions: $M \subset N \Rightarrow$ get inclusions
 $T_p M \subset T_p N$
 $p \in M$
 for $M \subset \mathbb{R}^k$ embedded:
 $T_p M \subset T_p \mathbb{R}^k$ (p.e.M)
 standard

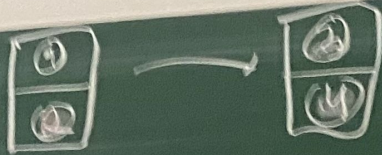
TW 4 $\gamma \in \text{Curves}_p(M)$ has $\gamma'(0) \in T_p \text{deriv } M$
 defined by $f \mapsto (f \circ \gamma)'(0)$
 \uparrow
 \mathbb{R}
 \uparrow
 $C^\infty(M)$



CONSEQUENCE: $F: M \rightarrow N$ is immersion/submersion at p
 $\Leftrightarrow (dF)_p: T_p M \rightarrow T_{F(p)} N$ is injective/surjective

$\frac{dx}{dt} \Big|_p$

Remarks on $I_p: T_p M \rightarrow T_p^{\text{dens}} M$: (5) it takes



For $\gamma \in \text{Curve}(M)$

$$\frac{d\gamma}{dt}(0) \xrightarrow{I_p} \gamma'(0)$$

In \mathbb{R}^n

$$\left(\frac{\partial}{\partial x_i}\right)_p \xrightarrow{I_p} \left(\frac{\partial}{\partial x_i}\right)_p$$

Compatible with taking differentials

$$\left(\frac{\partial}{\partial x_i}\right)_p \rightarrow \left(\frac{\partial}{\partial x_i}\right)_p \in T_p^{\text{dens}} M$$

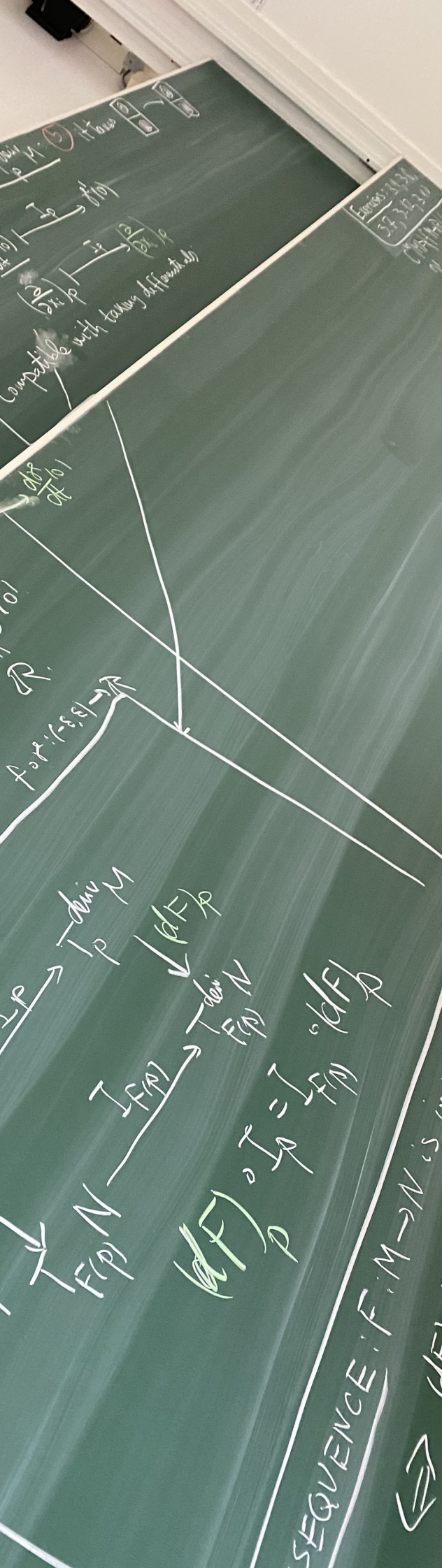
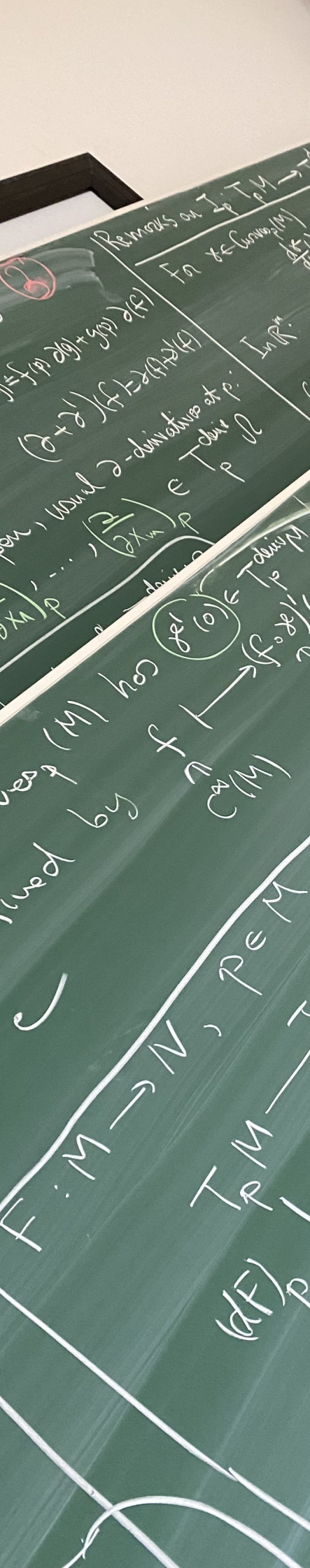
THEOREM (7) M, \mathbb{R}, p
 $I_p: T_p M \rightarrow T_p^{\text{dens}} M$
 IS A LINEAR ISOMORPHISM

$f: \text{open} \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$
 $d(fg) = f(p)dg + g(p)df$
 $d(fg) = (f(p)dg) + (g(p)df)$
 $(\partial + \partial)(f+g) = (\partial f) + (\partial g)$

\mathbb{R}^m open, usual ∂ -derivatives at p .
 $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}) \in T_p \mathbb{R}^m$

Remarks on $T_p M \rightarrow T_p M$
 For $\alpha \in \text{Curves}_p(M)$
 $\frac{d\alpha}{dt} \Big|_0 \in T_p M$

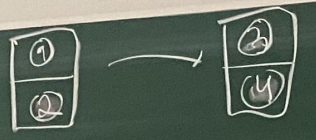
$\frac{d}{dt} \Big|_0$
 Computed with taking differentials



SEQUENCE: $F: M \rightarrow N$'s commutative Subsequence
 $\Leftrightarrow (dF)_p: T_p M \rightarrow T_p N$

any γ
 $f(p) \partial g + g(p) \partial f$
 $\gamma'(t) f \in \gamma'(t) + \gamma'(t) f$
 -derivatives at p
 $\in T_p \mathbb{R}^n$

Remarks on $I_p: T_p M \rightarrow T_p^{deriv} M$: $\textcircled{5}$ it takes



For $\gamma \in \text{Curv}_{\text{reg}}(M)$
 $\frac{d\gamma}{dt}(0) \xrightarrow{I_p} \gamma'(0)$

In \mathbb{R}^m : $\left(\frac{\partial}{\partial x_i}\right)_p \xrightarrow{I_p} \left(\frac{\partial}{\partial x_i}\right)_p$

Compatible with taking differentials

and $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{deriv} M$

Rk: Using χ chart around p (on arbitrary M) $\left(\frac{\partial}{\partial x_i}\right)_p$ - basis of $T_p M$
 sent to the $\left(\frac{\partial}{\partial x_i}\right)_p$
 Hence $I_p = \text{id}$

For $M = \mathbb{R}^m$ is the proposition!

THEOREM (7) M, p, χ
 $I_p: T_p M \rightarrow T_p^{deriv} M$
 IS A LINEAR ISOMORPHISM

$\Leftrightarrow \left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{deriv} M$
 is basis ??

$(f \circ \chi^{-1}) \circ \chi = f$
 $\gamma \circ \chi^{-1}(0) = (f \circ \gamma)'(0)$

Exercises: 3.1, 3.6, 3.7, 3.12, 3.10

$$\frac{dx^i}{dt} \Big|_{t=0}$$

Claim: To prove the theorem it suffices to prove (6)

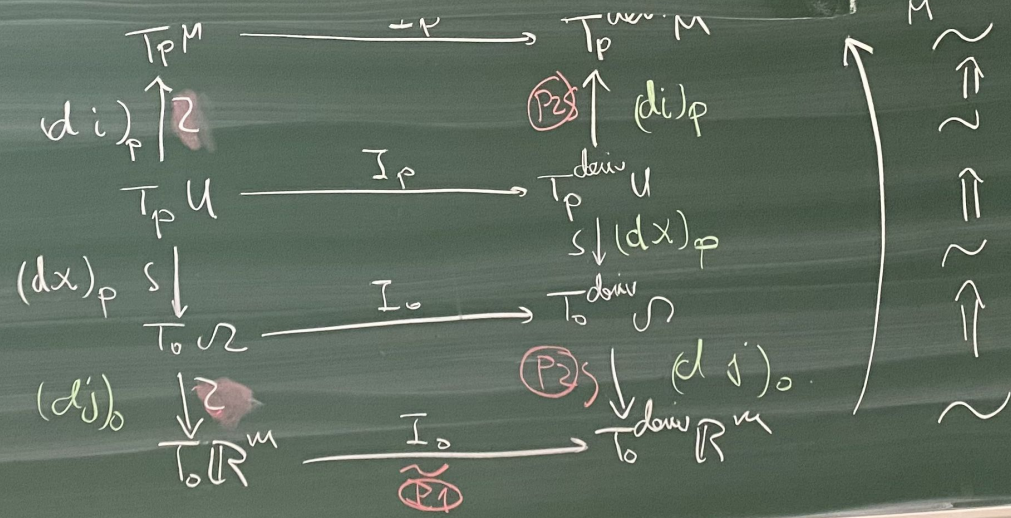
(P1) $\left(\frac{\partial}{\partial x^1}\right)_0, \dots, \left(\frac{\partial}{\partial x^m}\right)_0$ is a basis of $T_0 \text{ deriv } \mathbb{R}^m$

(P2) On any M , for any open $U \subset M$, $p \in U$, the map induced by inclusion, $(di)_p: T_p \text{ deriv } U \rightarrow T_p \text{ deriv } M$ is an isomorphism

Proof that (P1) & (P2) \Rightarrow the theorem: Let M, p -general. Choose chart $\mathcal{U}: U \xrightarrow{i} \mathbb{R}^m$ open around p , with $x(p)=0$.

$$i: U \rightarrow M$$

$$j: \mathbb{R}^m \rightarrow \mathbb{R}^m$$



DONE



$$\frac{\partial f}{\partial x^m} \Big|_{(0)}$$

$p, v \in \mathbb{R}^m$
 vector to M at p : a function
 charts $\mathcal{X} \rightarrow \mathbb{R}^m$, denoted $\mathcal{X} \mapsto \mathcal{X}^i$
 Marsden & Hughes
 $\mathcal{X}^i = (c^i)_{\mathcal{X}, p}$, where $c = \mathcal{X} \circ \mathcal{X}^{-1}$
 to space
 induces $T_p M \rightarrow \mathbb{R}^m$, $v \mapsto v^i$
 basis, denoted $\left(\frac{\partial}{\partial x_i}\right)_p$ for $T_p M$.

$M = \mathcal{U} \subset \mathbb{R}^m$ open, can use $\mathcal{X} = \text{Id} \Rightarrow$
 $T_p \mathcal{U} \xrightarrow{\text{Id}} \mathbb{R}^m$, $v \mapsto v^i$
 $\left(\frac{\partial}{\partial x_i}\right)_p$ - basis of $T_p \mathbb{R}^m$
 T_p sends $v \in T_p M$
 $\hookrightarrow v^i \in T_p \mathbb{R}^m$
 which is
 $\mathcal{D}_v C^{\infty}(M) \rightarrow \mathbb{R}^m$
 $\oplus \rightarrow \mathcal{D}_v(f) = \left(\frac{df}{dx^i}\right)_{\mathcal{X}, p}$

App 10.5
 $T_p M$

Proof of (P1) that $\left(\frac{\partial}{\partial x_i}\right)_0, \dots, \left(\frac{\partial}{\partial x_m}\right)_0$ is
 a basis of $T_0 \text{der} \mathbb{R}^m$

① Linearly independent? Assume $\lambda_1, \dots, \lambda_m \in \mathbb{R}$
 s.t. $\lambda_1 \left(\frac{\partial}{\partial x_1}\right)_0 + \dots + \lambda_m \left(\frac{\partial}{\partial x_m}\right)_0 = 0$ in $T_0 \text{der} \mathbb{R}^m$

To prove: $\lambda_1 = \dots = \lambda_m = 0$
 \rightarrow means: $(\forall) f \in C^{\infty}(\mathbb{R}^m): \lambda_1 \frac{df}{dx_1}(0) + \dots + \lambda_m \frac{df}{dx_m}(0) = 0$
 Choose $f = p_i, f(x) = x_i$. For any $i, \lambda_i = 0$

② Generating? Start with arbitrary $\partial \in T_0 \text{der} \mathbb{R}^m$
 To do: look for $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ s.t.

$$\partial = \lambda_1 \left(\frac{\partial}{\partial x_1}\right)_0 + \dots + \lambda_m \left(\frac{\partial}{\partial x_m}\right)_0$$

i.e.

$$\partial(f) = \lambda_1 \frac{df}{dx_1}(0) + \dots + \lambda_m \frac{df}{dx_m}(0) \quad (\forall) f$$

For $f = x_i$ this would mean $\partial(p_i) = \lambda_i$. Therefore define $\lambda_i = \partial(x_i)$

Remarks
 For $\gamma \in \text{Curve}_{\text{loop}}(M)$
 $v = \frac{d\gamma}{dt}(0) \in T_p$

In \mathbb{R}^m : $\left(\frac{\partial}{\partial x_i}\right)_p \in T_p$
 Compatible with taking def.

Rk: Using \mathcal{X} chart around p
 on the theorem

THEOREM: $(\forall) M, p, \mathcal{X}$
 $I_p: T_p M \rightarrow T_p \text{der}$
 IS A LINEAR ISOMORPHISM

Left to do: Show: $(\forall) f \in C^{\infty}(\mathbb{R}^m)$

Use that $\int \frac{dh}{dt} dt = h(t) - h(0)$

for all $h: I \rightarrow \mathbb{R}$

Apply it to $h(t) = f(t, x)$:

$f(t) - f(0) = \int_0^t \dots$
 formula for g_i

Left to do: show: (1) $f \in C^1(\mathbb{R}^m)$ one has $\frac{df}{dt}(0)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \dots + \frac{\partial f}{\partial x_m} \frac{dx_m}{dt}$$

Key: Any f can be written as

$$f(x) = f(0) + x_1 g_1(x) + \dots + x_m g_m(x)$$

 for some $g_1, \dots, g_m \in C^0(\mathbb{R}^m)$

i.e. $f = f(0) + p_1 g_1 + \dots + p_m g_m$
 as function
 $\Rightarrow \frac{df}{dt} = \frac{df(0)}{dt} + \frac{d}{dt}(p_1 g_1 + \dots + p_m g_m)$
 deriv rule $\Rightarrow \left(\frac{dp_1}{dt} g_1 + p_1 \frac{dg_1}{dt} + \dots \right)$
 $= \frac{dp_1}{dt} g_1(0) + \dots + \frac{dp_m}{dt} g_m(0)$

But $\frac{df}{dx_i}(0) = g_i(0)$

$$\frac{dp_1}{dt} \frac{df}{dx_1}(0) + \dots + \frac{dp_m}{dt} \frac{df}{dx_m}(0)$$

$\frac{dx}{dt}(0)$

(P2)
 th

$i: U \rightarrow M$

$T_p M$
 \uparrow
 $d i)_p$
 $T_p U$
 $(dx)_p$

$j: V \rightarrow \dots$

Use that

$$\int_0^x \frac{dh}{dt}(t) dt = h(x) - h(0)$$

for all $h: [a,b] \rightarrow \mathbb{R}$.

Apply it to $h(t) = f(t(x))$:

$$f(x) - f(0) = \sum x_i \text{ formulas for } g_i$$

9

Left to do: show (\mathbb{R}^n) one has

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

Key: Any f can be written as

$$f(x) = f(0) + x_1 g_1(x) + \dots + x_n g_n(x)$$

for some $g_i \in C^0(\mathbb{R}^n)$

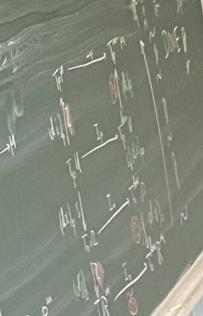
$$\text{i.e. } f = f(0) + p_1 x_1 + \dots + p_n x_n$$

$$\Rightarrow df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

$$\text{deriv rule} \Rightarrow \left(\frac{\partial f}{\partial x_1} g_1 + g_1 \frac{\partial f}{\partial x_1} + \dots \right)$$

$$\text{But } \frac{\partial f}{\partial x_i} = g_i(0) + \dots$$

On any f for any small h , $f(x+h) - f(x)$ is approx $df_x(h)$



VECTOR FIELDS

(9)

$M = \text{manifold}$. A set theoretical vector field on M is any map X :

$$M \ni p \longmapsto X_p \in T_p M$$

Smoothness of X ? Use charts $\chi: U \rightarrow \mathbb{R}^m$.

and the induced basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ for any $p \in U$

\Rightarrow can write: $X_p = \text{coeff}_1^X(p) \left(\frac{\partial}{\partial x_1}\right)_p + \dots + \text{coeff}_m^X(p) \left(\frac{\partial}{\partial x_m}\right)_p$

where $\text{coeff}^i: U \rightarrow \mathbb{R}$ require all these coefficients to be smooth (\forall) charts

M is any s.t. v.f. X that is smooth.

To prove: $\lambda_1 \dots \lambda_m = 0$
 \rightarrow no ans. $(\forall) f \in C^\infty(\mathbb{R}^m): \lambda_1 \frac{\partial f}{\partial x_1} + \dots + \lambda_m \frac{\partial f}{\partial x_m} = 0$
 Compatible with taking derivatives

Left to do: show: $(\forall) f \in C^\infty(\mathbb{R}^m)$ one has
 $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x_1} \frac{\partial x_1}{\partial t} + \dots + \frac{\partial f}{\partial x_m} \frac{\partial x_m}{\partial t}$

Key: Any f can be written as
 $f(x) = f(0) + x_1 g_1(x) + \dots + x_m g_m(x)$
 for some $g_i: \mathbb{R}^m \rightarrow \mathbb{R}$

\rightarrow i.e. $f = f(0) + p_1 g_1 + \dots + p_m g_m$
 on function

$\Rightarrow \frac{\partial f}{\partial t} = f(0) \cdot \frac{\partial t}{\partial t} + \frac{\partial}{\partial t} (p_1 g_1 + \dots + p_m g_m)$
 deriv rule $\Rightarrow \left(\frac{\partial p_1}{\partial t} g_1(0) + p_1 \frac{\partial g_1}{\partial t} + \dots \right)$
 $= \frac{\partial p_1}{\partial t} g_1(0) + \dots + \frac{\partial p_m}{\partial t} g_m(0)$

But $\frac{\partial f}{\partial x_i}(0) = g_i(0) = \left. \frac{\partial f}{\partial x_i} \right|_{x=0}$

