$M, p, m = \dim(M)$.

- a tangent vector to M at p: a function ①

$$v : \left\{ \begin{array}{l} \text{charts } \chi \text{ of} \\ M \text{ around } p \end{array} \right\} \longrightarrow \mathbb{R}^m, \text{ denoted } \chi \longmapsto v^\chi$$

Such that, $(\forall)\ \chi, \chi'$: $v^{\chi'} = (dc)_{\chi(p)}(v^\chi)$, where $c = \chi' \circ \chi^{-1}$.

All v's: T_pM $\xrightarrow{I_p}$ $T_p^{deriv} M$

- TW1: T_pM vector space

Rk: any χ around p induces $T_pM \xrightarrow{\sim} \mathbb{R}^m$, $v \longmapsto v^\chi$

Therefore, also basis, denoted $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ for T_pM.

In particular, for $M = \Omega \subseteq \mathbb{R}^m$ open: can use $\chi = Id \Rightarrow$

$$\Rightarrow \boxed{\text{Standard}_p : T_p\Omega \xrightarrow{\sim} \mathbb{R}^m, v \mapsto v^{Id}}$$

$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ - basis of $T_p\Omega$

- a derivation of M at p: any ②

$$\partial : C^\infty(M) \to \mathbb{R}$$

linear & deriv. rule $\partial(fg) = f(p)\partial(g) + g(p)\partial(f)$

- TW1: same ~ $(\partial + \partial')(f) = \partial(f) + \partial'(f)$

Rk: For $M = \Omega \subseteq \mathbb{R}^m$ open, usual ∂-derivatives

$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{deriv}\Omega$

Prop: This is a basis of $T_p^{deriv}\Omega$

More generally, $(\forall)$ M, $(\forall)$ chart χ around p

... the derivations

$$f \longmapsto \frac{\partial f_\chi}{\partial x_i}(\chi(p))$$

$f \in C^\infty(M)$, $\frac{\partial f_\chi}{\partial x_i}(\chi(p)) \in \mathbb{R}$

I_p: sends $v \in T_pM$ to $\partial_v \in T_p^{deriv}M$

which is

$$\partial_v : C^\infty(M) \to \mathbb{R},$$

$$f \longmapsto \partial_v(f) := (df^\chi)_{\chi(p)}(v^\chi) = (df^\chi)_{\chi(p)}\left(\frac{d\gamma^\chi}{dt}(0)\right) = (f_\chi \circ \dots$$

choose χ

• **TW 1**: $\gamma \in \mathrm{Curves}_p(M)$ has $\frac{d\gamma}{dt}(0) \in T_p M$ ... defined by ②

$\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$

$$\boxed{\left(\frac{d\gamma}{dt}(0)\right)^x := \frac{d\gamma^x}{dt}(0)} \quad (\in \mathbb{R}^m)$$

usual deriv in $\mathbb{R}^m$

• **TW 2**: Smooth maps $F : M \to N$ induce linear maps $\qquad$ $\ni p$

$(dF)_p : T_p M \to T_{F(p)} N$ & usual properties

Applied to inclusions $i : \underline{M} \hookrightarrow \underline{N} \Rightarrow$ get inclusions

$T_p M \subseteq T_p N$ $\quad (\forall p \in M)$

• **TW 3**: nothing new in $\mathbb{R}^m$: for $M \subseteq \mathbb{R}^L$ embedded: $(p \in M)$

$$\begin{array}{ccc} T_p M & \hookrightarrow & T_p \mathbb{R}^L \\ \vdots\ \downarrow \cong ? & & \downarrow \cong \text{ standard}_p \\ T_p^{geom} M & \hookrightarrow & \mathbb{R}^L \end{array}$$

TW 1: $\gamma \in \mathrm{Curves}_p(M)$ has ...

defined by $f \mapsto$ ...

$C^\infty(M)$

TW 2: ...

$F : M \to N$, $p \in M$

$$\begin{array}{ccc} T_p M & \xrightarrow{I_p} & T_p^{deriv} M \\ (dF)_p \downarrow & & \downarrow (dF)_p \\ T_{F(p)} N & \xrightarrow{I_{F(p)}} & T_{F(p)}^{deriv} N \end{array}$$

$(dF)_p \circ I_p = I_{F(p)} \circ (dF)_p$

CONSEQUENCE: $F : M \to N$ is immersion/submersion at ...

$\Leftrightarrow (dF)_p : T_p M \to T_{F(p)} N$ is injective/surjective ...

a function ①

$\to \mathbb{R}^m$, denoted $x \mapsto v^x$

(v^x), where $c = x' \circ x^{-1}$

All v's: T_pM $\xrightarrow{I_p}$ $T_p^{deriv}M$

$\to \mathbb{R}^m$, $v \mapsto v^2$

$(\ldots)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p$ for T_pM.

open: can use $x = Id \Rightarrow$

$\to \mathbb{R}^m$, $v \mapsto v^{Id}$

basis of $T_p\Omega$

②

- a derivation of M at p: any

$$\partial : C^\infty(M) \to \mathbb{R}$$

linear & deriv. rule: $\partial(fg) = f(p)\,\partial(g) + g(p)\,\partial(f)$

$(\partial + \partial')(f) = \partial(f) + \partial'(f)$

- TW1: same ✓

Rk: For $M = \Omega \subseteq \mathbb{R}^m$ open, usual ∂-derivatives at p:

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{deriv}\Omega$$

Prop: This is a basis of $T_p^{deriv}\Omega$

More generally, $(\forall)$ M, $(\forall)$ chart x around p: $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{deriv}M$

... the derivations

$$f \longmapsto \frac{\partial f_x}{\partial x_i}(x(p))$$

$$f \in C^\infty(M), \quad \frac{\partial f_x}{\partial x_i}(x(p)) \in \mathbb{R}$$

I_p sends $v \in T_pM$ to $\partial_v \in T_p^{deriv}M$

which is

$\partial_v : C^\infty(M) \to \mathbb{R}$,

$f \longmapsto \partial_v(f) \overset{\text{choose } x}{:=} (df^x)_{x(p)}(v^x) = (df^x)_{x(p)}\left(\frac{d\gamma^x}{dt}(0)\right) = \left(\underbrace{f_x \circ \gamma^x}\right)'(0) = (f \circ \gamma)'(0)$

$(f \circ x^{-1}) \circ (x \circ \gamma) = f \circ \gamma$

For $\gamma \in Curves_p(M)$ ⊢

$\frac{d\gamma}{dt}(0)$ ⊢

In $\mathbb{R}^m$: $\left(\frac{\partial}{\partial x_i}\right)_p$ ⊢

Compatible with taking

THEOREM: $(\forall)$ M, $\forall$, p

$I_p : T_pM \longrightarrow$

IS A LINEAR

... has $\frac{d\gamma}{dt}(0) \in T_pM$ ... defined by (2)

$(\ldots)' := \frac{d\gamma}{dt}(0)$ $(\in \mathbb{R}^m)$ usual deriv in $\mathbb{R}^m$

... maps $F: M \to N$ induce linear maps

$(dF)_p : T_pM \to T_{F(p)}N$ & usual properties

... $M \subseteq N \Rightarrow$ get inclusions

$T_pM \subseteq T_pN \quad \forall p \in M$

... on $\mathbb{R}^m$: for $M \subseteq \mathbb{R}^L$ embedded:

$$\begin{array}{ccc} T_pM & \hookrightarrow & T_p\mathbb{R}^L \\ \wr\,? & & \wr \text{ standard} \\ T_p^{geom}M & \hookrightarrow & \mathbb{R}^L \end{array} \qquad (p \in M)$$

(4) TW 4: $\gamma \in \mathrm{Curves}_p(M)$ has $\gamma'(0) \in T_p^{deriv}M$ → $\frac{d\gamma}{dt}(0)$

defined by $f \longmapsto (f \circ \gamma)'(0)$, $\quad f \in C^\infty(M)$, $(f\circ\gamma)'(0) \in \mathbb{R}$.

TW 2:

$\boxed{\begin{array}{l} F: M \to N,\ p \in M \qquad f\circ\gamma : (-\varepsilon,\varepsilon) \to \mathbb{R} \\[1ex] \begin{array}{ccc} T_pM & \xrightarrow{I_p} & T_p^{deriv}M \\ \downarrow (dF)_p & & \downarrow (dF)_p \\ T_{F(p)}N & \xrightarrow{I_{F(p)}} & T_{F(p)}^{deriv}N \end{array} \\[1ex] (dF)_p \circ I_p = I_{F(p)} \circ (dF)_p \end{array}}$

$\boxed{\begin{array}{l} \text{CONSEQUENCE: } F: M \to N \text{ is immersion / submersion at } p \\ \Leftrightarrow (dF)_p : T_pM \to T_{F(p)}N \text{ is injective / surjective} \end{array}}$

Remarks on $I_p : T_pM \longrightarrow T_p^{deriv}M$: 5 it takes $\boxed{\begin{matrix}1\\2\end{matrix}} \longrightarrow \boxed{\begin{matrix}3\\4\end{matrix}}$

For $\gamma \in \mathrm{Curves}_p(M)$

$$\frac{d}{dt}(\gamma)(0) \overset{I_p}{\longmapsto} \gamma'(0)$$

In $\mathbb{R}^n$: $\left(\frac{\partial}{\partial x_i}\right)_p \overset{I_p}{\longmapsto} \left(\frac{\partial}{\partial x_i}\right)_p$

Compatible with taking differentials

$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p \in T_p^{deriv}M$

THEOREM $(\forall) M, \forall p$,

$I_p : T_pM \longrightarrow T_p^{deriv}M$

IS A LINEAR ISOMORPHISM

$\frac{\partial}{\partial t}\cdot(\gamma)(0) = (f\circ\gamma)'(0)$

$= f \circ \gamma$

$T_p^{deriv}\mathbb{R}$

derivatives at p

ℓ: any

$\dots \to \mathbb{R}$

$\dots$le: $\partial(fg) = f(p)\,\partial(g) + g(p)\,\partial(f)$

$(\partial + \partial')(f) = \partial(f) + \partial'(f)$

$\dots$e ✓

$\dots \to \mathbb{R}^m$ open, usual ∂-derivatives at p: $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{deriv}\Omega$

Remarks on $I_p : T_pM \to T_p^{deriv}M$: (5) it's an iso

For $\gamma \in \mathrm{Curves}_p(M)$: $\frac{d\gamma}{dt}(0) \xmapsto{I_p} \gamma'(0)$

In $\mathbb{R}^m$: $\left(\frac{\partial}{\partial x_i}\right)_p \xmapsto{I_p} \left(\frac{\partial}{\partial x_i}\right)_p$

Compatible with taking differentials

$\dots \in \mathrm{Curves}_p(M)$ has $\gamma'(0) \in T_p^{deriv}M$ defined by $f \longmapsto (f\circ\gamma)'(0)$, $f \in C^\infty(M)$, $(f\circ\gamma)'(0) \in \mathbb{R}$, $f\circ\gamma : (-\varepsilon,\varepsilon) \to \mathbb{R}$

$\frac{d\gamma}{dt}(0)$

$\dots$W2: ✓

$F : M \to N$, $p \in M$

$$\begin{array}{ccc} T_pM & \xrightarrow{I_p} & T_p^{deriv}M \\ \downarrow (dF)_p & & \downarrow (dF)_p \\ T_{F(p)}N & \xrightarrow{I_{F(p)}} & T_{F(p)}^{deriv}N \end{array}$$

$$(dF)_p \circ I_p = I_{F(p)} \circ (dF)_p$$

$\dots$SEQUENCE: $F : M \to N$ is immersion/submersion $\dots$

$\Longleftrightarrow (dF)_p : T_pM \to T\dots$

Exercises: 3.4, 3.6, 3.7, 3.12, 3.10

Remarks on $I_p : T_pM \longrightarrow T_p^{deriv} M$: ⑤ it takes $\boxed{\begin{matrix}1\\2\end{matrix}} \longrightarrow \boxed{\begin{matrix}3\\4\end{matrix}}$

For $\gamma \in \mathrm{Curv}_{\mathrm{loop}}(M)$

$$\frac{d\gamma}{dt}(0) \xmapsto{I_p} \gamma'(0)$$

In $\mathbb{R}^m$: $\boxed{\left(\frac{\partial}{\partial x_i}\right)_p \xmapsto{I_p} \left(\frac{\partial}{\partial x_i}\right)_p}$

Compatible with taking differentials

... around p: $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{deriv} M$

Rk: Using χ chart around p (on arbitrary M): $\left(\frac{\partial}{\partial x_1}\right)_p, \dots$ basis of T_pM

... sent to the $\left(\frac{\partial}{\partial x_1}\right)_p, \dots$

For $M = \mathbb{R}^m$ is the proposition!

Hence $I_p = \text{iso} \iff$

$\iff$ $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{deriv} M$ is basis ??.

THEOREM $(\forall) M, \forall p$,
$I_p : T_pM \longrightarrow T_p^{deriv} M$
IS A LINEAR ISOMORPHISM

... any ③

$f(p)\,\partial(g) + g(p)\,\partial(f)$

$\partial')(f) = \partial(f) + \partial'(f)$

-derivatives at p:

$\in T_p^{deriv} \Omega$

$(f\circ\chi^{-1})\circ(\chi\circ\gamma) = f\circ\gamma$

$(\dots\circ\gamma^{\chi})'(0) = (f\circ\gamma)'(0)$

Exercises: 3.1, 3.6, 3.7, 3.12, 3.10

Rk.: Using a chart around p (on arbitrary M): $\left(\frac{\partial}{\partial x_1}\right)_p, \dots$ basis of T_pM ... $M=\mathbb{R}^m$ is ...

<u>Claim</u>: To prove the theorem it suffices to prove: (6)

(P1) $\left(\frac{\partial}{\partial x_1}\right)_0, \dots, \left(\frac{\partial}{\partial x_m}\right)_0$ is a basis of $T_0^{deriv}\mathbb{R}^m$

(P2) On <u>any</u> M, for <u>any</u> open $U \subseteq M$, $p \in U$, the map induced by inclusion, $(di)_p : T_p^{deriv} U \to T_p^{deriv} M$ is an isomorphism

<u>Proof that (P1) & (P2) $\Rightarrow$ the theorem</u>: Let M, p – general. Choose chart $x : U \to \Omega$ open around p, with $x(p) = 0$.

$U \subseteq M$, $\Omega \subseteq \mathbb{R}^m$

$i : U \to M$

$j : \Omega \to \mathbb{R}^m$

$$\begin{tikzcd}
T_pM \arrow[r, "I_p"] & T_p^{deriv} M \\
T_pU \arrow[u, "(di)_p", "\wr"'] \arrow[r, "I_p"] \arrow[d, "(dx)_p"', "\wr"] & T_p^{deriv} U \arrow[u, "(di)_p"', "\text{(P2)}\ \wr"] \arrow[d, "(dx)_p", "\wr"'] \\
T_0\Omega \arrow[r, "I_0"] \arrow[d, "(dj)_0"', "\wr"] & T_0^{deriv}\Omega \arrow[d, "(dj)_0", "\text{(P2)}\ \wr"'] \\
T_0\mathbb{R}^m \arrow[r, "I_0", "\wr\ \text{(P1)}"'] & T_0^{deriv}\mathbb{R}^m
\end{tikzcd}$$

$\wr \Uparrow \wr \Uparrow \wr \Uparrow \wr$

DONE $\square$

(1)

$M, p, m = \dim(M)$

...ector to M at p: a function

... {charts χ of M around p} $\longrightarrow \mathbb{R}^m$, denoted $\chi \longmapsto v^\chi$

... χ': $v^{\chi'} = (dc)_{\chi(p)}(v^\chi)$, where $c = \chi' \circ \chi^{-1}$

...tor space

...p induces $T_pM \xrightarrow{\sim} \mathbb{R}^m$, $v \longmapsto v^\chi$

...asis, denoted $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ for T_pM.

$M = \Omega \subseteq \mathbb{R}^m$ open. Can use $\chi = Id \Rightarrow$

$\dots_p: T_p\Omega \xrightarrow{\sim} \mathbb{R}^m$, $v \mapsto v^{Id}$

$\dots, \left(\frac{\partial}{\partial x_m}\right)_p$ - basis of $T_p\mathbb{R}^m$

All v's: T_pM

$T_p^{deriv}M$

I_p

I_p sends $v \in T_pM$ to $\partial_v \in T_p^{deriv}M$ which is

$\partial_v: C^\infty(M) \to \mathbb{R}$,

$f \longmapsto \partial_v(f) := (df)^\chi_{\chi(p)}$

Proof of (P1) (that $\left(\frac{\partial}{\partial x_1}\right)_0, \dots, \left(\frac{\partial}{\partial x_m}\right)_0$ is a basis of $T_0^{deriv}\mathbb{R}^m$)

(1) Linearly independent? Assume $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ s.t.

$$\lambda_1\left(\frac{\partial}{\partial x_1}\right)_0 + \dots + \lambda_m\left(\frac{\partial}{\partial x_m}\right)_0 = 0 \quad \text{in } T_0^{deriv}\mathbb{R}^m$$

$C^\infty(\mathbb{R}^m) \to \mathbb{R}$

To prove: $\lambda_1 = \dots = \lambda_m = 0$

means: $(\forall) f \in C^\infty(\mathbb{R}^m)$: $\lambda_1 \frac{\partial f}{\partial x_1}(0) + \dots + \lambda_m \frac{\partial f}{\partial x_m}(0) = 0$

Choose $f = pr_i$: $f(x) = x_i$. For any i: $\lambda_i = 0$

(2) Generating? Start with $\boxed{\text{arbitrary } \partial} \in T_0^{deriv}\mathbb{R}^m$

To do: look for $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ s.t.

$$\partial = \lambda_1\left(\frac{\partial}{\partial x_1}\right)_0 + \dots + \lambda_m\left(\frac{\partial}{\partial x_m}\right)_0$$

i.e.

$$\partial(f) = \lambda_1 \frac{\partial f}{\partial x_1}(0) + \dots + \lambda_m \frac{\partial f}{\partial x_m}(0) \quad (\forall) f$$

For $f = x_i$ this would mean

$\partial(pr_i) = \lambda_i$. Therefore define $\boxed{\lambda_i = \partial(x_i)}$

Left to do: show: $(\forall) f \in C^\infty(\mathbb{R}^m)$ ...

(7)

Remarks ...

For $\gamma \in Curves_p(M)$, $v = \frac{d\gamma}{dt}(0) \xmapsto{I_p}$

In $\mathbb{R}^m$: $\boxed{\left(\frac{\partial}{\partial x_i}\right)_p \xmapsto{I_p}}$

Compatible with taking dif...

Rk. on the theorem: Using χ chart around ...

THEOREM: $(\forall) M, \forall p$,

$I_p: T_pM \longrightarrow T_p^{der}$

IS A LINEAR ISOMO...

... also bases, denoted $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ for ...

Choose $f = pr$ ...

$\frac{d\gamma}{dt}(0)$

Use that $\int_0^1 \frac{dh}{dt}(t)\,dt = h(1) - h(0)$

for all $h : [0,1] \to \mathbb{R}$.

Apply it to $h(t) = f(tx)$:

$f(1) - f(0) = \dots$

$\Rightarrow$ formula for g_i

(8)

Left to do: show: $(\forall) f \in C^\infty(\mathbb{R}^m)$ one has

$$\partial(f) = \partial(pr_1)\left(\frac{\partial f}{\partial x_1}\right)_0 + \dots + \partial(pr_m)\left(\frac{\partial f}{\partial x_m}\right)_0$$

Key: Any f can be written as

$$f(x) = f(0) + x_1 \cdot g_1(x) + \dots + x_m \cdot g_m(x)$$

for some $g_1, \dots, g_m \in C^\infty(\mathbb{R}^m)$

i.e. $f = f(0) + pr_1 \cdot g_1 + \dots + pr_m \cdot g_m$ as function

$$\Rightarrow \partial(f) \overset{\partial \text{ is lin}}{=} \underbrace{f(0) \cdot \partial(1)}_{0} + \partial(pr_1 \cdot g_1 + \dots + pr_m \cdot g_m)$$

deriv rule

$$= \left(\underbrace{pr_1(0)}_{0} \cdot \partial(g_1) + g_1(0)\,\partial(pr_1)\right) + \dots$$

$$= \partial(pr_1) \cdot g_1(0) + \dots + \partial(pr_m)\, g_m(0)$$

But $\frac{\partial f}{\partial x_i}(0) = g_i(0)$

$$\partial(pr_1)\frac{\partial f}{\partial x_1}(0) + \dots + \partial(pr_m)\frac{\partial f}{\partial x_m}(0)$$

P2

th...

$i : U \to M$

$$T_pM$$
$$(di)_p \uparrow \cong$$
$$T_pU$$

$(dx)_p$

$j : \Omega \to \dots$

Use that

$$\int_0^1 \frac{\partial h}{\partial t}(t)\,dt = h(1) - h(0)$$

for all $h : [0,1] \to \mathbb{R}$.

Apply it to $h(t) = f(tx)$:

$f(x) - f(0) = \underbrace{\ \ \ \ }_{x_i}$ formulas for g_i

Left to do: show: $(\forall) f \in C^\infty(\mathbb{R}^m)$ one has

$$\partial(f) = \partial(pr_1)\left(\frac{\partial f}{\partial x_1}\right)_0 + \dots + \partial(pr_m)\left(\frac{\partial f}{\partial x_m}\right)_0$$

Key: Any f can be written as

$$f(x) = f(0) + x_1 \cdot g_1(x) + \dots + x_m \cdot g_m(x)$$

for some $g_1, \dots, g_m \in C^\infty(\mathbb{R}^m)$

i.e. $f = f(0) + pr_1 \cdot g_1 + \dots + pr_m \cdot g_m$ as function

∂ is linear

$$\Rightarrow \partial(f) = \underbrace{f(0) \cdot \partial(1)}_{0} + \partial(pr_1 \cdot g_1 + \dots + pr_m \cdot g_m)$$

deriv rule

$$= \left(\underbrace{pr_1(0)}_{0} \cdot \partial(g_1) + g_1(0)\,\partial(pr_1)\right) + \dots$$

$$= \partial(pr_1) \cdot g_1(0) + \dots + \partial(pr_m)\, g_m(0)$$

$$= \partial(pr_1)\frac{\partial f}{\partial x_1}(0) + \dots + \partial(pr_m)\frac{\partial f}{\partial x_m}(0)$$

But $\frac{\partial f}{\partial x_i}(0) = g_i(0)$

VECTOR FIELDS

9

M = manifold. A set theoretical vector field on M is any map X:

$$M \ni p \longmapsto X_p \in T_pM$$

Smoothness of X? Use charts $x: U \to \Omega$, $U \subset M$, $\Omega \subset \mathbb{R}^m$, and the induced basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ for any $p \in U$

$\Rightarrow$ can write: $X_p = \mathrm{coeff}^1(p) \cdot \left(\frac{\partial}{\partial x_1}\right)_p + \dots + \mathrm{coeff}^m(p) \left(\frac{\partial}{\partial x_m}\right)_p$

where $\mathrm{coeff}^i : U \to \mathbb{R}$ ← require all these coefficients to be smooth $(\forall)$ charts x

... on M is any s.t.v.f. X that is smooth.

Left to do: Show: $(\forall) f \in C^\infty(\mathbb{R}^m)$ one has

$$\partial(f) = \partial(pr_1)\left(\frac{\partial f}{\partial x_1}\right)_0 + \dots + \partial(pr_m)\left(\frac{\partial f}{\partial x_m}\right)_0$$

8

Key: Any f can be written as

$$f(x) = f(0) + x_1 \cdot g_1(x) + \dots + x_m \cdot g_m(x)$$

for some $g_1, \dots, g_m \in C^\infty(\mathbb{R}^m)$

i.e. $f = f(0) + pr_1 \cdot g_1 + \dots + pr_m \cdot g_m$ as functions

$$\Rightarrow \partial(f) = \underbrace{f(0) \cdot \partial(1)}_{0} + \partial(pr_1 \cdot g_1 + \dots + pr_m \cdot g_m)$$

deriv rule $\Rightarrow = \underbrace{pr_1(0)}_{0} \cdot \partial(g_1) + g_1(0)\,\partial(pr_1) + \dots$

$= \partial(pr_1) \cdot g_1(0) + \dots + \partial(pr_m) \cdot g_m(0)$

But $\frac{\partial f}{\partial x_i}(0) = g_i(0)$ — $= \partial(pr_1)\frac{\partial f}{\partial x_1}(0) + \dots + \partial(pr_m)\frac{\partial f}{\partial x_m}(0)$