Reminder: two faces of tgt vectors to $M$ at $p$: ①

$v \in T_pM$ : then we have $v^x \in \mathbb{R}^m$ $(\forall)$ $x$ = chart around $p$ (...)

$\partial_v \in T_p^{deriv}M$ : then we have $\boxed{\partial_v : C^\infty(M) \to \mathbb{R}}$ linear

& $\partial_v(fg) = f(p)\partial_v(g) + g(p)\partial_v(f)$

The two: identified via iso:

$$\boxed{I_p : \underset{v}{\underset{\psi}{T_pM}} \xrightarrow{\sim} \underset{\partial_v}{\underset{\psi}{T_p^{deriv}M}}}, \quad v \longmapsto \partial_v \text{ where } \boxed{\partial_v(f) \underset{\uparrow}{=} (df^x)_p(v^x)}$$

$v = X_p$ $\quad$ $x \cdots$

Important:

- given a chart $(U, x)$ $\Rightarrow$ for all $p \in U$ an induced

$$\boxed{\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \text{ - basis of } T_pM}$$

in $T_pM$ : those which, w.r.t. $x$ give the standard $e_1, \dots, e_m \in \mathbb{R}^m$

in $T_p^{deriv}M$ : the usual partial deriv. in $x$.

- we have differentials ...

$$(dF)_p : T_pM \longrightarrow T_{F(p)}N$$

2

## Vector fields on M (p no longer fixed!)

- a set theoretical vector field X : any

$$M \ni p \longmapsto \boxed{X_p \in T_pM} \subseteq T_p\mathbb{R}^L$$

- Smoothness: use any chart $(U, x)$.

For $p \in U$ use, and write, for $p \in U$:

$$X_p = \underbrace{\text{coeff}_1(p)}_{\in \mathbb{R}} \cdot \left(\frac{\partial}{\partial x_1}\right)_p + \dots$$

i.e. we have coefficient functions

$$\text{coeff}_1, \dots, \text{coeff}_m : U \to \mathbb{R}$$

Require all to be smooth, for all $x$.

**Def**: A vector field on M is any X as above, which is smooth. Denote by $\boxed{\mathfrak{X}(M)}$ the set of all vector fields on M

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# OPERATIONS INVOLVING $\mathfrak{X}(M)$:

- ADDITION: $(X+Y)_p := X_p + Y_p$
- MULTIPLICATION BY SCALARS: $(\lambda \cdot X)_p := \lambda \cdot X_p$
- — " — BY SMOOTH FUNCTIONS $(f \cdot X)_p = f(p) \cdot X_p$

I.E. OPERATION
$$\begin{aligned} C^\infty(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (f, X) &\longmapsto f \cdot X \end{aligned}$$

$C^\infty(M)$-module

- Action of $\mathfrak{X}(M)$ on $C^\infty(M)$:

$$\begin{aligned} \mathfrak{X}(M) \times C^\infty(M) &\to C^\infty(M) \\ (X, f) &\longmapsto \mathcal{L}_X(f) \end{aligned}$$

ENOUGH FOR EXPLICIT FORMULAS:

Ex: On $\mathbb{R}^m$: $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \in \mathfrak{X}(\mathbb{R}^m)$

$\Rightarrow f_1 \frac{\partial}{\partial x_1} + \dots + f_m \frac{\partial}{\partial x_m} \in \mathfrak{X}(\mathbb{R}^m)$ $(\forall)$ $f_1, \dots$ smooth functions

These are all (on $\mathbb{R}^m$).

Ex: When $M \subseteq \mathbb{R}^L$, $\frac{\partial}{\partial x_i}$ are usually not tangent to $M$. However, combinations $\boxed{f_1 \frac{\partial}{\partial x_1} + \dots}$ can be tangent to $M$, hence giving an $\in \mathfrak{X}(M)$.

Ex: $x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$ $\quad \mathrm{pr}_i = x_i$

On $M = S^1$ this is tangent to $S^1 \Rightarrow$

$\Rightarrow$ makes sense $x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathfrak{X}(S^1)$

$S^1 \ni p = (a,b) \longmapsto a\left(\frac{\partial}{\partial y}\right)_p - b\left(\frac{\partial}{\partial x}\right)_p \in T_pS^1$.

5

$$\begin{cases} \mathcal{L}_X : C^\infty(M) \longrightarrow C^\infty(M) \\ \mathcal{L}_X(f)(p) := \partial_{X_p}(f) = \end{cases}$$ Lie derivatives along $X \in \mathfrak{X}(M)$

**Lemma**: For a set-theoretical $X$:
$X$ = smooth $\Longleftrightarrow \mathcal{L}_X(f) \in C^\infty(M)$ $\forall f \in C^\infty(M)$.

**Def**: A derivation on $C^\infty(M)$: any linear map $L : C^\infty(M) \to C^\infty(M)$ satisfying $L(f \cdot g) = L(f) \cdot g + f \cdot L(g)$.

**Th**: $\boxed{I : \mathfrak{X}(M) \longrightarrow \mathrm{Deriv}(C^\infty(M))}$, $X \longmapsto \mathcal{L}_X$
is a bijection

For $L_1, L_2 \in \mathrm{Deriv}(C^\infty(M))$, is $L_1 \circ L_2 \in \mathrm{Deriv}(\cdot)$?

Q FOR THE BREAK

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~~pf~~ (of surjectivity)

Start with $L : \underline{C^\infty(M)} \to C^\infty(M) \xrightarrow{ev_p} \mathbb{R}$.

derivation

For $p \in M$: an induced

$$L_p : C^\infty(M) \to \mathbb{R},\ f \mapsto L(f)(p).$$

This is a derivation at $p \Rightarrow$

$\Rightarrow \exists\ X_p \in T_pM$ s.t. $L_p = \partial_{X_p}$.

$\Rightarrow$ we get a $\underline{\text{set-theoretical } X}$

with $\underline{\mathcal{L}_X}$ precisely $\underline{L}$

By Lemma $\Rightarrow X \in \mathfrak{X}(M)$ with $\mathcal{L}_X = L$ !

## OPERATIONS INVOLVING $\mathfrak{X}(M)$: 7

- **ADDITION**: $(X+Y)_p := X_p + Y_p$
- **MULTIPLICATION BY SCALARS** $(\lambda\cdot X)_p := \lambda\cdot X_p$
- —"— BY **SMOOTH FUNCTIONS** $(f\cdot X)_p = f(p)\cdot X_p$

I.E. OPERATION

$$C^\infty(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$$
$$(f,X)\longmapsto f\cdot X$$

- LIE DERIVATIVES ALONG X

$$\mathcal{L}_X : C^\infty(M)\to C^\infty(M)$$

8

$\mathfrak{X}(M)$ = a vector space

$\mathfrak{X}(M)$ = a $C^\infty(M)$-module

Example: On $\mathbb{R}^m$, because the vector fields $\frac{\partial}{\partial x_1},\dots$ involve the $\left(\frac{\partial}{\partial x_1}\right)_p,\dots,\left(\frac{\partial}{\partial x_m}\right)_p\in T_p^{\text{deriv}}\mathbb{R}^m$

⇙

the Lie derivatives induced by those are the usual operation of partial derivations

$$X = f_1\frac{\partial}{\partial x_1}+\dots\in\mathfrak{X}(\mathbb{R}^m)$$

$$\mathcal{L}_X(f) = f_1\frac{\partial f}{\partial x_1}+\dots$$

$\mathcal{L}_X : C^\infty(M)$

$\mathcal{L}_X(f)(p)$:

Lemma: Fo

X=smooth

Def: A c

linear

satisfy

Th:

I: $\mathfrak{X}(M$

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**Def**: The resulting vector field (given $X, Y$) is called the Lie bracket of $X$ and $Y$ and is denoted $[X,Y] \in \mathfrak{X}(M)$.

$$\boxed{\mathcal{L}_{[X,Y]}(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f)) \quad (\forall) f.}$$

Ex: In $\mathbb{R}^m$, start with $X = \sum f_i \frac{\partial}{\partial x_i}$, $Y = \sum g_i \frac{\partial}{\partial x_i}$ and compute..... compute..... see what you get for $[X,Y]$.

$[X,Y]$

$\mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$

# OPERATIONS INVOLVING $\mathfrak{X}(M)$: 12

a) ADDITION: $(X+Y)_p := X_p + Y_p$

b) MULTIPLICATION BY SCALARS: $(\lambda \cdot X)_p := \lambda \cdot X_p$

c) — " — BY SMOOTH FUNCTIONS $(f \cdot X)_p = f(p) X_p$

I.E. OPERATION $C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$(f, X) \longmapsto f \cdot X$$

d) LIE DERIVATIVES ALONG X

$$\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$$

e) LIE BRACKET OF VECTOR FIELDS:

$$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

- skew symmetric: $[Y, X] = -[X, Y]$
- bilinear: $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$
- $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ JACOBI IDENTITY

$\mathfrak{X}(M)$ is

- vector space
- $C^\infty(M)$-module
- Lie algebra

These all interact with each other 13

$\mathcal{L}_X(f)$

$[X, Y]$

multiply by a g:

$\mathcal{L}_{gX}(f) = g \, \mathcal{L}_X(f)$

$\mathcal{L}_X(gf) = \mathcal{L}_X(g) f + g \, \mathcal{L}_X(f)$

$[X, gY] = \ldots$ ?

page 94.

f). Diffeomorphisms $\boxed{F : M \to N}$

induce maps (isomorphisms)

$$F_* : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(N).$$

$$X \qquad ?F_*(X)?$$

For $q \in N$:

$F_*(X)_q \in T_q N$ to be defined

Use $(dF)_p : T_p M \longrightarrow T_q N$

for $\boxed{p = F^{-1}(q)}$, $X_p \in T_p M$, $q = F(p)$

I.E.: DEF

$F_*(X$

Rk: We

of type

There: F:

$F^*$:

Very oft

does not

But for

about F

$C^\infty(M$

$\mathcal{L}_X \downarrow$

$C^\infty($

$\mathfrak{X}(M)$ is

vector space

$C^\infty(M)$-module

Lie algebra

These all interact with each other (13)

$\mathcal{L}_X(f)$ | multiply by a $g$:

$\mathcal{L}_{gX}(f) = g\,\mathcal{L}_X(f)$

$[X,Y]$ | $\mathcal{L}_X(gf) = \mathcal{L}_X(g)f + g\,\mathcal{L}_X(f)$

$[X, gY] = \ldots$ ?

page 94.

f). Diffeomorphisms $\boxed{F: M \to N}$ induce maps (isomorphisms)

$$F_*: \mathfrak{X}(M) \longrightarrow \mathfrak{X}(N).$$

$$X \mapsto \ ?F_*(X)?$$

For $q \in N$:

$F_*(X)_q \in T_qN$ to be defined

Use $(dF)_p: T_pM \to T_qN$ for $\boxed{p = F^{-1}(q)}$

$X_p \mapsto F(p) = q$

I.E.: DEFINE (14)

$$F_*(X)_q := (dF)_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right).$$

Rk: We will see more constructions of type $F_*$(something on M)

$F^*$(something on N)

something = functions

There: $F: M \to N$ induces a map $f \mapsto f \circ F$

$$\boxed{F^*: C^\infty(N) \to C^\infty(M)}, \quad F^*(f) = f \circ F$$

Very often (in general constr.) this does not require F to be a diffeo. But for vector fields it does. What about $F_*$ on derivations?

$$\begin{array}{ccc} C^\infty(M) & \xleftarrow[\sim]{F^*} & C^\infty(N) \\ \mathcal{L}_X \downarrow & & \downarrow ?? \\ C^\infty(M) & \xleftarrow[\sim]{F^*} & C^\infty(N) \end{array}$$

$F$ = diffeo

?? ← precisely $\mathcal{L}_{F_*(X)}$

Corollary: $F_*$ preserves the

$[F_*(X), F$

I.E.: DEFINE

$$F_*(X)_q := (dF)_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right).$$

Rk: We will see more constructions of type $F_*$(something on $M$)

$F^*$(something on $N$)

"functions"

Thm: $F: M \to N$ induces a map $f: N \to \mathbb{R}$

$$\boxed{F^*: C^\infty(N) \to C^\infty(M)}, \quad F^*(f) = f \circ F$$

Very often (in general consts) this does not require $F$ to be a diffeom. But for vector fields it does. What about $F_*$ on derivations?

$$\begin{tikzcd} C^\infty(M) \arrow[d, "\mathcal{L}_X"'] & C^\infty(N) \arrow[l, "F^*"', "\cong"] \arrow[d, "??"] \\ C^\infty(M) & C^\infty(N) \arrow[l, "F^*"', "\cong"] \end{tikzcd}$$

$F$ = diffeo

$\leftarrow$ precisely $\mathcal{L}_{F_*(X)}$



**Corollary**: $F_*: \mathfrak{X}(M) \to \mathfrak{X}(N)$ preserves the Lie brackets:

$$[F_*(X), F_*(Y)] = F_*([X,Y])$$