HOMEWORK 1; GIVEN ON SEPT 14, 2022

Exercise 1. Let $M = S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

(1) Show that M is a smooth submanifold of $\mathbb{R}^2$. In what follows we will denote by $\mathcal{A}^{\mathrm{can}}_{S^1}$ the canonical smooth (maximal) atlas on M, i.e. the one consisting of all *smooth charts* of M in the sense of Chapter I (as they are mentioned also in Theorem 1.38).

(2) Let $\mathcal{P}$ be the collection consisting of four charts of S^1,

$$(U_n, \mathrm{pr}_n), (U_s, \mathrm{pr}_s), (U_e, \mathrm{pr}_e), (U_v, \mathrm{pr}_v),$$

given by

$$\begin{aligned} U_n &= \{(x, y) \in S^1 : y > 0\}, \quad \mathrm{pr}_n(x, y) = x, \\ U_s &= \{(x, y) \in S^1 : y < 0\}, \quad \mathrm{pr}_s(x, y) = x, \\ U_e &= \{(x, y) \in S^1 : x > 0\}, \quad \mathrm{pr}_e(x, y) = y, \\ U_v &= \{(x, y) \in S^1 : x < 0\}, \quad \mathrm{pr}_v(x, y) = y. \end{aligned}$$

Indicate these charts in a picture and prove that $\mathcal{P}$ is a smooth atlas.

(3) Let $\mathcal{S}$ be the collection consisting of two charts (U_N, σ_N), (U_S, σ_S) of S^1, given by

$$\begin{aligned} U_N &= S^1 \setminus \{(0, 1)\}, \sigma_N(x, y) = \frac{x}{1 - y}, \\ U_S &= S^1 \setminus \{(0, -1)\}, \sigma_S(x, y) = \frac{x}{1 + y}. \end{aligned}$$

Indicate these charts in a picture and prove that $\mathcal{S}$ is a smooth atlas.

(4) Show that $\mathcal{P}^{\max} = \mathcal{S}^{\max}$- i.e. that $\mathcal{P}$ and $\mathcal{S}$ induce the same smooth structure on S^1.

(5) Construct yet another atlas, denoted $\mathcal{T}$, that induces the same smooth structure on S^1, consisting now of two charts (U_E, θ_E), (U_V, θ_V) of S^1, with

$$U_E = S^1 \setminus \{(1, 0)\}, \quad U_V = S^1 \setminus \{(-1, 0)\},$$

and where θ_E and θ_V are related to the angle variable θ in the polar coordinate representation $x = \cos\theta, y = \sin\theta$. In other words, define carefully/explicitly θ_E and θ_V, prove that they form a smooth atlas $\mathcal{T}$ on S^1, then show that $\mathcal{T}$ induces the same smooth structure on S^1 as $\mathcal{P}$ and $\mathcal{S}$.

(6) Finally, show that these smooth structures on S^1 also coincide with the canonical smooth structure on S^1, induced from the fact that S^1 is a smooth submanifold of $\mathbb{R}^2$ (i.e. the $\mathcal{A}^{\mathrm{can}}_{S^1}$ from the first part).