HOMEWORK 2; GIVEN ON SEPT 21, 2022

Recall that, in general, the points in the (real) projective space $\mathbb{P}^n$ are denoted $[x_0 : x_1 : \ldots : x_n]$, hence there is an obvious map

$$H_m : S^m \to \mathbb{P}^m, \quad H_m(x_0 : x_1 : \ldots : x_m) = [x_0 : x_1 : \ldots : x_m].$$

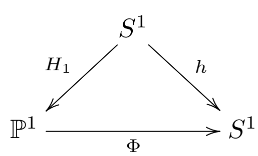
We endow $\mathbb{P}^m$ with the smooth structure discussed in the class (we only discussed one, by providing an atlas made of $m+1$ charts), and S^m with the canonical smooth structure, for which we provided several descriptions in the class (just choose whatever description we prefer).

Exercise 1. Please do the following:

1. Show that H_m is smooth, specifying what description of the smooth structure on S^m you chose to use.
2. For $m = 1$ we also consider the map

   $$h : S^1 \to S^1, \quad h(x, y) := (x^2 - y^2, 2xy).$$

   Please provide a geometric description of h.
3. Show that also h is smooth, but it is not a diffeomorphism.
4. Show that there exists and is unique a map $\Phi : \mathbb{P}^1 \to S^1$ such that $h = \Phi \circ H_1$ i.e. a commutative diagram:

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5. Show that Φ is smooth, and it is actually a diffeomorphism.
6. What do you think it happens if we replace $\mathbb{R}$ by $\mathbb{C}$: is $\mathbb{CP}^1$ diffeomorphic to some sphere?