HOMEWORK 3; GIVEN ON SEPT 28, 2022

Exercise 1. Let $f : \mathbb{P}^2 \to \mathbb{R}^4$ be the function given by

$$f([x:y:z]) = (xy,\ yz,\ zx,\ y^2 - z^2) \quad \text{for } (x,y,z) \in S^2.$$

We also compose f with the projection $\mathbb{R}^4 \to \mathbb{R}^3$ on the first three coordinates,

$$g := \mathrm{pr} \circ f : \mathbb{P}^2 \to \mathbb{R}^3;$$

the image of g is know as "the Roman surface", or the "Steiner surface" (discovered by Steiner in Rome in 1844, according to Wikipedia) and it is not difficult to see that it can also be described as

$$R := \left\{ (X,Y,Z) \in \mathbb{R}^3 : |X|, |Y|, |Z| \leq \frac{1}{2},\ (XY)^2 + (YZ)^2 + (ZX)^2 = XYZ, \right\},$$

Please do the following:

(1) check that f is well-defined and write the general formula for f (not only for (x,y,z) in the sphere).
(2) compute the representation of f with respect to the charts χ^i of $\mathbb{P}^2$ that we discussed last week (and the identity chart for $\mathbb{R}^4$).
(3) show that f is an embedding (hence: yes, $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$!).
(4) While R is not an embedded submanifold of $\mathbb{R}^3$ (see below), show that $R_0 = \{(X,Y,Z) \in R : XYZ \neq 0\}$ is, and that there is an open $\mathbb{P}^2_0 \subset \mathbb{P}^2$ to which g restricts to become a diffeomorphism between $\mathbb{P}^2_0$ and R_0.
(5) show that g is an immersion everywhere except for 6 points.
(•) bonus question: show that, even after removing those 6 points $\{p_1, \ldots, p_6\}$, and applying g, the resulting subspace

$$g(\mathbb{P}^2 \setminus \{p_1, \ldots, p_6\}) \subset \mathbb{R}^3$$

is not an embedded submanifold.

Comment: One can actually show that $\mathbb{P}^2$ cannot be embedded in $\mathbb{R}^3$. The g above can be seen as an attempt to find an immersion of $\mathbb{P}^2$ in $\mathbb{R}^3$ (non-injective, of course). You may be surprised to hear that such immersions actually exist. Finding an explicit one is quite a bit more difficult but also very interesting, and gives rise to Boy's surface in $\mathbb{R}^3$ ... but I let you google this one ...