HOMEWORK 4 (OCTOBER 5, 2022)

This exercise is to help you work with tangent vectors, as (abstractly) defined in the lectures, when looking inside $\mathbb{R}^n$ and embedded submanifolds. Therefore, we will use the tangent vectors introduced (abstractly) in the lectures, such as:

- The canonical basis

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p \in T_p\mathbb{R}^n. \tag{1}$$

  Note that this allows us to write/represent an arbitrary tangent vector vector $v \in T_p\mathbb{R}^n$ uniquely as

$$v = \lambda_1 \cdot \left(\frac{\partial}{\partial x_1}\right)_p + \ldots + \lambda_n \left(\frac{\partial}{\partial x_n}\right)_p, \tag{2}$$

  with $\lambda_i \in \mathbb{R}$.
- For $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$, its "speed" as an element

$$\frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n \quad (p = \gamma(0)) \tag{3}$$

Via the standard identification

$$\text{standard}_p : T_p\mathbb{R}^n \to \mathbb{R}^n, \quad v \mapsto v^{\text{Id}}$$

sending v to its value at the identity chart, the canonical basis (1) corresponds to the standard basis $e_1, \ldots, e_n$ of the vector space $\mathbb{R}^n$, and the abstract speed (3) corresponds to the usual derivative. But we work with the more abstract objects. Please also keep in mind that, given two tangent vectors $v, w \in T_pM$, at some point $p \in M$, to check that

$$v = w$$

it suffices to check that, for some chart χ around p, $v^\chi = w^\chi$.

Next, for an embedded submanifold $M \subset N$, we have discussed how to interpret the resulting inclusion $T_pM \subset T_pN$, for $p \in M$. In particular, for embedded submanifolds $M \subset \mathbb{R}^n$, one has

$$T_pM \subset T_p\mathbb{R}^n \quad (p \in M).$$

While arbitrary tangent vectors to $\mathbb{R}^n$ look like (2), the question is: when does such a vector (2) actually belong to T_pM? For instance, when M is just a point (a 0-dimensional submanifold!), the answer is: only when all λ_i vanish!

The main tool you have at hand is to use "speeds of curves": to show that $v \in T_p\mathbb{R}^n$ belongs to T_pM you have to find a curve γ in M such that $v = \frac{d\gamma}{dt}(0)$ (... well, there is also the regular value theorem, that you can use in extreme cases ...).

Exercise: Please do the following:

(1) For arbitrary curves $\gamma : I \to M$ in a manifold M, defined on some interval $I \subset \mathbb{R}$, make sense of (define) the speeds at arbitrary times:
$$\frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M.$$

(2) When $M \subset \mathbb{R}^n$, using what we asked you to keep mind, show that
$$\frac{d\gamma}{dt}(t) = \gamma_1'(t)\left(\frac{\partial}{\partial x_1}\right)_{\gamma(t)} + \ldots + \gamma_n'(t)\left(\frac{\partial}{\partial x_n}\right)_{\gamma(t)},$$
where $\gamma_i : I \to \mathbb{R}$ are the components of γ and $\gamma_i'(t) \in \mathbb{R}$ the usual derivatives.

(3) Similarly, show that, for any $h : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ smooth, and $p \in \mathbb{R}^n$, the differential between the abstract tangent spaces,
$$(dh)_p : T_p\mathbb{R}^n \longrightarrow T_{h(p)}\mathbb{R}^k$$
is still given by "the usual formula's", i.e. it sends
$$\left(\frac{\partial}{\partial x_i}\right)_p \mapsto \sum_j \frac{\partial h^j}{\partial x_i}(p)\left(\frac{\partial}{\partial x_j}\right)_{h(p)},$$
where h^j is the j-th component of h, and $\frac{\partial}{\partial x_j}$ are the usual partial derivatives.

(4) If $M \subset \mathbb{R}^n$ is an embedded submanifold and h is as above, we can consider the restriction $h|_M : M \to \mathbb{R}^k$ and its differential at an arbitrary point $p \in M$:
$$(dh|_M)_p : T_pM \to T_{h(p)}\mathbb{R}^k.$$
On the other hand, remembering that inclusions $T_pM \subset T_p\mathbb{R}^n$ (look them up!), we can restrict $(dh)_p$ to T_pM. Show that the two are equal:
$$(dh|_M)_p = (dh)_p|_{T_pM}.$$

(4) For the sphere $S^2 \subset \mathbb{R}^3$, show that for any $q = (a, b, c) \in S^2$,
$$X_q^1 := c\left(\frac{\partial}{\partial y}\right)_q - b\left(\frac{\partial}{\partial z}\right)_q,$$
$$X_q^2 := a\left(\frac{\partial}{\partial z}\right)_q - c\left(\frac{\partial}{\partial x}\right)_q,$$
$$X_q^3 := b\left(\frac{\partial}{\partial x}\right)_q - a\left(\frac{\partial}{\partial y}\right)_q$$
belong to T_qS^2.

(5) When is X_q^1, X_q^2 a basis of T_qS^2. And what about X_q^2, X_q^3?

(6) The three tangent vectors will be send by the differential $(df)_q$ of the map
$$f : S^2 \to \mathbb{R}^2, \quad f(x, y, z) = (xy, xz)$$
to three tangent vectors to $\mathbb{R}^2$ (at what point?). Compute them!

(7) Assuming you have done the previous point: now do it again, but differently.

(8) Find all the points at which f fails to be a submersion.