HOMEWORK 6 (TO BE HANDED IN BY OCT 26, 2022)

Exercise 1. Each of the following is worth 1.5 points (hence lots of bonus points!):

1). Show that, on any manifold M which is connected [1] , if $f \in C^\infty(M)$ has the property that $df = 0$, then f must be constant.

2). Consider the vector fields $V^1, V^2, V^3 \in \mathfrak{X}(S^3)$ from the previous homework, and define the similar 1-forms on S^3, given by the following formulas[2]:

$$\begin{cases} \theta_1 := 2\left(-ydx + xdy + tdz - zdt\right), \\ \theta_2 := 2\left(-zdx - tdy + xdz + ydt\right), \qquad \text{(as 1-forms on } S^3\text{)} \\ \theta_3 := 2\left(-tdx + zdy - ydz + xdt\right). \end{cases}$$

Show that $\theta_i(V^j) = \delta_i^j$ (i.e. 1 if $i = j$ and 0 otherwise).

3). Show that, for any function $f \in C^\infty(S^3)$, one has

$$df = L_{V^1}(f) \cdot \theta_1 + L_{V^2}(f) \cdot \theta_2 + L_{V^3}(f) \cdot \theta_3 \qquad \text{(as 1-forms on } S^3\text{)}.$$

4). Considering also the vector field on $V \in \mathfrak{X}(S^3)$ from the last homework, show that the smooth functions on S^3

$$h_1 := \theta_1(V), \quad h_2 := \theta_2(V), \quad h_3 := \theta_3(V)$$

are precisely the components of the Hopf map from the last homework.

5). Show that:

$$\begin{cases} dh_1 = h_2 \cdot \theta_3 - h_3 \cdot \theta_2, \\ dh_2 = h_3 \cdot \theta_1 - h_1 \cdot \theta_3, \qquad \text{(as 1-forms on } S^3\text{)} \qquad (*) \\ dh_3 = h_1 \cdot \theta_2 - h_2 \cdot \theta_1 \end{cases}$$

6). Show that $L_V(h_1) = L_V(h_2) = L_V(h_3) = 0$.

7). Show that the 1-form $\theta_1 \in \Omega^1(S^3)$ cannot be written as the pull-back via the Hopf map $h : S^3 \to S^2$, i.e. as $\theta_1 = h^*(\eta)$ for some $\eta \in \Omega^1(S^2)$.
(Hint: if $\theta = h^*(\eta)$ for some η, what can you say about $\theta(V)$?).

8). Show that the 1-form $\theta_1 \in \Omega^1(S^3)$ cannot be written as df with $f \in C^\infty(S^3)$.
(Hint: If $\theta = df$ for some f, what can you say about $\theta([V^2, V^3])$? Or $\theta(V^1)$?).

9). Considering xdy as a 1-form on S^2, compute $h^*(xdy)$ and write it as a sum $f_1 \cdot \theta_1 + f_2 \cdot \theta_2 + f_3 \cdot \theta_3$, with $f_1, f_2, f_3 : S^3 \to \mathbb{R}$ (to be computed).

10). Show that, if M is any connected manifold and $h = (h_1, h_2, h_3) : M \to \mathbb{R}^3$ is a smooth function with the property that there exist 1-forms $\theta_1, \theta_2, \theta_3 \in \Omega^1(M)$ such that the formulas (*) hold true, then there exists $r \in \mathbb{R}_{\geq 0}$ such that h takes values in S^2_r, the sphere of radius r.

[1] that is: for any $x, y \in M$ there exists a curve $\gamma : I \to M$, with $0, 1 \in I$ and $\gamma(0) = x$, $\gamma(1) = y$

[2] where we recall that, according to our conventions/terminology, when writing 1-forms on a larger space, $\mathbb{R}^4$ in this case, and saying that we view them as forms on the smaller space, S^3 in this case, we mean that we consider their restriction to S^3