HOMEWORK 7 (OCTOBER 26, 2022)

In the last two homeworks you have seen the vector fields on S^3 given by

$$V^1 = \frac{1}{2}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t}\right),$$
$$V^2 = \frac{1}{2}\left(-z\frac{\partial}{\partial x} - t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + y\frac{\partial}{\partial t}\right),$$
$$V^3 = \frac{1}{2}\left(-t\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t}\right),$$

which you proved to provide a basis for the tangent spaces of S^3 and to satisfy

$$[V^1, V^2] = V^3, \quad [V^2, V^3] = V^1, \quad [V^3, V^1] = V^2, \qquad (*)$$

and the 1-forms on S^3 given by

$$\begin{cases} \theta_1 = 2\left(-ydx + xdy + tdz - zdt\right), \\ \theta_2 = 2\left(-zdx - tdy + xdz + ydt\right), \\ \theta_3 = 2\left(-tdx + zdy - ydz + xdt\right). \end{cases}$$

that you proved to be dual to V^1, V^2, V^3. And there was also the vector field

$$V := \frac{1}{2}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}\right)$$

for which you proved that $\theta_i(V) = h_i$ are the components of the Hopf map, for which you also proved $dh_1 = h_2 \cdot \theta_3 - h_3 \cdot \theta_2$, $dh_2 = h_3 \cdot \theta_1 - h_1 \cdot \theta_3$, $dh_3 = h_1 \cdot \theta_2 - h_2 \cdot \theta_1$.

Exercise 1. Please do the following:

1). (1pt) Show by direct computation (involving the dx, etc from $\mathbb{R}^4$) that

$$d\theta_1 = -\theta_2 \wedge \theta_3, \quad d\theta_2 = -\theta_3 \wedge \theta_1, \quad d\theta_3 = -\theta_1 \wedge \theta_2. \qquad (**)$$

2). (1pt) Show that a 2-form η on S^3 is zero if and only if $\eta(V^i, V^j) = 0$ for all i, j.

3). (1.5pt) Prove again the formulas (**), but now using V^1, V^2, V^3 and (*).

4). (1pt) Deduce that θ_1, θ_2 and θ_3 are not exact.

5). (1pt) Compute $L_{V_1}(\theta_2)$ by direct computation (involving the dx, etc from $\mathbb{R}^4$).

6). (1pt) Now compute $L_{V_1}(\theta_2)$ using Cartan's magic formula.

7). (1.5pt) Prove now that $L_V(\theta_1) = L_V(\theta_2) = L_V(\theta_3) = 0$ (hopefully you see how to do it without much computations now!).

8). (1pt) Show that $\theta_1 \wedge \theta_2 \wedge \theta_3$ is a volume form on S^3. Hopefully you do not start computing like crazy, but you remember that a top-degree form is a volume form means that it is nowhere vanishing, and you try to find some vector fields X^1, X^2 and X^3 such that your 3-form, when evaluated on (X^1, X^2, X^3) is a function that does not vanish anywhere.

9). (1.5pt) Compute the flow of V at time $\frac{\pi}{4}$, $\phi_V^{\frac{\pi}{4}} : S^3 \to S^3$.

10). (1.5pt) Compute $(\phi)^*(\theta_1)$ where $\phi : S^3 \to S^3$ sends (x, y, z, t) to $\frac{\sqrt{2}}{2}(x - y, x + y, z - t, z + t)$.

11). (1.5pt) Now use the outcome from 7) and the description of L_V via the flow of V to deduce that $\epsilon \mapsto (\phi_V^\epsilon)^* \theta_1$ does not depend on ϵ, hence $(\phi_V^\epsilon)^* \theta_1 = \theta_1$ for all ϵ.