MANIFOLDS(WISB342) EXAM (NOVEMBER 8, 2022)

Exercise 1 (1pt). You know already, from one of the homeworks, that $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$. Show now that $\mathbb{P}^2$ can be embedded in S^4. (Hint: do not look for complicated formulas).

Exercise 2 (2pt). Let $f : M \to N$ be a smooth map between two manifolds of dimensions m and n, respectively, let $N_0 \subset N$ be a (smooth, embedded) submanifold and we are interested in the pre-image

$$M_0 = f^{-1}(N_0) := \{x \in M : f(x) \in N_0\}.$$

The RVT (regular value theorem) tells us that if N_0 consists of a single point and we require f to be a submersion at all points in M_0, then M_0 is a submanifold of M of dimension $n - m$, whose tangent spaces are given by the kernels of the differentials of f:

$$T_x M_0 = \{v \in T_x M : (df)_x(v) = 0\} \quad (\text{for } x \in M_0).$$

Here we want to generalise the RVT to more general submanifolds N_0. To that end, we replace the submersion condition by the condition that "f is transverse to N_0" by which we mean: for each $x \in M_0$ one has $(df)_x(T_x M) + T_{f(x)} N_0 = T_{f(x)} N$ or, more explicitly: any element $w \in T_{f(x)} N$ can be written as

$$w = (df)_x(v) + w_0, \quad \text{with } v \in T_x M, w_0 \in T_{f(x)} N_0.$$

a) Further assuming that $N_0 = g^{-1}(z)$ for some submersion $g : N \to P$ and $z \in P$ into yet another manifold P, show that, indeed, M_0 is an submanifold of M.
b) Describe the dimension of M_0 in terms of the dimensions of M, N and N_0.
c) Describe the tangent spaces of M_0 in terms of the ones of M, N, N_0 and the differential of f.
d) Finally, show that the conclusions above hold in general (without assuming g).

Exercise 3 (7pts). Consider the following curve in $\mathbb{R}^3$:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = (t^2, t^3, t).$$

a) Show that the following is a submanifold of $\mathbb{R}^3$ containing γ:

$$M = \{(x, y, z) \in \mathbb{R}^3 : y = xz\}.$$

b) Show that the following defines a vector field on M

$$V := 2z\frac{\partial}{\partial x} + (x + 2z^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

c) Show that γ is an integral curve of V.
d) Here are three more subspaces of $\mathbb{R}^3$ that contain γ:

$$\{(x, y, z) \in \mathbb{R}^3 : x = z^2\}, \quad \{(x, y, z) \in \mathbb{R}^3 : x^3 = y^2\}, \quad \{(x, y, z) \in \mathbb{R}^3 : y = z^3\}.$$

Among them, only one is not a submanifold of $\mathbb{R}^3$. Which one, and why is it not?
e) Find vector fields $X, Y \in \mathfrak{X}(M)$ which, at each point in M, give a basis of the tangent space of M.
f) Compute $[X, Y]$.
g) 1pt Compute the flow of V, describing explicitly all the diffeomorphisms ϕ_V^t induced by V.
h) Find a non-zero 1-form $\theta \in \Omega^1(\mathbb{R}^3)$ such that $\theta|_M = 0$.
i) Find a 1-form on M which is not exact, i.e. cannot be written as df for some $f \in C^\infty(M)$.
j) Show that $\mu = (dx \wedge dz)|_M$ is a volume form.
k) For $\omega = e^y(x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy)|_M \in \Omega^2(M)$ find $f \in C^\infty(M)$ such that $\omega = f \cdot \mu$.
l) Compute $i_V(\omega)$.
m) Also compute $L_V(\omega)$, writing the result in the form $g \cdot \mu$ (with g a function explicitly computed).

Exercise 4 (1pt). Let's wonder whether a manifold M can admit a volume form μ of type $\mu = \theta \wedge \theta$, for some other differential form θ on M. Show that if this happens then the dimension of M is divisible by 4. Then show that this can happen already on $M = \mathbb{R}^4$ (provide an explicit example!).

(SOLUTIONS (WRITTEN IN A RUSH).)

I

Exercise 1: From homework $\Rightarrow$ an embedding $\varphi: \mathbb{P}^2 \to \mathbb{R}^4$.

On the other hand, the stereographic proj. $\Rightarrow$ a diffeo $\psi: \mathbb{R}^4 \xrightarrow{\sim} S^4 \setminus$ north pole

$\Rightarrow \exists$ an embedding $\psi \circ \varphi$ of $\mathbb{P}^2$ in $S^4 \setminus \{p_N\}$, hence also in S^4.

Exercise 4: Say θ has degree k: $\theta \in \Omega^k(M)$.

Then $\mu \in \Omega^{2k}(M)$ hence $\dim(M) = 2k$.

$\theta \wedge \theta = (-1)^k \theta \wedge \theta \Rightarrow \mu = (-1)^k \mu$, but $\mu \neq 0$ $\Rightarrow k =$ even

$\Rightarrow \dim(M) \vdots 4$.

On $\mathbb{R}^4$ can take $\theta = dx \wedge dy + dz \wedge dt$, with $\theta \wedge \theta = 2\, dx \wedge dy \wedge dz \wedge dt$

Exercise 2: Under the assumption $N_0 = g^{-1}(z)$ we have:

- $M_0 = \{x \in M : f(x) \in N_0\} = \{x \in M : g(f(x)) = z\} = (g \circ f)^{-1}(z)$.
- we also claim that $g \circ f : M \to P$ is a submersion at all $x \in M$ i.e. the composition $T_x M \xrightarrow{(df)_x} T_{f(x)} N \xrightarrow{(dg)_{f(x)}} T_z P$, $(d\, g\circ f)_x$, is surjective (chain rule).

pf of claim: $u \in T_z P \Rightarrow u = (dg)_{f(x)}(w)$ for some $w \in T_{f(x)} N$ (since $(dg)_{f(x)}$ is surj). By hypothesis, $w = (df)_x(v) + w_0$ as in the statement. Hence $u = (dg)_{f(x)}((df)_x(v)) + (dg)_{f(x)}(w_0)$, where $w_0 \in TN_0 = \ker dg$, so $(dg)_{f(x)}(w_0) = 0$

$\Rightarrow$ we found v s.t.: $u = (d\, g\circ f)_x(v)$ ✓.

Therefore, by the RVT, for $g \circ f$ (and f):

a). $M_0 =$ smooth submanifold.

b). $\dim M_0 = \dim M - \dim P$. But $\dim N_0 = \dim N - \dim P \Rightarrow$ $\boxed{\dim M_0 = m - n + \dim N_0}$

c). $T_x M_0 = \{v : (d\, g\circ f)_x(v) = 0\} = \{v : (dg)_{f(x)}((df)_x(v)) = 0\} =$

$= \{v : (df)_x(v) \in \operatorname{Ker}(dg)_{f(x)}\}$, where $\operatorname{Ker}(dg)_{f(x)} = T_{f(x)} N_0$

$\Rightarrow$ $\boxed{T_x M_0 = \{v : (df)_x(v) \in T_{f(x)} N_0\}}$!

For d): remember that being a submanifold (and the tangent spaces) is local: can be checked in neighbd. of points. Moreover, any submanifold $N_0 \subseteq N$ is, locally, of type $g^{-1}(z)$ as above. In more detail: for $x \in M_0$, $y = f(x) \in N_0$, choose U chart of N adapted to N_0. Then apply the first part to $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ (with $U \cup U \cap N_0$) and $g : U \to ?$ the projection you get looking at the def. of adapted chart (...)

[II]

Exercise 3 :

a). Write $M = F^{-1}(0)$ where $F: \mathbb{R}^3 \to \mathbb{R}$, $F(x,y,z) = -y + xz$.

The differential of F at an arbitrary $\underbrace{(x,y,z)}_{p}$ is:

$$(dF)_{x,y,z} : T_{(x,y,z)}\mathbb{R}^3 \longrightarrow \mathbb{R}.$$

$$u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p + w\left(\frac{\partial}{\partial z}\right)_p \longmapsto u \cdot \frac{\partial F}{\partial x}(x,y,z) + \dots$$

$$= u \cdot z + v \cdot (-1) + w \cdot x$$

$$= uz + wx - v$$

but I am not going to write $(\;)_p$ anymore.

This is clearly surjective ($-\frac{\partial}{\partial y}$ is sent to $1 \in \mathbb{R}$!)

$\Rightarrow M$ is smooth and

$$T_{(x,y,z)}M = \left\{ u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} : uz + wx = v \right\}.$$

b). $V_{x,y,z} = \underbrace{2z}_{u}\frac{\partial}{\partial x} + \underbrace{(x+2z^2)}_{v}\frac{\partial}{\partial y} + \underbrace{1}_{w=1}\frac{\partial}{\partial z}$. to be

tangent to $M \iff uz + wx = v \iff 2z \cdot z + 1 \cdot x = x + 2z^2$ ☑

c). We have to show that

$$\dot{\gamma}(t) = X_{\gamma(t)}. \Rightarrow X_{(t^2,t^3,t)} = 2t\frac{\partial}{\partial x} + \underbrace{(t^2+2t^2)}_{3t^2}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

$$\underbrace{(t^2)'}_{2t}\frac{\partial}{\partial x} + \underbrace{(t^3)'}_{3t^2}\frac{\partial}{\partial y} + \underbrace{t'}_{1}\frac{\partial}{\partial z}. \quad \text{OK!}$$

d). If you ~~try to~~ apply the regular value theorem, you see the first and last are ~~not smooth~~ submanifolds. Hence the problematic one seems to be $M' = \{(x,y,z) : x^3 = y^2\}$

To show it is indeed problematic one shows that one cannot find ~~a~~ nice charts around $0 = (0,0,0) \in M'$.

Indeed, if $\chi : U \xrightarrow{\sim} \mathbb{R}^2$ would be a diffeo, $U \subset M'$

(which ~~would~~ exist if M' was submanifold!) then

$$U \setminus \{0\} \xrightarrow{\sim} \mathbb{R}^2 \setminus \text{a point}.$$

But $U \setminus \{0\} = U_+ \cup U_-$ where $U_+ = \{(x,y,z) \in M' : y > 0\}$, $U_- = \{(x,y,z) \in M' : y < 0\}$

are both open hence $U \setminus \{0\}$ is not connected while $\mathbb{R}^2 -$ a point is.

III

e). In a) we described the T_pM for $p=(x,y,z)$, arriving at the equation $v = uz + wx$ so that:

$$T_{\underset{(x,y,z)}{p}} M = \left\{ u\left(\frac{\partial}{\partial x}\right)_p + (uz + wx)\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} : u, w \in \mathbb{R} \right\}$$

(I will omit it from now on)

$$= \left\{ u\cdot\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\right) + w\cdot\left(x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) : u, w \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\{X, Y\} \text{ where } \begin{cases} X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}. \\ Y = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \end{cases}$$

f). Various ways. For instance use the def: $(\forall) f$ - smooth $\in C^\infty(\mathbb{R}^3)$

$$L_{[X,Y]}(f) = L_X(L_Y(f)) - L_Y(L_X(f)) =$$

$$= L_X\left(x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) - L_Y\left(\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y}\right) =$$

$$= \frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) + z\frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) - x\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial f}{\partial y} + x\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial x \partial z} + zx\frac{\partial^2 f}{\partial y^2} + z\frac{\partial^2 f}{\partial y \partial z} - x\frac{\partial^2 f}{\partial y \partial x} - xz\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial f}{\partial y} - z\frac{\partial^2 f}{\partial z \partial y} = 0$$

$$\Rightarrow \boxed{[X,Y] = 0}$$

can be done nicer using the properties of $[\cdot,\cdot]$!!

g). Integral curves $\gamma(t) = (x(t), y(t), z(t))$ are solutions of

$$\dot\gamma(t) = X_{\gamma(t)} = 2z(t)\frac{\partial}{\partial z} + (x(t) + 2z(t)^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

i.e. of: $\begin{cases} \dot x(t) = 2z(t) \\ \dot y(t) = x(t) + 2z(t)^2 \\ \dot z(t) = 1. \end{cases}$ Prescribing $\begin{cases} x(0) = a \\ y(0) = b \\ z(0) = c. \end{cases}$ we

have: $\begin{cases} \dot z(t) = 1 \\ z(0) = c \end{cases} \Rightarrow \boxed{z(t) = t + c}$. Then $\begin{cases} \dot x(t) = 2z(t) = 2t + 2c \\ x(0) = a. \end{cases}$

$\Rightarrow \boxed{x(t) = t^2 + 2ct + a}$. Finally,

$$\begin{cases} \dot y(t) = x(t) + 2z(t)^2 = t^2 + 2ct + a + 2t^2 + 4ct + 2c^2 = 3t^2 + 6ct + a + 2c^2 \\ y(0) = b. \end{cases}$$

$$\Rightarrow \boxed{y(t) = t^3 + 3ct^2 + (a + 2c^2)t + b}$$

Hence $\varphi_V^t(a,b,c) = \varphi_V(t,a,b,c) = (t^2 + 2ct + a,\ t^3 + 3ct^2 + (a+2c^2)t + b,\ t + c)$

IV

h). On M, $f = y - xz$ is zero, hence $df|_M = 0$. Hence one can take

$$\theta = df = dy - x\,dz - z\,dx \in \Omega^1(\mathbb{R}^3).$$

i). Take for instance $\xi = x\, dy\,|_M$. If ξ was exact, we would have $d\xi = 0$ i.e.

$$(dx \wedge dy)\,|_M = 0.$$

But $(dx\wedge dy)|_M (X, Y) = \ldots = 1\cdot x - z \cdot 0 = x \leftarrow$ not the zero function!

CONTRAD.

j). We have to show that $\mu_p \neq 0$ $\forall p \in \mathbb{R}^3$.

Let $p = (x, y, z)$ and compute:

$$\mu_p(X_p, Y_p) = \ldots = 1\cdot 1 - 0 \cdot 0 = 1.$$

(in various ways – e.g. with the formula for wedge)

Hence $\mu_p \neq 0$ $\forall p$.

k). $\omega = e^y \cdot (x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)|_M$

Remember that, on M: $dy = x\,dz + z\,dx$.

$$\Rightarrow \omega = e^y \Big(x\,(x\,dz + z\,dx)\wedge dz + y\, dz\wedge dx + z\, dx \wedge (x\,dz + z\,dx) \Big)\Big|_M$$

($dz \wedge dz = 0$, $dx \wedge dx = 0$)

$$= e^y \left(xz\, dx \wedge dz + y\, dz \wedge dx + zx\, dx \wedge dz \right)\Big|_M =$$

($dz \wedge dx = -dx \wedge dz$)

$$= e^y\,(2xz - y)\, dx \wedge dz\,|_M = y e^y\, dx \wedge dz\,|_M.$$

($xz = y$ on M)

Hence $f(x,y,z) = y e^y$ (on M).

l). $i_V(\omega) = f\, i_V(dx \wedge dz) = f\, i_V(dx) \wedge dz - f\, dx \wedge i_V(dz)$.

$V = 2z \frac{\partial}{\partial x} + (x + 2z^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ ($i_V(dx) = 2z$, $i_V(dz) = 1$)

$$= f\,(2z\,dz - dx)$$
$$= y e^y\,(2z\,dz - dx)$$

m). $L_V(\omega) = d(i_V(\omega)) + i_V(d\omega)$. ($d\omega = 0$ as a 3-form on a 2-manifold.)

$$= d\big(ye^y \cdot (2z\,dz - dx)\big) = d(ye^y) \wedge (2z\,dz - dx) + ye^y \wedge d(2z\,dz - dx) =$$

$$= (ye^y + e^y)\, dy \wedge (2z\,dz - dx) + ye^y \wedge (\ldots) \quad (\ldots = 0)$$

($dy \to x\,dz + z\,dx$)

$$= (y+1)e^y \cdot (2xz\, dz \wedge dz - x\, dz \wedge dx + 2z^2\, dx \wedge dz - z\, dx \wedge dx)$$

$$= (y+1)e^y \cdot (2z^2 + x)\, dx \wedge dz. \Rightarrow g = (y+1)e^y\,(x + 2z^2).$$

($dx \wedge dz = \mu$)

OR, better: $L_V(\omega) = L_V(f)\mu + f L_V(\mu)$ and compute $L_V(\mu)$ with Cartan $\Rightarrow 0$. $\Rightarrow L_V(\omega) = L_V(f)\mu$. $= (y+1)e^y(x+2z^2)$