Marius Crainic

Manifolds 2023-2024

http ://www.staff.science.uu.nl/∼crain101/manifolds-2023/

Version September 14, 2023

Springer

Contents

Chapter 1
Reminders on Topology and Analysis

1.1 Reminder 1: Topology; topological manifolds

Here is a very brief reminder on the basic notions from Topology. For those which are not so familiar with these basics, one may skip the later parts of this section (most notably the part on partitions of unity) and return to it later on, when necessary.

1.1.1 The objects of Topology

First of all, the main objects of Topology: **a topological space** is a set X endowed with a **topology**, i.e. a collection $\mathscr{T}$ of subsets of X (called **the opens of the topological space**, or simply **opens in** X) such that $\emptyset$ and X are open in X, arbitrary unions of opens are open and finite intersections of opens are open. We usually omit $\mathscr{T}$ from the notations, and we simply say that X is a topological space; hence that means that X is a set and we can talk about the subsets of X that are open (in X).

A topology on X allows us to make sense of the central phenomena of Topology: ”two points being close to each other”. First of all we can make sense of neighborhoods in a topological space X: given $x \in X$, a **neighborhood** (in X) of x is any subset $V \subset X$ that contains at least an open neighborhood of x, i.e. an open U with $x \in U$. In turn, this allows us to talk about convergence: a sequence $(x_n)_{n\geq 1}$ of elements of X **converges** (in the topological space X) to $x \in X$ if for any neighborhood V of x there exists an integer n_V such that $x_n \in V$ for all $n \geq n_V$.

The notion of neighborhoods also allows to talk also about an important property one requires on topological spaces in order to exclude pathological examples- Hausdorffness: a topological space X called **Hausdorff** if for any $x, y \in X$ distinct, there are neighborhoods U of x and V of y such that $U \cap V = \emptyset$. Hence, intuitively, this means that ”if two are distinct, then they cannot be too close to each other” (yes, not having this sounds pathological but, since this condition is not automatic, it is often imposed precisely to avoid ”strange/pathological spaces).

1.1.2 The morphisms/isomomorphisms of Topology

The relevant maps (the only ones that really matter) in Topology are the continuous ones: a map $f : X \to Y$ between topological spaces is called **continuous** if for any U-open in Y, its pre-image $f^{-1}(U)$ is open in X. ”Isomorphism” between topological spaces are known under the name of **homeomorphisms**: they are the bijections $f : X \to Y$ with the property that both f as well as f^{-1} are continuous.

In the language of ”Category Theory”, Topology is the category whose objects are topological spaces, and whose morphisms (between objects) are the continuous maps.

Remark 1.1. Note that, while proving that two topological spaces are homeomorphic (i.e there exists a homeomorphism between them) is relatively easy in principle (one just has to produce ONE single homeomorphism between them- and for that it is often enough to follow ones intuition), proving that two spaces are not homeomorphic is much harder. One way to proceed is by understanding the specific "topological properties" of the spaces under discussion (such as Hausdorffness, compactness, etc); if one of them has such a topological property and the other one does not, then they cannot be homeomorphic. A more advanced approach consists of constructing topological invariants of algebraic nature (such as numbers, groups, etc)- and that is what Algebraic Topology is about.

1.1.3 Metric topologies; bases

One of the largest class of topological spaces are metric spaces (X,d): any metric $d : X\times X \to \mathbb{R}$ induces a topology $\mathscr{T}_d$ on X: a subset $U \subset X$ is open iff for any $x \in U$ there exists $r > 0$ such that U contains the d-ball of center x and radius r:

$$B_d(x,r) := \{y \in X : d(x,y) < r\}. \tag{1.1.1}$$

In general, a topological space X is called **metrizable** if there is a metric d on X such that the original topology on X coincides with $\mathscr{T}_d$ (note also that, if such a d exists, in general it is far from being unique; e.g. already $2d$, $3d$, $\frac{d}{d+1}$ would do the same job). One of the most interesting questions about topological spaces is to decide whether they are metrizable or not; **metrizability theorems** aim at finding simple topological conditions that imply metrizability.

Considering the Euclidean metric d on $\mathbb{R}^m$, or on any subset $A \subset \mathbb{R}^m$, we see that A is endowed with a canonical topology- called **the Euclidean topology** on the subset $A \subset \mathbb{R}^m$ (exercise: show that also the square metric induces the same topology). Note that, in this case, the resulting notion of convergence (and continuity) coincides with the one from Analysis.

Remark 1.2. Continuing the previous remark, let us point out that showing that two Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ of different dimensions $m \neq n$ are not homeomorphic is non-trivial. When $m = 1$, this can be done using the notion of connectedness but, for $m,n \geq 2$, one has to appeal to tools from Algebraic Topology.

We now return to general metric spaces. A metric d on X allows us to talk about the open balls $B_d(x,r)$ for $x \in X$, $r \in \mathbb{R}_+$ (see (1.1.1)), giving rise to the collection of open balls induced by d:

$$\mathscr{B}_d = \{B_d(x,r) : x \in X, r \in \mathbb{R}_+\}.$$

This is not a topology on X, but $\mathscr{T}_d$ is the smallest topology containing $\mathscr{B}_d$.

Recall also (simple exercise) that any family $\mathscr{B}$ of subsets of X gives rise to a topology $\mathscr{T}(\mathscr{B})$ on X, defined as the smallest one containing $\mathscr{B}$. It is called **the topology generated by** $\mathscr{B}$. In general, the members of $\mathscr{T}(\mathscr{B})$ are arbitrary unions of finite intersections of members of $\mathscr{T}$.

Depending on the properties of $\mathscr{B}$, the members of $\mathscr{T}(\mathscr{B})$ may have simpler descriptions. The most common case is when $\mathscr{B}$ is a **topology basis**, i.e. satisfies the following axioms: any $x \in X$ is contained in at least one member B of $\mathscr{B}$ and, for any $B_1, B_2 \in \mathscr{B}$ and any $x \in B_1 \cap B_2$, there exists $B \in \mathscr{B}$ containing x with $B \subset B_1 \cap B_2$. In this case, for $U \subset X$, the following are equivalent:

(0) U belongs to $\mathscr{T}(\mathscr{B})$.
(1) for any $x \in U$ there exists $B \in \mathscr{B}$ s.t. $x \in B \subset U$.
(2) U is a union of members of $\mathscr{B}$.

For instance, for any metric d on X, the collection $\mathscr{B}_d$ is a topology basis, and (1) is precisely the original definition of $\mathscr{T}_d$.

One can change a bit the point of view and, starting with a topology $\mathscr{T}$ on X, look for collections $\mathscr{B}$ generating $\mathscr{T}$, i.e. such that $\mathscr{T} = \mathscr{T}(\mathscr{B})$. Of course, one possibility is to take $\mathscr{B} = \mathscr{T}$, but this is the least interesting one. The more interesting choices are the ones for which $\mathscr{B}$ is smaller- e.g. countable. And here is the precise terminology: given a topological space X, a **basis for the topological space** X is any collection $\mathscr{B}$ of subsets of X with the property it is a topology basis and $\mathscr{T} = \mathscr{T}(\mathscr{B})$. As above, for a collection $\mathscr{B}$ of subsets of X, the following are equivalent:

(0) $\mathscr{B}$ is a basis for the space X.
(1) for any open U in X and any $x \in U$ there exists $B \in \mathscr{B}$ s.t. $x \in B \subset U$.
(2) any open in X is a union of members of $\mathscr{B}$.

(in particular, each of the conditions (1) and (2) imply that $\mathscr{B}$ is a topology basis).

Repeating what we said before, but with a slightly different wording, we have that for any metric d on X, the metric topology admits $\mathscr{B}_d$ as basis. Another possible basis for the space X (endowed with the topology $\mathscr{T}_d$), slightly smaller, is

$$\mathscr{B}_d = \{B_d(x, \frac{1}{n}) : x \in X, n \in \mathbb{N}\}.$$

For the Euclidean metric d_{Eucl} on $\mathbb{R}^m$ we can do even better:

$$\mathscr{B}_{\mathbb{Q}} := \{B_{d_{\text{Eucl}}}(q, \frac{1}{n}) : q \in \mathbb{Q}^m, n \in \mathbb{N}\}$$

is still a basis for the Euclidean topology on $\mathbb{R}^m$, but it it "much smaller": it is countable.

In general, one says that a topological space X is **second countable** if it admits a basis $\mathscr{B}$ which is countable.

1.1.4 Topological manifolds

The second countability condition is a very subtle one and turns out to be of capital importance in establishing some central results in Topology and Geometry- such as metrizability and embedding theorems. In particular, it is part of the basic axioms for the notion of manifolds. For now:

Definition 1.3. A **topological m-dimensional manifold** is a topological space X satisfying the following:

(TM0): any point $x \in X$ admits a neighborhood X which is homeomorphic to an open subset of $\mathbb{R}^m$.
(TM1): it is Hausdorff.
(TM2): it is second countable.

A homeomorphism

$$\chi : U \to \Omega \subset \mathbb{R}^m$$

from an open subset U of X to an open subset Ω in $\mathbb{R}^m$ is called a **m-dimensional topological chart for** X, and U is called the domain of the chart- so that axiom (TM0) can also be read as:

(TM0): X can be covered by (domains) of m-dimensional topological charts.

You should convince yourself (or remember) why some of the usual examples of topological spaces such as spheres, tori, Moebius band, etc are topological manifolds. Also, in all these examples, one should concentrate first on the condition (TM0) (... as the labelling indicates). Note however that, while the notion of dimension is intuitively clear (at least in all examples), handling it theoretically is not such a piece of cake; see Remark 1.2. This is due to the fact that there is no obvious topological characterization of the (intuitive notion) of dimension. This will be much less of a problem as soon as we move to (differentiable) manifolds.

Remark 1.4. Since the notion of "topological space" is built on the notion of "open", so are most of the basic definitions in Topology- such as continuity, Hausdorffness, compactness, etc etc. However, under rather mild assumptions, such definitions can be rephrased more intuitively, using sequences. The main "mild assumption" that we have in mind here is that of "first countability"; please see the basic course on Topology. This condition is weaker even than the second countability condition. For instance, metric topologies are always first countable but may fail to be second countable. For our purpose, it is enough to know that either of the conditions (TM0) or (TM2) implies 1st countability (and, if you look at the definitions, you will see that this statement is completely trivial).

What is interesting to know here is that, when restricting to spaces X which are first countable, many of the basic notions can be reformulated in terms of sequences. E.g.:

- X is Hausdorff iff any convergent sequence in X has at most one limit.
- $f : X \to Y$ is continuous iff it is sequential continuous i.e.: if $(x_n)_{n\geq 1}$ is a sequence converging in X to $x \in X$, then $(f(x_n))_{n\geq 1}$ converges in Y to $f(x)$.

1.1.5 Inside a topological space

Recall that, given a space X, a subset $A \subset X$ is said to be **closed in** X if its complement $X \setminus A$ is open. Of course, knowing the closed subsets of X is equivalent to knowing the open ones- hence one could have introduced the notion of topology completely in terms of closed subsets (which would then be the axioms?). Opens are preferred because some of the the most important properties can be described more directly in terms of opens (and perhaps also because they are closer in spirit to the notion of "ball" in a metric space). However, closed subsets often have some very nice properties- e.g. when talking about compactness.

Given the axioms of a topology (namely the fact that arbitrary unions of opens is open or, equivalently, that arbitrary intersections of closeds is closed), it follows that for any subset A of a topological space X one can talk about:

- the largest open contained in A- and this is called **the interior of** A (in the space X), and denoted $\mathrm{Int}(A)$.
- the smallest closed containing A- and this is called **the closure of** A (in the space X), and denoted $\mathrm{Cl}(A)$,

Recall also that, under the first countability axiom (in particular, for topological manifolds), the closure has a particularly nice description in terms of sequences:

$$\mathrm{Cl}(A) = \{x \in X : \exists \text{ a sequence in } A \text{ converging to } x\}.$$

1.1.6 Construction of topological spaces

We have already seen two (related) ways of constructing topologies on a set X: the metric topology $\mathscr{T}_d$ induced by any metric d on X, and the topology $\mathscr{T}(\mathscr{B})$ generated by any family $\mathscr{B}$ of subsets of X (with the particularly nice situation when $\mathscr{B}$ is a topology basis).

There are various other important constructions of topologies out of the old ones. For instance, given any two topological spaces X and Y, the Cartesian product

$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$

carries a canonical topology, called **the product topology**. There is a slight complication: while we would like that the products of opens is open,

$$\mathscr{B}_{X\times Y} := \{U \times V : U - \text{open in } X, V - \text{open in } Y\}$$

is not a topology on $X \times Y$; instead, it is a topology basis, and the product topology is defined as the topology generated by $\mathscr{B}_{X\times Y}$. Equivalently, and more conceptually, it is the smallest topology on $X \times Y$ with the property that the projections

$$\mathrm{pr}_X : X \times Y \to X, \mathrm{pr}_Y : X \times Y \to Y$$

are continuous.

The last description is more conceptual because it follows a general philosophy that one should apply when looking for topologies: require that the most interesting maps that you have around to be continuous, and look for "the best (least boring)" topology that does that (usually "the best" means "the largest" or "the smallest").

Another example of this philosophy is **the induced topology**: given a topological space X, any subset $A \subset X$ carries a canonical, induced, topology: it is the smallest topology with the property that the canonical inclusion

$$i : A \to X, \quad i(a) = a$$

is continuous (why would looking for the largest topology with this property be "boring"?). Explicitly, the opens in A (endowed with the topology induced from X) are the intersections $A \cap U$ of A with opens U of X.

Yet another example is that of **quotient topology**. In some sense, it is the other extreme compared to the previous example. While before we started with an inclusion $I : A \to X$, we now start with a surjection

$$\pi : X \to Y,$$

where X is a topological space and Y is just a set (on which we would like to induce a topology). This time, looking for "the most interesting" topology on Y, we are lead to looking at the largest topology on Y with the property that π is continuous (why?). We obtain the quotient topology on Y: a subset $U \subset Y$ is an open of this topology if an donly if $\pi^{-1}(U)$ is open in X (check that this is, indeed, a topology on Y).

The terminology "quotient" comes from the fact that, typically, the situation of having a surjection $\pi : X \to Y$ arises when starting with X and an equivalence relation R on X. Then, with the intuition that we want to glue the points of X that are equivalent (w.r.t. the equivalence relation R), we obtain the quotient space

$$Y = X/R$$

(abstractly made of R-equivalence classes $[x]_R = \{y \in X : (x,y) \in R\}$ of points $x \in X$) together with the canonical projection

$$\pi_R : X \to X/R, \quad \pi_R(x) = [x]_R.$$

Therefore, starting with an equivalence relation R on a topological space X, we see that the resulting quotient X/R carries a canonical (quotient) topology.

One of the most interesting examples of quotient topologies is the canonical topology on the projective space

$$\mathbb{P}^m := \{l : l - \text{line through the origin in } \mathbb{R}^{m+1}\}$$

(i.e. the set of all 1-dimensional vector subspaces of $\mathbb{R}^{m+1}$). We can put ourselves in the previous situation by considering

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{P}^m, x \mapsto l_x,$$

where l_x is the line through the origin and x (i.e. the vector subspace $\mathbb{R} \cdot x$ spanned by x). In terms of equivalence relations, we deal with the equivalence relation on $\mathbb{R}^{m+1} \setminus \{0\}$ given by:

$$x \sim y \Longleftrightarrow l_x = l_y \Longleftrightarrow y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}.$$

Using the Euclidean topology on $\mathbb{R}^{m+1} \setminus \{0\}$ we obtain a natural topology on $\mathbb{P}^m$; endowed with this topology, $\mathbb{P}^m$ is called **the projective space** (of dimension m). You should convince yourself that convergence in this topology corresponds to the intuitive idea of "lines getting close to each other".

Finally, given the notion of induced topology, one can make use of that of embeddings: an **embedding** of a topological space X into a topological space Y is any map $i : X \to Y$ that is continuous, injective and, when interpreted as a continuous map $i : X \to i(X)$ and we endow $i(X) \subset Y$ with the topology induced from Y, it is a homemorphism (note that the last map is automatically continuous and bijective, but that does not imply that its inverse is continuous as well!). Next to metrizability theorems (see above), one of the most interesting problems in Topology/Geometry is that of deciding whether a space X can be embedded in a Euclidean space; results in this direction are usually labelled as **embedding theorems**. Looking at the notion of topological manifold, it is worth pointing out that, due also to the axioms (TM1) and (TM2), it follows that any topological manifold is metrizable and can be embedded in some Euclidean space!

1.1.7 Topological properties

As we have pointed out in Remark 1.1, to distinguish topological spaces from each other (or to understand better each specific one), it is useful to isolate the various topological properties that spaces may have. By a topological property we mean any property that can be described by only using the notion of opens or, equivalently, any property that is preserved via homeomorphisms. We have already mentioned several such properties: Hausdorffness and second countability. Here we recall a few more.

The first one is that of connectedness: a space X is called **connected** if it cannot be written as $X = U \cup V$ with U, V-disjoint non-empty opens in X. Or, equivalently, if the only subsets of X that are both open and closed are $\emptyset$ and X. In general, if X is not connected, it can be "broken" into connected pieces; more precisely, recall that a **connected component** of a space X is any connected subset $C \subset X$ which (when endowed with the induced topology) is connected, and which is maximal (w.r.t. the inclusion) with this property. Then the set of connected components defines a partition of X by closed subspace. In examples, the partition into connected components is usually easy to guess intuitively; here is a simple exercise that can be used as a recipe to confirm such guesses: assume that we manage to write X as

$$X = X_1 \cup \ldots \cup X_k, \quad \text{with } X_i \cap X_j = \emptyset \text{ for } i \neq j.$$

Assume also that all the X_is are open or, equivalently (why?), that all the X_is are closed. Then $\{X_1, \ldots, X_k\}$ must coincide with the partition into connected components.

Remark 1.5. Of course, the number of connected components may sometimes be infinite (even non-countable). Note however that, for topological manifolds M, due to the second countability axiom, the number of connected components if always at most countable (and finite if M is compact). Actually, one often restricts the attention to connected manifolds.

Another important topological property is that of compactness. While this is a property that one usually encounters in the first courses in Analysis (compacts in $\mathbb{R}^m$ being the subsets $A \subset \mathbb{R}^m$ that are closed and bounded), the fact that this is a topological property (i.e. can be described by appealing only to the notion of opens in A, without any reference to the Euclidean metric or to the way that A sits inside $\mathbb{R}^m$) is not at all obvious. That makes the resulting general definition less intuitive and a bit hard to digest at first: a topological space X is said to be **compact** if for any open cover

$$\mathscr{U} = \{U_i : i \in I\}$$

of X (i.e. each U_i is open in X, their union is X, and I is an indexing set), one can extract a finite subcover, i.e. there exists $i_1, \ldots, i_k \in I$ such that $\{U_{i_1}, \ldots, U_{i_k}\}$ is still a cover of X- i.e.

$$X = U_{i_1} \cup \ldots \cup U_{i_k}.$$

Here is the list of the most important properties of compactness:

1. Compact inside Hausdorff is closed: if X is a topological space, $A \subset X$ is endowed with the induced topology (see above) then:
$$A - \text{compact}, X - \text{Hausdorff} \Longrightarrow A - \text{is closed in} X.$$
2. Closed inside compact is compact: if X is a topological space, $A \subset X$ is endowed with the induced topology (see above) then:
$$A - \text{is closed in } X, X - \text{compact} \Longrightarrow A - \text{is compact}$$
3. Any compact Hausdorff space is automatically normal:
$$X - \text{compact} \Longrightarrow X - \text{normal}.$$
Recall here that a topological space X is said to be **normal** if for any $A, B \subset X$ closed disjoint subsets, one can find opens in X, U containing A and V containing B, such that $U \cap V = \emptyset$.
4. Product of compacts is compact: if X and Y are compact spaces then $X \times Y$, endowed with the product topology (see above), is compact:

$$X, Y - \text{compact} \Longrightarrow X \times Y - \text{compact}.$$

5. Continuous applied to compact is compact: if $f : X \to Y$ is continuous and $A \subset X$ (with the induced topology) is compact, then so is $f(A) \subset Y$:

$$f : X \to Y \text{ continuous}, A - \text{compact inside } X \Longrightarrow f(A) - \text{compact}.$$

6. In particular: quotients of compacts are compacts.
7. A continuous bijection from a compact space to a Hausdorff one is automatically a homeomorphism:

$$(\text{continuous } f) : (\text{compact space } X) \to \ (\text{Hausdorff space } Y) \Longrightarrow f \text{ is a homeomorphism}.$$

More generally: a continuous injection from compact to Hausdorff is automatically an embedding (see above).

Exercise 1.6. Assuming that you already know that the unit interval $[0,1]$ (endowed with the Euclidean topology) is compact, use the properties listed above to deduce that: for subsets of $\mathbb{R}^m$ endowed with the Euclidean topology:

$$A \subset \mathbb{R}^m \text{ is compact} \Longleftrightarrow A \text{ is closed and bounded in } \mathbb{R}^m.$$

A related topological property is the local version of compactness: one says that a space X is **locally compact** if any point $x \in X$ admits a compact neighborhood. If X is also Hausdorff, it follows that any point in X admits "arbitrarily small compact neighborhoods": for any neighborhood U of x in X there exists a compact neighbhborhood of x, contained in U. In general, Hausdorff locally compact spaces can be compactified by adding one extra-point. More on the 1-point compactification can be found in the lecture note on Topology.

For topological manifolds, axiom (MT0) ensures that they are automatically locally compact. But also axioms (MT1), (MT2) interact nicely with local compactness: they ensure the existence of "exhaustions". This is Theorem 4.37 in the notes on Topology:

Theorem 1.7. *Any locally compact, Hausdorff, 2nd countable space X admits an exhaustion, i.e. a family $\{K_n : n \in \mathbb{Z}_+\}$ of compact subsets of X such that $X = \cup_n K_n$ and $K_n \subset \mathring{K}_{n+1}$ for all n.*

Proof. Let $\mathscr{B}$ be a countable basis and consider $\mathscr{V} = \{B \in \mathscr{B} : \overline{B} - \text{compact}\}$. Then $\mathscr{V}$ is a basis: for any open U and $x \in X$ we choose a compact neighborhood N inside U; since $\mathscr{B}$ is a basis, we find $B \in \mathscr{B}$ s.t. $x \in B \subset N$; this implies $\overline{B} \subset N$ and then $\overline{B}$ must be compact; hence we found $B \in \mathscr{V}$ s.t. $x \in B \subset U$. In conclusion, we may assume that we have a basis $\mathscr{V} = \{V_n : n \in \mathbb{Z}_+\}$ where $\overline{V}_n$ is compact for each n. We define the exhaustion $\{K_n\}$ inductively, as follows. We put $K_1 = \overline{V}_1$. Since $\mathscr{V}$ covers the compact K_1, we find i_1 such that

$$K_1 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_1}.$$

Denoting by D_1 the right hand side of the inclusion above, we put

$$K_2 = \overline{D}_1 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_1}.$$

This is compact because it is a finite union of compacts. Since $D_1 \subset K_2$ and D_1 is open, we must have $D_1 \subset \mathring{K}_2$; since $K_1 \subset D_1$, we have $K_1 \subset \mathring{K}_2$. Next, we choose $i_2 > i_1$ such that

$$K_2 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_2},$$

we denote by D_2 the right hand side of this inclusion, and we put

$$K_3 = \overline{D}_2 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_2}.$$

As before, K_3 is compact, its interior contains D_2, hence also K_2. Continuing this process, we construct the family K_n, which clearly covers X. 😀

1.1.8 The algebra of continuous functions

Given a topological space X, an ”observable on X” has a precise meaning: it is a continuous function

$$f : X \to \mathbb{R}.$$

The set of all such continuous functions is denoted by

$$\mathscr{C}(X).$$

One of the simplest but most fundamental ideas in various parts of Geometry is that of understanding a space X via the associated ”object” $\mathscr{C}(X)$. This will allow one to consider ”more relevant observables”: e.g. for subspaces $X \subset \mathbb{R}^m$, one can consider only fs that are smooth, or polynomials. Or even to handle ”spaces” which, although are quite intuitive, are not topological spaces in the strict sense of the word. All together, this point of view gives rise to several directions in Geometry: Differential Geometry (where the key-word is ”smooth” instead of ”continuous”), Algebraic Geometry (where the key-word is ”polynomial”, or ”complex analytic”), Noncommutative Geometry (where X does not even make sense, but $\mathscr{C}(X)$ does).

Of course, what makes these work is the rich structure that $\mathscr{C}(X)$ posses- making the ”object” $\mathscr{C}(X)$ (a priory just a set) into a more interesting mathematical object. We recall here the most important part of the algebraic structure present on $\mathscr{C}(X)$: it is an algebra. Recall here:

Definition 1.8. A (real) **algebra** is a vector space A over $\mathbb{R}$ together with an operation

$$A \times A \to A, \ \ (a,b) \mapsto a \cdot b$$

which is unital in the sense that there exists an element $1 \in A$ such that

$$1 \cdot a = a \cdot 1 = a \quad \forall\, a \in A,$$

and which is $\mathbb{R}$-bilinear and associative, i.e., for all $a, a', b, b', c \in A$, $\lambda \in \mathbb{R}$,

$$(a+a') \cdot b = a \cdot b + a' \cdot b, \ a \cdot (b+b') = a \cdot b + a \cdot b',$$

$$(\lambda a) \cdot b = \lambda (a \cdot b) = a \cdot (\lambda b),$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c\ .$$

We say that A is commutative if $a \cdot b = b \cdot a$ for all $a, b \in A$.

Similarly one talks about complex algebras: then A is a vector space over $\mathbb{C}$ and $\lambda \in \mathbb{C}$.

For a topological space X, the algebra structure on $\mathscr{C}(X)$ is defined simply by pointwise addition and multiplication: for $f, g \in \mathscr{C}(X)$ and $\lambda \in \mathbb{R}$, $f+g, f \cdot g, \lambda \cdot f \in \mathscr{C}(X)$ are given by:

$$(f+g)(x) = f(x) + g(x), \quad (f \cdot g)(x) = f(x)g(x), \quad (\lambda \cdot f)(x) = \lambda f(x).$$

And, considering the space $\mathscr{C}(X, \mathbb{C})$ of $\mathbb{C}$-valued continuous functions on X, one obtains a complex algebra.

The fact that, under certain assumptions, a topological space X can be recovered from the algebra $\mathscr{C}(X)$, is the content of the Gelfand-Naimark theorem. While we refer to the basic course on Topology for the full statement and details, here is the very brief summary:

Theorem 1.9 (informative version of Gelfand Naimark theorem). *There is a way to associate to any algebra A a topological space $X(A)$ (called the spectrum of A) so that, when applied to $A = \mathscr{C}(X)$- the algebra of continuous functions on a compact Hausdorff space X, one recovers X (i.e. $X(\mathscr{C}(X))$ is homeomorphic to X).*

Remark 1.10 (Some details). The spectrum $X(A)$ of an algebra A is defined as the set of characters on A, i.e. maps

$$\chi : A \to \mathbb{R}$$

which preserve the algebra structure, i.e. which are linear, multiplicative ($\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in A$) and send the unit of A to $1 \in \mathbb{R}$. The topology on $X(A)$ is "the best" for which all the evaluation maps

$$\mathrm{ev}_a : X(A) \to \mathbb{R}, \quad \chi \mapsto \chi(a) \quad \text{(one for each } a \in A)$$

are continuous.

For instance, when $A = \mathscr{C}(X)$ for a compact Hausdorff space X, then any point $x \in M$ gives rise to a character χ_x on $\mathscr{C}(X)$, namely the evaluation at x, and the resulting map

$$X \to X(A), \quad x \mapsto \chi_x$$

is the one the realises the desired homeomorphism (of course, there are things to prove along the way). Let us give here a direct argument showing that, if X is a compact space, then any character on $\mathscr{C}(X)$,

$$\chi : \mathscr{C}(X) \to \mathbb{R},$$

is necessarily of type χ_x for some $x \in M$ (proving that the previous map is surjective). □

And, with the mind at the fact that we may want to consider more restrictive conditions than continuity (e.g. smoothness), here is the resulting relevant abstract notion:

Definition 1.11. Given an algebra A (over the base field $\mathbb{R}$ or $\mathbb{C}$), a **subalgebra** of A is any vector subspace $B \subset A$, containing the unit 1 of A and such that

$$b \cdot b' \in B \quad \forall\, b, b' \in B.$$

When we want to be more specific about the base field, we talk about real or complex subalgebras.

For instance, for $X \subset \mathbb{R}^m$, when looking at smooth or polynomial functions, we obtain a sequence of subalgebras:

$$\mathscr{C}^{\text{polyn}}(X) \subset \mathscr{C}^{\infty}(X) \subset \mathscr{C}(X).$$

Finally, when looking at a subset

$$\mathscr{A} \subset \mathscr{C}(X),$$

(subalgebra or not), there are several interesting properties that turn out to be interesting- and we say that:

(1) $\mathscr{A}$ is **point separating** if for any $x, y \in X$ distinct there exists $f \in \mathscr{A}$ such that $f(x) \neq f(y)$ or, equivalently, if there exists $f \in \mathscr{A}$ such that $f(x) = 0$ and $f(y) = 1$.
(2) $\mathscr{A}$ is **normal** if for any two disjoint closed subset $A, B \subset X$, there exists $f \in \mathscr{A}$ such that $f|_A = 0$, $f|_B = 1$.
(3) $\mathscr{A}$ is **closed under sums** if $f + g \in \mathscr{A}$ whenever $f, g \in \mathscr{A}$.
(3) $\mathscr{A}$ is **closed under quotients** if $f/g \in \mathscr{A}$ whenever $f, g \in \mathscr{A}$ and g is nowhere vanishing.

For instance, the Stone-Weierstrass theorem (which will not be used in the rest of the course) says that, if X is a compact Hausdorff space, then any point-separating sub-algebra $\mathscr{A} \subset \mathscr{C}(X)$ is dense in $\mathscr{C}(X)$; with particular cases of the type: real valued continuous functions on $[0,1]$ (or other similar spaces) can be approximated by polynomial functions.

Note that, for a general topological space X, even the entire $\mathscr{A} = \mathscr{C}(X)$ need not be point separating or normal. Actually, it is a rather simple exercise to check that the point separation of $\mathscr{C}(X)$ implies that X must be Hausdorff, while the normality of $\mathscr{C}(X)$ implies that the topological space X must be normal (i.e., as recalled above: any two disjoint closed subsets $A, B \subset X$ can be separated topologically: there exist opens $U, V \subset X$ containing A and B, respectively, with $U \cap V = \emptyset$). What is far less obvious (actually one of the most non-trivial basic results in Topology) is the converse, known as the Urysohn lemma: if a topological space X is Hausdorff and normal then $\mathscr{C}(X)$ is normal; more precisely, for any two disjoint closed subsets $A, B \subset X$ there exists

$$f : X \to [0,1] \text{ continuous and such that } f|_A = 0, f|_B = 1.$$

This will not be used later in the course; we mention it here just for completeness.

1.1.9 Partitions of unity

Finally, one more basic topic from Topology- but this time one that is difficult to appreciate (and perhaps even to digest) without entering the realm of Differential Geometry and/or Analysis: partitions of unity. To be able to talk about partitions of unity that are not just continuous (as we will be interested only on smooth functions), we can place ourselves in the following setting: X is a topological space and

$$\mathscr{A} \subset \mathscr{C}(X)$$

is a given vector subspace; we will be looking at partitions of unity that belong to $\mathscr{A}$. For the main definition, we first need to recall the notion of support: given $\eta : X \to \mathbb{R}$ continuous, **the support of** η **in** X, denoted $\text{supp}_X(\eta)$ or simply $\text{supp}(\eta)$ is the closure in X of the set $\eta \neq 0$ of points of X on which η does not vanish:

$$\text{supp}_X(\eta) := \overline{\{\eta \neq 0\}} = \overline{\{x \in X : \eta(x) \neq 0\}}^X.$$

Given an open $U \subset X$, we say that η **is supported in** U if $\text{supp}(\eta) \subset U$. This condition allows one to promote functions that are defined only on U, $f : U \to \mathbb{R}$, to functions on X, at least after multiplying by η; namely, $\eta \cdot f$, a priory defined only on U, if extended to X by declaring it to be zero outside U, the resulting function

$$\eta \cdot f : X \to \mathbb{R}$$

will be continuous (check this and, by looking at examples, convince yourselves that this does not work if the condition $\text{supp}(\eta) \subset U$ is replaced by the weaker one that $\{\eta \neq 0\} \subset U$).

We now move to partitions of unity; we start with the finite ones.

Definition 1.12. Let X be a topological space, $\mathscr{U} = \{U_1, \ldots, U_n\}$ a finite open cover of X. A continuous partition of unity subordinated to $\mathscr{U}$ is a family of continuous functions $\eta_i : X \to [0,1]$ satisfying:

$$\eta_1 + \ldots + \eta_k = 1, \ \ \text{supp}(\eta_i) \subset U_i.$$

Given $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\{\eta_i\}$ is an $\mathscr{A}$-partition of unity if $\eta_i \in \mathscr{A}$ for all i.

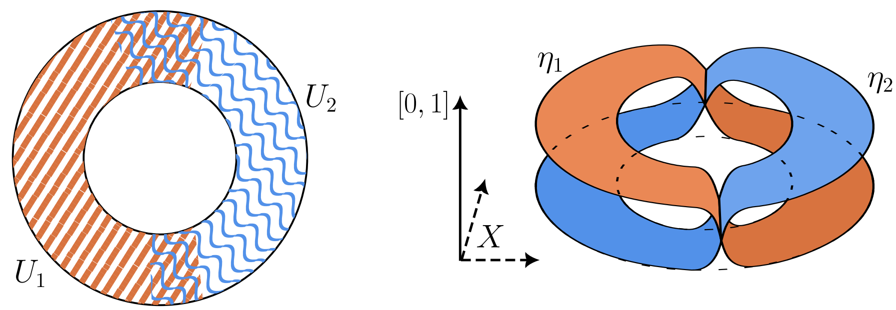


Fig. 1.1 On the left, an annulus X is covered by two open sets U_1 and U_2. The graph on the right shows two functions $\eta_i : X \to [0,1]$ that form a partition of unity subordinate to this cover.

Theorem 1.13. *Let X be a topological space and assume that $\mathscr{A} \subset \mathscr{C}(X)$ is normal and is closed under sums and quotients. Then, for any finite open cover $\mathscr{U}$, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.*
In particular if X is Hausdorff and normal, by the Uryshon Lemma (to ensure that $\mathscr{A} := \mathscr{C}(X)$ is normal), any finite open cover $\mathscr{U}$ admits a continuous partition of unity subordinated to $\mathscr{U}$.

Proof (sketch; for more details, see the lecture notes on Topology). The main ingredients are:

(St1) the remark made above that the normality of $\mathscr{A}$ implies that X is a normal space (simple exercise).
(St2) the fact that, in a normal space X, whenever we have $A \subset U$ with A-closed in X and U-open in X, one can find a smaller open V such that

$$A \subset V \subset \overline{V} \subset U$$

(short proof, but a bit tricky).
(St3) the shrinking lemma: for any finite open cover $\mathscr{U} = \{U_1, \ldots, U_k\}$ of a normal space X one can find another cover $\mathscr{V} = \{V_1, \ldots, V_n\}$ such that

$$\overline{V}_i \subset U_i \quad \forall\, i \in \{1, \ldots, k\}.$$

(this follows by applying the previous step inductively, starting with $U = U_1$ $A = X \setminus (U_2 \cup \ldots \cup U_k)$.

Now the proof of the theorem. Apply the shrinking lemma twice and choose open covers $\mathscr{V} = \{V_i\}$, $\mathscr{W} = \{W_i\}$, with $\overline{V}_i \subset U_i$, $\overline{W}_i \subset V_i$. For each i, we use the separation property of $\mathscr{A}$ for the disjoint closed sets $(\overline{W}_i, X - V_i)$. We find $f_i : X \to [0,1]$ that belongs to $\mathscr{A}$, with $f_i = 1$ on $\overline{W}_i$ and $f_i = 0$ outside V_i. Note that

$$f := f_1 + \ldots + f_k$$

is nowhere zero. Indeed, if $f(x) = 0$, we must have $f_i(x) = 0$ for all i, hence, for all i, $x \notin W_i$. But this contradicts the fact that $\mathscr{W}$ is a cover of X. From the properties of $\mathscr{A}$, each

$$\eta_i := \frac{f_i}{f_1 + \ldots + f_k} : X \to [0,1]$$

is continuous. Clearly, their sum is 1. Finally, $\mathrm{supp}(\eta_i) \subset U_i$ because $\overline{V}_i \subset U_i$ and $\{x : \eta_i(x) \neq 0\} = \{x : f_i(x) \neq 0\} \subset V_i$. 😀

And here is a nice application of the existence of (finite) partitions of unity:

Theorem 1.14. *Any compact topological manifold M can be embedded in some Euclidean space $\mathbb{R}^m$.*

Proof. Cover M by opens that are homeomorphic to $\mathbb{R}^d$, where d is the dimension of M. Using that M is compact, we find an open cover $\mathscr{U} = \{U_1, \ldots, U_n\}$ together with homeomorphisms $\chi_i : U_i \to \mathbb{R}^d$. Since M is compact it is also normal hence we find a partition of unity $\{\eta_1, \ldots, \eta_n\}$ subordinated to $\mathscr{U}$. Each of the functions $\eta_i \cdot \chi_i : U_i \to \mathbb{R}^d$ is extended to M by declaring it to be zero outside U_i; by the previous comments, the resulting functions $\tilde{\chi}_i : M \to \mathbb{R}^d$ are continuous. Consider now

$$i = (\eta_1, \ldots, \eta_k, \tilde{\chi}_1, \ldots, \tilde{\chi}_k) : M \to \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{k \text{ times}} \times \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{k \text{ times}} = \mathbb{R}^{k(d+1)}.$$

One check directly that i is injective; since M is compact and $\mathbb{R}^{k(d+1)}$ is Hausdorff, by the properties recalled on compactness, i will be an embedding. 😀

Finite partitions of unity are useful mainly when working over compacts (so that one can ensure finite open covers). For the more general case one first has to make precise sense of "infinite sums $\sum_i \eta_i$". For that first recall that, given a topological space X, a family $\mathscr{S}$ of subsets of X is said to be **locally finite** (in X) if for any $x \in X$ there exists a neighborhood V of x which intersect only a finite number of members of $\mathscr{S}$. Given a family $\{\eta_i\}_{i \in I}$ (I some indexing set) of continuous functions $\eta_i : X \to \mathbb{R}$, we say that $\{\eta_i\}_{i \in I}$ **is locally finite** in X if their supports (in X) $\mathrm{supp}(\eta_i)$ form a locally finite family of subsets of X. Note that in this case the sum

$$\sum_{i \in I} \eta_i : X \to \mathbb{R}$$

can be defined pointwise (at any $x \in X$ only a finite number of terms do not vanish), and the resulting function is continuous. For a subset $\mathscr{A} \subset \mathscr{C}(X)$, we say that is **closed under locally finite sums** if for any locally finite family $\{\eta_i\}$ with $\eta_i \in \mathscr{A}$, $\sum_i \eta_i$ is again in $\mathscr{A}$.

With these, we can now talk about infinite partitions of unity:

Definition 1.15. Let X be a topological space, $\mathscr{U} = \{U_i : i \in I\}$ an open cover of X. A (continuous) partition of unity subordinated to $\mathscr{U}$ is a locally finite family of continuous functions $\eta_i : X \to [0, 1]$ satisfying:

$$\sum_{i \in I} \eta_i = 1, \quad \mathrm{supp}(\eta_i) \subset U_i.$$

Given $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\{\eta_i\}$ is an $\mathscr{A}$-partition of unity if $\eta_i \in \mathscr{A}$ for all i.

If we are pragmatic and we only care about what is directly applicable later on in this course, the result to have in mind is:

Theorem 1.16. *Let X be a Hausdorff, locally compact and 2nd countable space, $\mathscr{A} \subset \mathscr{C}(X)$ and assume that:*

- *$\mathscr{A}$ is closed under locally finite sums and under quotients and*
- *$\mathscr{A}$ satisfies: for any $x \in M$ and any open neighborhood U of x, there exists $f \in \mathscr{A}$ supported in U with $f(x) > 0$.*

Then, for any open cover $\mathscr{U}$ of X, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.

For the curious student, here is the more detailed discussion to which the previous theorem belongs (with the explanation of how the proof goes). The existence of partitions of unity subordinated to arbitrary open covers forces a topological property of X called paracompactness: we say that a topological space X is **paracompact** if for any open cover $\mathscr{U}$ of X, there exists a locally finite open cover $\mathscr{V}$ that is a refinement of $\mathscr{U}$ in the sense that any $V \in \mathscr{V}$ is included inside some $U \in \mathscr{U}$. The existence of arbitrary partitions of unity is ensured by the following:

Theorem 1.17. *Let X be a paracompact Hausdorff space and assume that $\mathscr{A} \subset \mathscr{C}(X)$ is normal, closed under locally finite sums and closed under quotients.*
Then, for any open cover $\mathscr{U}$ of X, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.

Since paracompact spaces are automatically normal, hence we can use Uryshon's lemma, it follows that in a paracompact Hausdorff space a for any open cover there exists a continuous partition of unity subordinated to the cover.

The proof of the previous theorem is almost identical with the one from the finite case- just that one now has to establish an infinite version of the shrinking lemma (and that is where paracompactness enters),

To apply the previous theorem, there are two points that may be difficult to check: the paracompactness of X and, when working with arbitrary $\mathscr{A}$, that $\mathscr{A}$ is normal. For the first one, the following comes in handy:

Theorem 1.18. *Any Hausdorff, locally compact and 2nd countable space is paracompact.*

In particular, topological manifolds are automatically paracompact. One can actually show that, under the axioms (TM0) and (TM1), the axiom (TM2) on second countability is equivalent to the fact that M is paracompact and has a countable number of connected components.

Proof. We use an exhaustion $\{K_n\}$ of X (Theorem 1.7). Let $\mathscr{U}$ be an open cover of X. For each $n \in \mathbb{Z}_+$ there is a finite family $\mathscr{V}_n$ which covers $K_n - \text{Int}(K_{n-1})$, consisting of opens V with the properties: $V \subset \text{Int}(K_{n+1}) - K_{n-1}$, $V \subset U$ for some $U \in \mathscr{U}$. Indeed, for any $x \in K_n - \text{Int}(K_{n-1})$ let V_x be the intersection of $\text{Int}(K_{n+1}) - K_{n-1}$ with any member of $\mathscr{U}$ containing x; since $K_n - \text{Int}(K_{n-1})$ is compact, just take a finite subcollection $\mathscr{V}_n$ of $\{V_x\}$, covering $K_n - \text{Int}(K_{n-1})$. Set $\mathscr{V} = \cup_n \mathscr{V}_n$; it covers X since each $K_n - K_{n-1} \subset K_n - \text{Int}(K_{n-1})$ is covered by $\mathscr{V}_n$. Finally, it is locally finite: if $x \in X$, choosing n and V such that $V \in \mathscr{V}_n$, $x \in V$, we have $V \subset \text{Int}(K_{n+1}) - K_{n-1}$, hence V can only intersect members of $\mathscr{V}_m$ with $m \leq n+1$ (a finite number of them!).

Finally, to check the normality of $\mathscr{A}$ needed in Theorem 1.17, the following comes in handy:

Theorem 1.19. *Let X be a Hausdorff paracompact space and $\mathscr{A} \subset \mathscr{C}(X)$ closed under locally finite sums and under quotients. If X is also locally compact, then the following are equivalent:*

1. *$\mathscr{A}$ is normal.*
2. *for any $x \in M$ and any open neighborhood U of x, there exists $f \in \mathscr{A}$ supported in U with $f(x) > 0$.*

In particular, for a topological manifold M, checking that a subset $\mathscr{A} \subset \mathscr{C}(M)$ is normal is a local matter- and that is very useful since, locally, topological manifolds look just like Euclidean spaces.

Proof. That 1 implies 2 is clear: apply the separation property to $\{x\}$ and $X - V$. Assume 2. We claim that for any $C \subset X$ compact and any open U such that $C \subset U$, there exists $f \in \mathscr{A}$ supported in U, such that $f|_C > 0$. Indeed, by hypothesis, for any $c \in C$ we can find an open neighborhood V_c of c and $f_c \in \mathscr{A}$ positive such that $f_c(c) > 0$; then $\{f_c \neq 0\}_{c \in C}$ is an open cover of C in X, hence we can find a finite subcollection (corresponding to some points $c_1, \ldots, c_k \in C$) which still covers C; finally, set $f = f_{c_1} + \ldots + f_{c_k}$.

To prove 1, let $A, B \subset X$ be two closed disjoint subsets. As terminology, $D \subset X$ is called relatively compact if $\overline{D}$ is compact. Since X is locally compact, any point has arbitrarily small relatively compact open neighborhoods. For each $y \in X - A$, we choose such a neighborhood $D_y \subset X - A$. For each $a \in A$, since $a \in X - B$, applying step (St2) from the proof of Theorem 1.13, we find an open D_a such that $a \in D_a \subset X - B$. Again, we may assume that $\overline{D}_a$ is relatively compact. Then $\{D_x : x \in X\}$ is an open cover of X; let $\mathscr{U} = \{U_i : i \in I\}$ be a locally finite refinement. We split the set of indices as $I = I_1 \cup I_2$, where I_1 contains those i for which $U_i \cap A \neq \emptyset$, while I_2 those for which $U_i \subset X - A$. Using the shrinking lemma (the infinite version of the one described in (St3) of he proof of Theorem 1.13) we can also choose an open cover of X, $\mathscr{V} = \{V_i : i \in I\}$, with $\overline{V}_i \subset U_i$. Note that, by construction, each U_i (hence also each V_i) is relatively compact. Hence, by the claim above, we can find $\eta_i \in \mathscr{A}$ such that

$$\eta_i|_{\overline{V}_i} > 0, \ \ \text{supp}(\eta_i) \subset U_i.$$

Finally, we define

$$f(x) = \frac{\sum_{i \in I_1} \eta_i(x)}{\sum_{i \in I} \eta_i(x)}$$

From the properties of $\mathscr{A}$, $f \in \mathscr{A}$. Also, $f|_A = 1$. Indeed, for $a \in A$, a cannot belong to the U_i's with $i \in I_2$ (i.e. those $\subset X - A$); hence $\eta_i(a) = 0$ for all $i \in I_2$, hence $f(a) = 1$. Finally, $f|_B = 0$. To see this, we show that $\eta_i(b) = 0$ for all $i \in I_1$, $b \in B$. Assume the contrary. We find $i \in I_1$ and $b \in B \cap U_i$. Now, from the construction of $\mathscr{U}$, $U_i \subset D_x$ for some $x \in X$. There are two cases. If $x = a \in A$, then the defining property for D_a, namely $D_a \cap B = \emptyset$, is in contradiction with our assumption ($b \in B \cap U_i$). If $x = y \in X - A$, then the defining property for D_y, i.e. $D_y \subset X - A$, is in contradiction with the fact that $i \in I_1$ (i.e. $U_i \cap A \neq \emptyset$).

1.2 Reminder 2: Analysis

The relationship between Analysis and Differential Geometry is subtle. On one hand, Differential Geometry relies on the very basics of Analysis. On the other hand, various notions/results from Analysis become much more transparent/intuitive once the geometric perspective/intuition is brought into picture. In some sense, in many cases, the geometric point of view indicates the (expected) results while analysis provides the tools to prove them.

1.2.1 $\mathbb{R}^n$

The basic playground for multivariate analysis is the standard Euclidean space

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}}.$$

Despite its simplicity, this "space" has many different (but related) structures- and often the problem with handling $\mathbb{R}^n$ comes from the fact that it may not be completely clear which of the structures present on $\mathbb{R}^n$ is relevant for the specific discussions. Here are some of the many interesting structures present on $\mathbb{R}^n$:

- it is a vector space. When we want to emphasize this structure, we will denoted by

  $$v = (v_1, \ldots, v_n) \in \mathbb{R}^n$$

  its elements and we will think of them as "vectors"/"directions". Intrinsic in this notation is the presence of yet another piece of structure: it is not just a vector space- it comes with a preferred (canonical) basis:

  $$e_1, \ldots, e_m \in \mathbb{R}^n;$$

  in coordinates, e_i has 1 on the i-th position and 0 everywhere else.
- it is a vector space endowed with an inner product:

  $$\langle v, w \rangle = \sum_{i=1}^{n} v_i \cdot w_i,$$

  hence it is also a normed vector space, with the norm:

  $$||v|| = \sqrt{\langle v, v \rangle}.$$

- it is a topological space- endowed with the standard Euclidean topology. When we want to emphasize this structure, we will denote by

  $$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

its elements and we will think of them as ”points”. For instance, when looking at a circle in $\mathbb{R}^2$, the vector space space structure on $\mathbb{R}^2$ is not so relevant, and we think of the circle as made by points rather then vectors. Also, when talking about the continuity of a function $f : \mathbb{R}^n \to \mathbb{R}$, the vector space structure of $\mathbb{R}^n$ is not relevant (though it may be useful).
Recall also that the topology on $\mathbb{R}^n$ is a shadow of yet another structure: $\mathbb{R}^m$ is also a metric space, with the standard Euclidean metric:

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

We say a ”shadow” because uses part of what the metric allows us top talks about: points being ”close to each other” (or, more precisely: convergence and continuity). In particular, there are several other natural metrics on $\mathbb{R}^n$ that induce the same Euclidean topology- e.g. the so called square metric

$$d'(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

- even when thinking about $\mathbb{R}^n$ as a topological space, so of its elements as points, each point $x \in \mathbb{R}^n$ can be represented using ”canonical coordinates”- used already above. Again, the coordinates are not so relevant/important: they are useful and can be used, but they are not intrinsic to the structure. For instance, a circle in $\mathbb{R}^2$ can be described using coordinates by the equation $x^2 + y^2 = 1$, but the circle itself can be drawn without any coordinate axes at our disposal.
Note that the standard coordinates we mentioned are the simplest illustration of the notion of ”chart”- to be discussed in a bit more detail below, and essential in defining the notion of manifold.
- it is a topological space ”on which analysis can be performed” (... i.e. a manifold).
- etc.

Of course, all these are inter-related but, in each situation, it is important to realize which of these structures really matter. In particular, whenever one encounters a definition or result, it is instructive to figure out whether the elements in $\mathbb{R}^m$ that show up play the role of points and which ones of vectors, and how much the definition/result depends on the coordinates. This is the first step towards a geometric understanding of Analysis.

1.2.2 The differential and the inverse function theorem

One can talk about various notions of derivatives of a function f at a point

$$x \in \mathbb{R}^n$$

whenever we have a function f defined on a neighborhood of x- so that the expressions $f(y)$ used below makes sense for all y near x or, equivalently, $f(x+v)$ is defined for small vectors v.

Typically one assumes that f is defined on an open subset $\Omega \subset \mathbb{R}^n$ and takes values in some other Euclidean space $\mathbb{R}^k$,

$$f : \Omega \to \mathbb{R}^k,$$

so that it makes sense to talk about derivatives of f at any point in its domain, $x \in \Omega$.

The most intrinsic notion of derivative is that of ”**total derivative**”, also called **the differential of** f (at the given point $x \in \Omega$). This notion arises when trying to approximate f, near x, by simpler (linear-like) functions. Understanding f near x is about understanding

$$v \mapsto f(x+v)$$

for $v \in \mathbb{R}^n$ near 0. It sends $v = 0$ to $f(x)$, hence the best one can hope for is to approximate

$$v \mapsto f(x+v) - f(x)$$

by functions that are linear in v. Think that we try to write the last expression as a function linear in v, plus one that is qudratic in v, etc (plus eventually an "error term"), but we are interested only in the linear term A. We see we are looking for a linear map

$$A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k)$$

with the property that

$$\lim_{v \to 0} \frac{f(x+v) - f(x) - A(v)}{||v||} = 0. \tag{1.2.1}$$

It is easy to see that, if such a map A exists, then it is unique. And that is what the differential of f at x is.

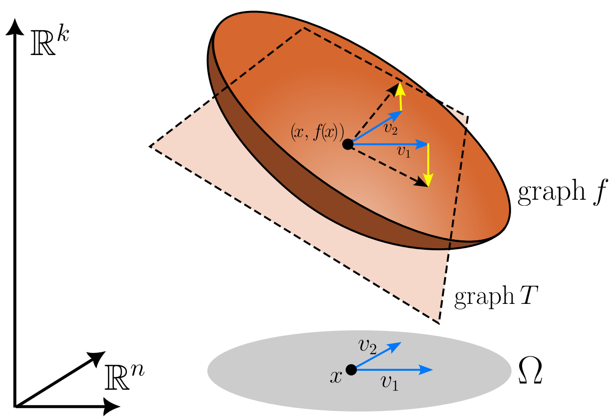


Fig. 1.2 Visualization of the total derivative for a function $f : \Omega \to \mathbb{R}^k$ defined over an open $\Omega \subseteq \mathbb{R}^n$, in this case with $n = 2$ and $k = 1$. Infinitesimal movements through a point $x \in \Omega$ are represented by the blue horizontal vectors v_i and the resulting infinitesimal movement in the codomain $\mathbb{R}^k$ is represented by the yellow vertical vectors $D_x f(v_i)$. They sum up to dashed vectors of the form $(v_i, D_x f(v_i)) \in \mathbb{R}^n \times \mathbb{R}^k$ that are tangent to the graph of f in the point $(x, f(x))$. The graph of the best affine approximation T of f in x (which sends any $\tilde{x} \in \mathbb{R}^n$ to $T(\tilde{x}) = f(x) + D_x f(\tilde{x} - x)$) is an affine plane spanned by the dashed vectors.

Definition 1.20. We say that $f : \Omega \to \mathbb{R}^k$ is **differentiable at** x if there exists a linear map $A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k)$ satisfying (1.2.1). The linear map A (necessarily unique) is called **the differential of** f **at the point** x (or the total derivative of f at x) and is denoted

$$D_x f = (Df)_x : \mathbb{R}^n \to \mathbb{R}^k.$$

We say that f is **differentiable** if it differentiable at all points x in its domain Ω. We say that f is **of class** C^1 if it is differentiable and the resulting map

$$Df : \Omega \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k), \quad x \mapsto (Df)_x$$

is continuous (recall here that, via the matrix representation of linear maps, $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k)$ can be interpreted as the Euclidean space $\mathbb{R}^{n \cdot k}$).

We say that f is of class C^2 if it differentiable and df is of class C^1; proceeding inductively, we can talk about f being of class C^l for any $l \in \mathbb{N}$. We say that f is **smooth** if it is of class C^l for all l.

Recall here also the chain rule that allows one to compute the differential of a composition of two functions:

Proposition 1.21 (the chain rule). *Given opens $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^k$ and functions*

$$\Omega \xrightarrow{f} \Omega' \xrightarrow{g} \mathbb{R}^l,$$

if f is differentiable at $x \in \Omega$ and g is differentiable at $f(x) \in \Omega'$, then $g \circ f$ is differentiable at x and

$$(D(g \circ f))_x = (Dg)_{f(x)} \circ (Df)_x.$$

Despite the fact that the differential $(Df)_x$ arises as "the linear approximation" of f near x, it contains a great deal of information of f near x- and that makes it extremely useful. Probably the best and most fundamental illustration is the inverse function theorem. Recall here that

Definition 1.22. A map $f : \Omega \to \Omega'$ between two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$ is said to be a **diffeomorphism** if it is bijective and both f and f^{-1} are smooth.

We say that f is a **local diffeomorphism** around $x \in \Omega$ if there exist opens $\Omega_x \subset \Omega$ and $\Omega'_{f(x)} \subset \Omega'$ with $x \in \Omega_x$, such that $f|_{\Omega_x} : \Omega_x \to \Omega'_{f(x)}$ is a diffeomorphism.

It is interesting to draw an analogy with Topology, where the main objects are topological spaces, the relevant maps are the continuous ones and two spaces are "isomorphic in Topology" (homeomorphic) if there exists a bijection f between them such that both f as well as f^{-1} are continuous. However, in topology is it usually very hard to prove that two given spaces are not homeomorphic (and one often has to appeal to methods from Algebraic Topology); for instance, just the simple the fact that $\mathbb{R}^n$ and $\mathbb{R}^k$ are homeomorphic only when $n = k$ is very hard to prove. In contrast, the similar statements for diffeomorphisms are much easier to prove thanks to the notion of differential. Indeed, using the chain rule, the following should be a rather easy exercise:

Exercise 1.23. Show that if a map $f : \Omega \to \Omega'$ between two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$ is a local diffeomorphism around $x \in \Omega$, then

$$(Df)_x : \mathbb{R}^n \to \mathbb{R}^k$$

is a linear isomorphism. Deduce that if two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$ are diffeomorphic, then $n = k$.

Although this clearly shows the usefulness of the differential, its great power is due to the inverse function theorem (and its immediate consequences, such as the implicit function theorem -see below). Indeed, we see that a condition on the differential of f at a single (given) point x tells us information about f around x:

Theorem 1.24 (The inverse function theorem). *Given a smooth map $f : \Omega \to \Omega'$ between two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$, if f is differentiable at a point $x \in \Omega$ and $(Df)_x$ is an isomorphism, then f is a local diffeomorphism around x.*

1.2.3 Directional/partial derivatives; the implicit function theorem

Note that, when talking about the differential $(Df)_x(v)$ (hence $f : \Omega \to \mathbb{R}^k$, with $\Omega \subset \mathbb{R}^n$ open), $x \in \Omega$ should be thought of as a point, while $v \in \mathbb{R}^n$ as a direction (vector). This becomes more apparent if we reformulate the total derivative as a directional derivative.

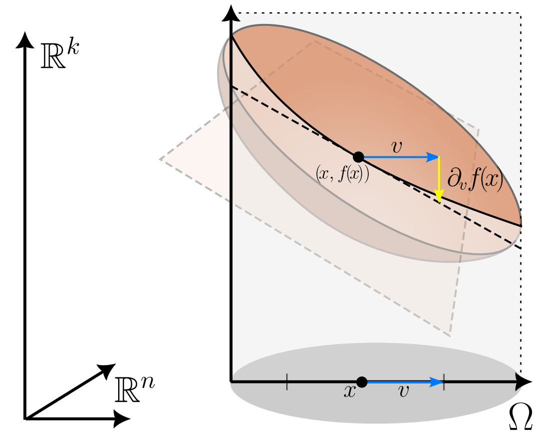


Fig. 1.3 The directional derivative $\partial_v f(x)$ of the function f from Fig. 1.2 can be seen to arise geometrically in the following way: The map $t \mapsto x + vt$ traces a line through Ω. The slice of the graph of f over this line is exactly the graph of the function $t \mapsto f(x+tv)$ if we label the horizontal axis by integer multiples of v. The directional derivative can now be defined as the normal derivative of this function since we have a one-dimensional domain.

Definition 1.25. With $f : \Omega \to \mathbb{R}^k$ and $x \in \Omega$ as above, and an arbitrary vector $v \in \mathbb{R}^n$, **the derivative of** f **at** x **in the direction** v is defined as the vector

$$\partial_v(f)(x) = \frac{\partial f}{\partial v}(x) := \frac{d}{dt}\Big|_{t=0} f(x+tv) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \in \mathbb{R}^k.$$

When this derivative exists, we say that f is differentiable at x in the v-direction.

The relationship with the total differential is immediate: just replace in (1.2.1) v (small enough) by tv with $v \in \mathbb{R}^n$ fixed (but arbitrary) and $t \in \mathbb{R}$ approaching 0; using that A is linear, we find that:

$$(Df)_x(v) = \frac{\partial f}{\partial v}(x).$$

This relationship is visualized in Fig. 1.3. In particular, if f is differentiable at x then it is differentiable in all directions. The converse is not true; however, one can show that if f is of class C^1 if and only if all the directional derivatives $\frac{\partial f}{\partial v}$ exist and are continuous (see also the discussion below on partial derivatives).

Applying the previous definition to $v \in \{e_1, \dots, e_n\}$, a vector in the standard basis of $\mathbb{R}^n$, we obtain the partial derivatives

$$\frac{\partial f}{\partial x_i}(x) := \frac{\partial f}{\partial e_i}(x) = \frac{d}{dy}\Big|_{y=x_i} f(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \in \mathbb{R}^k.$$

Its components are the partial derivatives of the components f_i of f:

$$\frac{\partial f}{\partial x_i}(x) = \left(\frac{\partial f_1}{\partial x_j}(x), \dots, \frac{\partial f_k}{\partial x_j}(x) \right) \quad (\text{where } f = (f_1, \dots, f_k)).$$

These partial derivatives contain the same information as $(Df)_x$, just in a less intrinsic way; however, they allow one to handle $(Df)_x$ more concretely, via matrices. For that recall that, due to the fact that the standard Euclidean spaces come with a preferred basis, linear maps

$$A : \mathbb{R}^n \to \mathbb{R}^k$$

can be represented as matrices

$$A = (A^i_j)_{1\leq i\leq k, 1\leq j\leq n} = \begin{pmatrix} A^1_1 & \dots & A^1_n \\ \dots & \dots & \dots \\ A^k_1 & \dots & A^k_n \end{pmatrix}$$

To make a distinction between the matrix A and the linear map A, one may want to denote by $\hat{A}$ the linear map, at least for a while. Then the relationship between the two is (by definition):

$$\hat{A}(e_j) = \sum_{i=1}^{k} A^i_j e_i$$

or, on a general vector $v = v^1 e_1 + \ldots + v^n e_n \in \mathbb{R}^n$, one has

$$\hat{A}(v) = \sum_{i=1}^{k} (\sum_{j=1}^{n} A^i_j v^j) e_i.$$

To write also this formula in terms of matrix multiplication, we interpret any $v \in \mathbb{R}^n$ as a row matrix and we denote by v^T its transpose (column matrix):

$$v^T = \begin{pmatrix} v^1 \\ \dots \\ v^n \end{pmatrix}$$

With this, the previous formula becomes

$$\hat{A}(v)^T = A \cdot v^T$$

It follows immediately that the standard multiplication of matrices,

$$(A \cdot B)^i_j = \sum_k A^i_k B^k_j,$$

corresponds to the composition of linear maps:

$$\widehat{AB} = \hat{A} \circ \hat{B}.$$

All together, there should be no confusion in identifying A with $\hat{A}$ even notationally.

In the case of the differential $(Df)_x$, to see the matrix representing it we write

$$(Df)_x(e_j) = \frac{\partial f}{\partial x_j}(x) = \left(\frac{\partial f_1}{\partial x_j}(x), \ldots, \frac{\partial f_k}{\partial x_j}(x)\right) = \sum_{i=1}^{k} \frac{\partial f_i}{\partial x_j}(x) e_j$$

i.e., in the matrix notation,

$$(Df)_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \dots & \dots & \dots \\ \frac{\partial f_k}{\partial x_1}(x) & \dots & \frac{\partial f_k}{\partial x_n}(x) \end{pmatrix}.$$

Note that, with this, the fact that $(Df)_x$ is an isomorphism is equivalent to the fact that the matrix above is invertible. More generally, the rank of $(Df)_x$ as a linear map coincides with the rank as a matrix.

With these:

Proposition 1.26. *A function $f : \Omega \to \mathbb{R}^k$ is of class C^1 if and only if all the partial derivatives $\frac{\partial f}{\partial x_i}$ exist and are continuous functions on Ω.*

In this case we can further look at partials derivatives of order two etc. Hence the higher, order l, partial derivatives are defined inductively:

$$\frac{\partial^l f}{\partial x_{i_1} \dots \partial x_{i_l}} = \frac{\partial}{\partial x_{i_1}} \left(\frac{\partial}{\partial x_{i_2}} \left(\dots \left(\frac{\partial f}{\partial x_{i_l}} \right) \right) \right).$$

The previous proposition that extends to a characterization of f being of class C^l; in particular, f is smooth if and only if all its higher partial derivatives exist. We will denote by

$$\mathscr{C}^\infty(\mathbb{R}^n)$$

the space (algebra!) of smooth functions on $\mathbb{R}^n$. A **smooth partitions of unity** on $\mathbb{R}^n$ (subordinated to an open cover) is any partition of unity whose members η_i are smooth- i.e. Definition 1.12 (finite case) and Definition 1.15 (general case) applied at $\mathscr{A} := \mathscr{C}^\infty(\mathbb{R}^n) \subset \mathscr{C}(\mathbb{R}^n)$.

Theorem 1.27. *Any open cover of $\mathbb{R}^n$ admits a smooth partition subordinated to it.*

Proof. We want to use Theorem 1.17 for $\mathscr{A} := \mathscr{C}^\infty(\mathbb{R}^n)$. This is clearly closed under quotients and, for the same reason that locally finite sums of continuous functions are continuous, it is closed under locally finite sums. We still have to check the last condition on $\mathscr{A}$ or, equivalently: for any $x \in \mathbb{R}^n$ and any ball centered at x, $B(x, \varepsilon)$, there exists a smooth functions $f : \mathbb{R}^n \to [0,1]$ such that $f(x) > 0$ and f is supported in the ball. It is clear that we may assume that $x = 0$. Also, by rescaling the argument of f (i.e. multiply it by a constant) we may assume that $\varepsilon = 1$. Then set $f(x) = g(x_1^2 + \dots + x_n^2)$ where $g : \mathbb{R} \to [0,1]$ is any smooth function with $g(0) > 0$ and $g = 0$ outside $[-\frac{1}{2}, \frac{1}{2}]$. That such a function exists should be clear by thinking of its graph. The following exercise provides and explicit formula. 😀

Exercise 1.28. Show that

$$g_0 : \mathbb{R} \to \mathbb{R}, \quad g_0(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is a smooth function. Then show that

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = g_0(x + \frac{1}{2}) g_0(\frac{1}{2} - x).$$

is a smooth function with the properties required at the end of the previous proof.

We now recall the implicit function theorem- which is one the important and rather immediate consequences of the inverse function theorems. The importance is rather geometric, as it arises when looking at curves, surfaces (or higher dimensional ... submanifolds) in $\mathbb{R}^n$. While such subspaces are usually given by equations of type $f(x_1, \dots, x_n) = 0$ (think e.g. of $x^2 + y^2 = 1$, defining the unit circle in the plane), one would like to express some of the coordinates x_i in terms of the others (or, equivalently, describe our subspace as a graph).

Theorem 1.29. *Let $f : \Omega \to R^k$ be a smooth map defined on an open $\Omega \subset \mathbb{R}^m \times \mathbb{R}^k$ whose elements we label as $(x,y) = (x_1, ..., x_m, y_1, ..., y_k)$. Furthermore let $(\tilde{x}, \tilde{y}) \in \Omega$ be a point where $f(\tilde{x}, \tilde{y}) = 0$ and the matrix*

$$\left(\frac{\partial f_i}{\partial y_j}(\tilde{x}, \tilde{y}) \right)_{1 \le i,j \le k}$$

is non-singular. Then there exists a function $par : U \to \mathbb{R}^k$ defined in a neighborhood U of $\tilde{x}$ such that for all (x,y) near $(\tilde{x}, \tilde{y})$, one has

$$f(x,y) = 0 \Longleftrightarrow y = par(x).$$

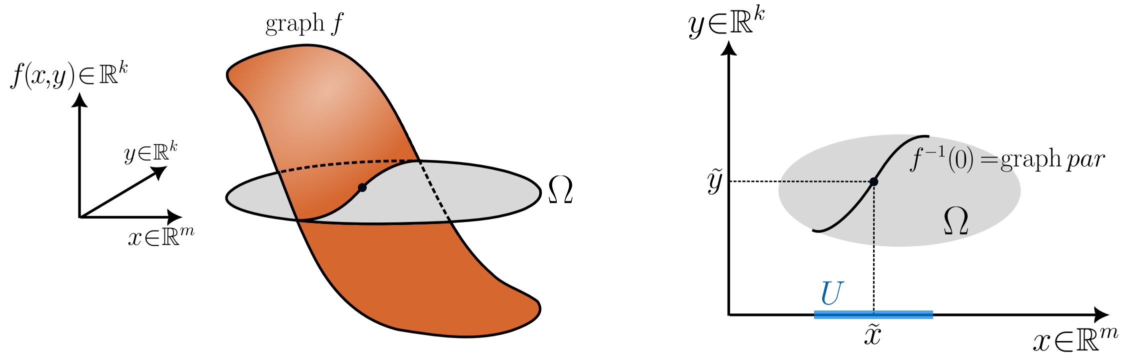


Fig. 1.4 The implicit function Theorem 1.29 illustrated for a concrete choice of $f:\Omega\to\mathbb{R}^k$. The theorem establishes that the preimage $f^{-1}(0)$ (under appropriate assumptions) can locally be written as a graph of a function $par: U\to\mathbb{R}^k$ over a subset of the variables. In other words, the condition that f vanishes *implicitly* defines the function par.

The matrix appearing in this theorem is exactly the Jacobian matrix of the map $y\mapsto f(\tilde{x},y)$ at $y=\tilde{y}$. You can convince yourself using Fig. 1.4 that its non-singularity is a neccessary condition to find a smooth function par: In the depicted situation, it corresponds exactly to a tangency of $f^{-1}(0)$ in the y-direction at $(\tilde{x},\tilde{y})$.

Remark 1.30. Note that this theorem is not completely canonical: It gives preference to the last k components of the arguments of f, i.e. depends on how we split the components of elements of Ω into x and y-components. For example, there is an obvious modification in which the starting assumption is that the Jacobian with respect to the *first* k components is regular, instead of the last ones. Such modifications are necessary even when looking at the simplest examples: E.g., for the unit circle where $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^2+y^2-1$, the condition

$$\frac{\partial f}{\partial y}(x,y)\neq 0$$

(necessary for the theorem) is valid at *almost* all the points (x,y) in the circle (and, indeed, we can always solve $y=\pm\sqrt{1-x^2}$), except for the points $(1,0)$ and $(-1,0)$ (and, indeed, there is a problem there: we would need a function two take two values simultaneously close to $y=0$ in order for it's graph to match $f^{-1}(0)$). However, at those points one can switch the roles of x and y- and, indeed, around those points which are problematic for $\pm\sqrt{1-x^2}$, one can write $x=\pm\sqrt{1-y^2}$.

A more intrinsic version of the theorem (and with exactly the same proof) can be obtained by requiring that $(Df)_x$ has maximal rank k, without specifying which minor is non-singular. the conclusion will be that there exists a permutation $\sigma\in S_{m+k}$ such that $f(p)=0$ near $\tilde{p}\in\mathbb{R}^{m+k}$ is equivalent to

$$\pi_y(\sigma\cdot p)=par(\pi_x(\sigma\cdot p)),$$

where π_x and π_y are the projections of $\mathbb{R}^{m+k}$ onto the first m and last k components, respectively, and we write $\sigma\cdot p$ for the result of permuting p by σ. But perhaps the most geometric formulation is what is know as the submersion theorem- see below.

Proof. Consider the map

$$F:\Omega\to\mathbb{R}^{m+k},\quad F(x,y):=(x,f(x,y)).$$

Then the non-singularity condition in the statement precisely means that $(DF)_{(\tilde{x},\tilde{y})}$ is non-singular. Hence, by the inverse function theorem, we find a smooth inverse G of F, defined near $F(\tilde{x},\tilde{y})$. Given the form of F, it follows that G is of a similar form:

$$G(x,z)=(x,g(x,z)).$$

That $G\circ F$ and $F\circ G$ are the identity maps (near $(\tilde{x},\tilde{y})$, and $F(\tilde{x},\tilde{y})$, respectively) translates into

$$g(x,f(x,y))=y\qquad\text{and}\qquad f(x,g(x,z))=z. \tag{1.2.2}$$

The first equation shows that

$$f(x,y) = 0 \Longrightarrow g(x,0) = y,$$

hence we have an obvious candidate $par(x) := g(x,0)$. Note that the assumption $f(\tilde{x},\tilde{y}) = 0$ guarantees that g is defined for (x,z) near $(\tilde{x},0)$. The fact that, indeed, $f(x, par(x)) = 0$ for x close to $\tilde{x}$ is just the second equation in (1.2.2) applied when $z = 0$. 😀

1.2.4 Local coordinates/charts

The standard coordinates in $\mathbb{R}^n$, despite being "obvious", are often not the best ones to use in specific problems. E.g.: often when dealing with (algebraic or differential) equations or computing integrals, one proceeds to a change of variables (i.e. passing to more convenient coordinates). Baby example: looking at the curve in $\mathbb{R}^2$ defined by

$$5x^2 + 2xy + 2y^2 = 1,$$

a change of coordinates of type

$$x = \frac{u+v}{3}, \quad y = \frac{u-2v}{3} \tag{1.2.3}$$

brings us to the simpler looking equation $u^2 + v^2 = 1$. A very common change of coordinates in $\mathbb{R}^2$ is the passing to polar coordinates:

$$x = r\cos(\theta), \quad y = r\sin(\theta). \tag{1.2.4}$$

To formalise such changes of coordinates, one talks about charts:

Definition 1.31. A **smooth chart of** $\mathbb{R}^n$ is a diffeomorphism

$$\chi = (\chi_1, \ldots, \chi_n) : U \to \Omega \subset \mathbb{R}^n$$

between an open $U \subset \mathbb{R}^n$ and an open $\Omega \subset \mathbb{R}^n$. The open U is called the **domain of the chart** and, for $p \in U$,

$$(\chi_1(p), \ldots, \chi_n(p))$$

are called the **coordinates of** p **w.r.t. the chart** (U,χ) and we also say that (U,χ) is a smooth chart around p.

For instance the change of coordinates (1.2.3) is about the chart

$$\chi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \chi(x,y) = (2x+y, x-y) \tag{1.2.5}$$

so that, in the new coordinates, a point $p = (x,y)$ will have the coordinates (w.r.t. χ)

$$u(x,y) = 2x+y, \quad v(x,y) = x-y.$$

Similarly for the polar coordinates where, computing the inverse of $(r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$, one finds the chart

$$\chi(x,y) = \left(\sqrt{x^2+y^2}, arctg(\frac{y}{x})\right).$$

1.2.5 Changing coordinates to make functions simpler (the immersion/submersion theorem)

In general, given a smooth function

$$f : \mathbb{R}^n \to \mathbb{R}^k$$

and a point $p \in \mathbb{R}^n$, whenever we have two new charts χ and χ' around p and $f(p)$, respectively, one can represent the function f using the new resulting coordinates: **the representation of f w.r.t. the charts χ and χ'** is

$$f_{\chi}^{\chi'} = \chi' \circ f \circ \chi^{-1}.$$

Of course, for the standard charts (the identity maps) one obtains back f. If just χ', or just χ, is the standard chart then we use the notations f_χ and $f^{\chi'}$, respectively.

For instance, for the function

$$f(x,y) = 5x^2 + 2xy + 2y^2,$$

with respect to the new chart (1.2.5) one obtains $f_\chi(u,v) = u^2 + v^2$.

In general, it is interesting to try to write smooth functions in the simplest possible way, modulo change of coordinates. The simplest types of functions for which this is possible are the most "non-singular" ones. More precisely, given

$$f : U \to \mathbb{R}^k$$

a smooth map defined on an open $U \subset \mathbb{R}^n$ and given $x \in U$, the "non-singular behaviour" that we require is that

$$(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$$

has maximal rank. It is interesting to consider the cases $n \geq k$ and $n \leq k$ separately. The first case brings us to the more canonical version of the implicit function theorem:

Theorem 1.32 (the submersion theorem). *Assume that f is a* ***submersion*** *at a given point $p \in U$ in the sense that $(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is surjective. Then there exists a smooth chart χ of $\mathbb{R}^n$ around p such that, around $\chi(p)$, $f_\chi = f \circ \chi^{-1}$ is given by*

$$f_\chi(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k).$$

Proof. Since the matrix representing $(Df)_p$ is of maximal rank, one of its maximal minors (an $k \times k$ matrix) is invertible; we may assume that the invertible minor is precisely the one made of the last k rows (why?)- which is also the hypothesis of the implicit function theorem (Theorem 1.29). Looking at the proof of the theorem, one remarks that the desired chart is $\chi = \tilde{f}$. 😀

A similar argument gives rise to the following:

Theorem 1.33 (the immersion theorem). *Assume that f is an immersion at a given point $p \in U$ in the sense that $(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is injective. Then there exists a smooth chart χ' of $\mathbb{R}^k$ around $f(p)$ such that, in a neighborhood p, $f^{\chi'} = \chi' \circ f$ is given by*

$$f^{\chi'}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \underbrace{0, \ldots, 0}_{k-n \text{ zeros}}). \tag{1.2.6}$$

More precisely, denoting $q = f(p)$, there exist:

- *a smooth chart $\chi' : U'_q \to \Omega'_q$ of $\mathbb{R}^k$ around q,*
- *a neighborhood Ω_p of p in $\mathbb{R}^n$, inside the domain of f*

such that (1.2.6) holds on Ω_p. Furthermore, one may choose χ' and Ω_p so that:

$$f(\Omega_p) = \{u \in U'_q : \chi'_{L+1}(u) = \ldots = \chi'_k(u) = 0\}.$$

Proof. Let us give a proof that makes reference to $(Df)_p$ as a linear map and not as a matrix. Since $(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is injective, we find a second linear map $B : \mathbb{R}^{k-n} \to \mathbb{R}^k$ such that

$$((Df)_p, B) : \mathbb{R}^n \times \mathbb{R}^{k-n} \to \mathbb{R}^k$$

is an isomorphism. Consider then

$$h : U \times \mathbb{R}^{k-n} \to \mathbb{R}^k, \quad h(x_1, x_2) = f(x_1) + B(x_2).$$

We see that h satisfies the hypothesis of the inverse function theorem at the point $(x, 0)$. Hence it is a diffeomorphism around a neighborhood of $(x, 0)$. We denote by

$$\chi' : U' \to \Omega'$$

its inverse. Note that:

1. U' is an open neighborhood of $f(p)$ in $\mathbb{R}^k$.
2. Ω' is an open neighborhood of $(p, 0)$ in $\mathbb{R}^k$, contained in $U \times \mathbb{R}^{k-n}$.
3. the intersection of $\Omega' \subset \mathbb{R}^k$ with $\mathbb{R}^n \times \{0\}$,

$$\Omega := \{u \in \mathbb{R}^n : (u, 0) \in \Omega'\},$$

   is an open neighborhood of p included in the domain of f.

Note that, since $\chi'(h(x_1, x_2) = 0$ for all $(x_1, x_2) \in \Omega'$ and $h(x, 0) = f(x)$, we have $\chi'(f(x)) = (x, 0)$ for all $x \in \Omega$. This proves the main part of the theorem; for the last part, note that we have, by the first part, that $f(\Omega)$ is inside the zero set of $\chi'_2 : U' \to \mathbb{R}^{k-n}$ (the second component of the chart χ' w.r.t. the decomposition $\mathbb{R}^k = \mathbb{R}^n \times \mathbb{R}^{k-n}$). For the reverse inclusion, let $x \in U'$ with $\chi'_2(x) = 0$; since $h \circ \chi' = \text{id}$ on U', we obtain

$$x = h(\chi'(x)) = h(\chi'_1(x), 0) = f(\chi'_1(x))$$

where, for the last equality, we used the explicit formula for h. Moreover, since $\chi'(x) \in \Omega'$ and since $\chi'(x) = (\chi'_1(x), 0)$, by the definition of Ω, we have $\chi'_1(x) \in \Omega$. With the previous equality in mind, we obtain $x \in f(\Omega)$. 😀

For later use let us introduce the notion of smoothness defined on arbitrary subsets $M \subset \mathbb{R}^n$.

Definition 1.34. Given $M \subset \mathbb{R}^n$ and a function $f : M \to \mathbb{R}^k$, we say that f **is smooth around** $p \in M$ if, in a neighborhood U of p in M, $f|_U$ admits a smooth extension to an open inside $\mathbb{R}^n$ containing U. When this happens around all $p \in M$, we say that f is **smooth**.

A **diffeomorphism** between $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ is any bijection $f : M \to N$ with both f and f^{-1} smooth.

And here is a nice application of the existence of smooth partitions of unity.

Exercise 1.35. Show that if $M \subset \mathbb{R}^n$ is a closed subset that any smooth function $f : M \to \mathbb{R}^k$ admits a smooth extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^k$. (Hint: $\mathbb{R}^n \setminus M$ is open; get an open cover of $\mathbb{R}^n$ out of one of M.)

1.2.6 Embedded submanifolds of $\mathbb{R}^L$

We now move to the notion of (smooth) embedded submanifolds of $\mathbb{R}^L$ [1]. In low dimensions, these are curves (1-dimensional) and surfaces (2-dimensional); for an arbitrary dimension m we will be talking about m-dimensional submanifolds of $\mathbb{R}^L$. For instance, the standard sphere

$$S^m = \{(x_0, \ldots, x_m) \in \mathbb{R}^{m+1} : \sum_i (x_i)^2 = 1\} \quad (L = m+1)$$

will be such a smooth m-dimensional submanifold. Already when looking at the simplest examples one sees that such subspaces may (naturally) be described in several different (but equivalent) ways. E.g., already for the unit circle in the plane, one has the standard descriptions:

- implicit (by equations): $x^2 + y^2 = 1$.
- parametric: $x = \cos(t), y = \sin(t)$ with $t \in \mathbb{R}$.

Accordingly, the notion of submanifold of $\mathbb{R}^L$ can be introduced in several ways that look differently (but which turn out to be equivalent).

We start with the definition that can be seen as just a small variation on the notion of topological manifold from Definition 1.3 just that, for $M \subset \mathbb{R}^L$ the axioms (TM1), (TM2) are automatically satisfied, and one can can further take advantage of the Euclidean space to talk about *smoothness* of charts- as in Definition 1.34.

Definition 1.36. An m-dimensional embedded submanifold of $\mathbb{R}^L$ is any subset $M \subset \mathbb{R}^L$ which, for each $p \in M$, satisfies **the** (m-dimensional) **manifold condition at** p in the following sense: there exists a topological chart of M (Definition 1.3)

$$\chi : U \to \Omega$$

($U \subset M$ open neighborhood of p, Ω open in $\mathbb{R}^m$) which is also a diffeomorphism (i.e. χ and χ^{-1} are smooth in the sense of Definition 1.34). These will also be called **smooth (*m*-dimensional) charts** for M.

Of course, when $M = \mathbb{R}^L$, the resulting notion of "smooth chart for $\mathbb{R}^L$" coincides with the one already introduced in Definition 1.31. For general M, a particularly nice class of smooth charts of M are the ones that can be obtained by restricting such charts of $\mathbb{R}^L$. More precisely, given a subset $M \subset \mathbb{R}^L$, a smooth chart of $\mathbb{R}^L$

$$\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^L \tag{1.2.7}$$

said to be **adapted to** M if it takes $U := M \cap \widetilde{U}$ into $\Omega := \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\})$:

[1] here L is an integer, possibly large, that will denote the dimension of the Euclidean space inside which our manifolds $M \subset \mathbb{R}^L$; we use here the letter L not only to suggest that L may be possibly large w.r.t. the dimension of M, but also to emphasise that the role of the dimension L is very different than that of the dimension m of M

$$\widetilde{\chi}|_U : U \to \Omega. \tag{1.2.8}$$

Equivalently: inside $\widetilde{U} \subset \mathbb{R}^L$, the points that belong to M are characterised by the equations $\widetilde{\chi}_i = 0$ for $i > m$:

$$M \cap \widetilde{U} = \{q \in \widetilde{U} : \widetilde{\chi}_{m+1}(q) = \ldots = \widetilde{\chi}_L(q) = 0.\}$$

Note also that Ω may be, and will be, interpreted as an open in $\mathbb{R}^m$; in this way, any smooth chart (1.2.7) of $\mathbb{R}^L$ that is adapted to M induces a smooth chart (1.2.8) of M.

Not every smooth chart χ of M is induced by an adapted smooth chart $\widetilde{\chi}$ of $\mathbb{R}^L$. However:

Proposition 1.37. *For $M \subset \mathbb{R}^L$ and $p \in M$, the manifold condition for M at p is equivalent to the existence of a smooth chart of $\mathbb{R}^L$ around p, that is adapted to M.*

The proof will be done together with the proof of the following theorem. This theorem describes submanifolds parametrically (think of $x = \cos(t), y = \sin(t)$ for the circle) and by equations (think of $x^2 + y^2 = 1$ for the circle), taking care of the precise conditions.

Theorem 1.38. *Given a subset $M \subset \mathbb{R}^L$, $p \in M$, the following are equivalent:*

1. *M satisfies the m-dimensional manifold condition at p.*
2. *M admits an m-dimensional **parametrization around** p- by which we mean a homeomorphism*

$$par : \Omega \to U \subset M$$

between an open $\Omega \subset \mathbb{R}^m$ and an open neighborhood U of p in M satisfying the regularity condition that, as a map from Ω to $\mathbb{R}^L$, par is an immersion.

3. *M can be described by an m-dimensional **implicit equation around** p- by which we mean a submersion*

$$eq : \widetilde{U} \to \mathbb{R}^{L-m}$$

defined on an open neighborhood $\widetilde{U}$ of p in $\mathbb{R}^L$ and which describes M near p by the equation $eq = 0$:

$$M \cap \widetilde{U} = \{q \in \widetilde{U} : eq(q) = 0\}.$$

Proof. For keeping track of notations note that, throughout the proof, we look around the given point $p \in M \subset \mathbb{R}^L$ and around the corresponding point

$$x = \chi(p) = par^{-1}(p) \in \Omega \subset \mathbb{R}^m.$$

Therefore, we will deal with

- neighborhoods U_p of p in M, and $\widetilde{U}_p$ of p in $\mathbb{R}^L$.
- neighborhoods Ω_x of x in $\mathbb{R}^m$.

The points in the neighborhoods of p will be denoted by q, while the ones in the neighborhoods of x by y; for them, we may be looking at similar neighborhoods U_q, $\widetilde{U}_q$ and Ω_y.

We first prove that (1) implies (2). We start with the the chart $\chi : U \to \Omega \subset \mathbb{R}^m$ defined in a neighborhood U of p in M. Setting $par = \chi^{-1}$ we have to check that par is a homeomorphism -which is clear by construction (it has the continuous χ as inverse)- and that, as a map $\Omega \to \mathbb{R}^L$, it is an immersion. For the last part use that the composition

$$\Omega \xrightarrow{par} U \xrightarrow{\chi} \Omega$$

is the identity on Ω and then apply the chain rule to deduce that, for each point $y \in \Omega$, $(D\chi)_{par(y)} \circ (D\,par)_y$ is the identity- hence, in particular, $(D\,par)_y$ will be injective.

We now prove that (2) implies both (1) as well as (3). Hence we start with a parametrization $par : \Omega \to U \subset M$; as above, we set $\chi = par^{-1} : U \to \Omega$. To get (1), we still have to check that χ is smooth in the sense of Definition

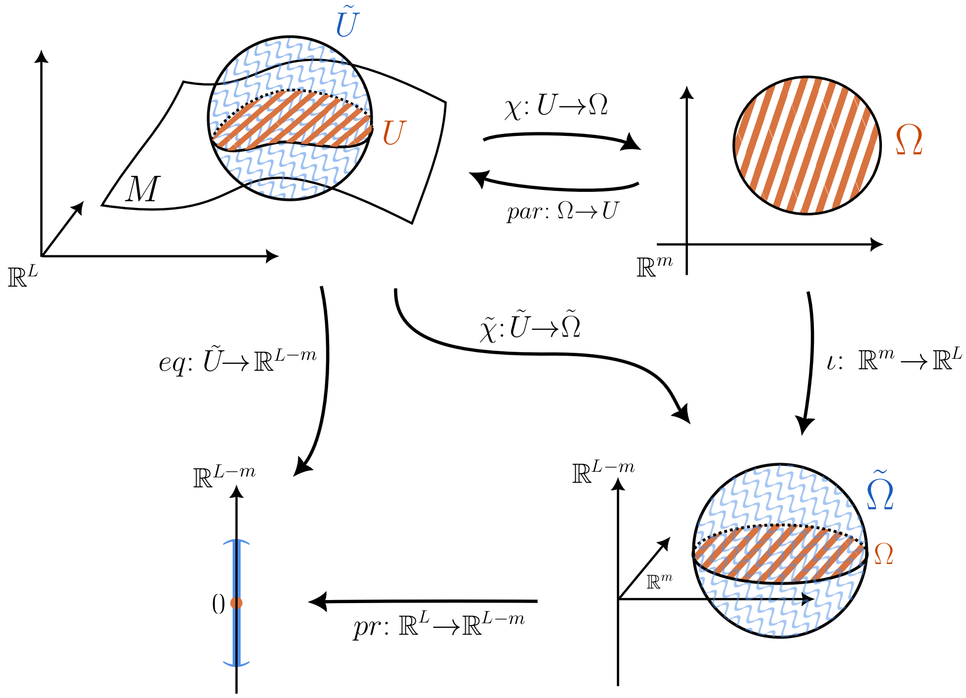


Fig. 1.5 The equivalent ways of phrasing the manifold condition in Theorem 1.38 involve (adapted) charts, parametrizations or implicit equations that are related as shown in this figure. Note that $U=\tilde{U}\cap M$ and $\Omega=\tilde{\Omega}\cap\mathbb{R}^m$. ι and pr are canonical inclusions and projections of the product space $\mathbb{R}^L=\mathbb{R}^m\times\mathbb{R}^{L-m}$ in the lower right. All these maps commute where they can be evaluated.

1.34: i.e., around any point $q\in U$, it is obtained by restricting a smooth map defined on an open $\widetilde{U}_q\subset\mathbb{R}^L$. For that we use the immersion theorem (Theorem 1.33) applied to $par:\Omega\to\mathbb{R}^L$ around

$$y=\chi(q)\in\Omega.$$

We find:

- an open neighborhood Ω_y of y in $\Omega\subset\mathbb{R}^m$
- a diffeomorphism $\widetilde{\chi}:\widetilde{U}_q\to\widetilde{\Omega}_q$ from an open neighborhood $\widetilde{U}_q\subset\mathbb{R}^L$ of p' to an open $\widetilde{\Omega}_q\subset\mathbb{R}^L$,

so that, on Ω_y, $\widetilde{\chi}\circ par$ becomes the inclusion on the first factors. We now write $\widetilde{\chi}=(\widetilde{\chi}_1,\widetilde{\chi}_2)$ where we use again the decomposition $\mathbb{R}^L=\mathbb{R}^m\times\mathbb{R}^{L-m}$. We deduce that $\widetilde{\chi}_1(par(z))=z$ for all $z\in\Omega_y$. Since $z=\chi(par(z))$ for all $z\in\Omega$, we deduce that $\widetilde{\chi}_1(r)=\chi(r')$ for all $r\in par(\Omega_q)$. In this way, on the neighborhood $par(\Omega_q)$ of q in U, χ is now the restriction of a smooth function defined on an open neighborhood of q in $\mathbb{R}^L$- namely $\widetilde{\chi}_1:\widetilde{U}_q\to\mathbb{R}^m$.

To prove (3) (still assuming (2)), we use $q=p$ in the previous reasoning and the resulting diffeomorphism $\widetilde{\chi}:\widetilde{U}_p\to\widetilde{\Omega}_p$; in principle, the desired function f will be $\widetilde{\chi}_2$, but we have to choose the domain of definition carefully. For that we use the last part of Theorem 1.33 which says that we may assume that

$$par(\Omega_x)=\{y\in\widetilde{U}_p:\widetilde{\chi}_{m+1}(y)=0,\ldots,\widetilde{\chi}_L(y)=0\}.$$

Since this is open in M, we can write it as $M\cap W_p$ for some open $W_p\subset\mathbb{R}^L$. Considering now

$$\widetilde{U}:=\widetilde{U}_p\cap W_p,\quad eq=\widetilde{\chi}_2|_{\widetilde{U}}:\widetilde{U}\to\mathbb{R}^{L-m}$$

one checks right away that $M \cap \widetilde{U}$ is the zero set of eq (why is eq a submersion?).

Exercise 1.39. Conclude now that $\widetilde{\chi}$ is actually an adapted chart.

We are now left with proving that (3) implies (1). Let $eq : \widetilde{U} \to \mathbb{R}^{L-m}$ satisfying the conditions from the hypothesis. Note that if we replace $\widetilde{U}$ by a smaller open neighborhood of p in $\mathbb{R}^L$ (and eq by its restriction), those conditions will still be satisfied. Therefore, using the submersion theorem applied to eq, we may assume that we also find a diffeomorphism $\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega}$ into an open subset of $\mathbb{R}^L$, such that $eq = \widetilde{\chi}_2$. This chart will then take the zero set of eq into the zero set of the second projection $\mathrm{pr}_2 : \widetilde{\Omega} \to \mathbb{R}^{L-m}$, i.e. into

$$\Omega := \{u \in \mathbb{R}^m : (u,0) \in \widetilde{\Omega}\}.$$

We deduce that the restriction of $\widetilde{\chi}$ to $U = M \cap \widetilde{U}$,

$$\chi := \widetilde{\chi}|_U : U \to \Omega \subset \mathbb{R}^m,$$

is a smooth chart of M (around p). 😀

Example 1.40. Returning to the circle S^1,

- $h(x,y) = x^2 + y^2 - 1$ serves as a (1-dimensional) implicit equation (around any point!)
- $p(t) = (\cos(t), \sin(t))$, when considered on sufficiently small intervals (on which it is injective) serves as paramatrization of S^1 around any point in S^1.
- as smooth (1-dimensional) charts one could use two projections $pr_1, pr_2 : S^1 \to \mathbb{R}$, restricted to the appropriate domains (so that they become homeomorphisms). Another possible choice of charts is given by the stereographic projections (see the lecture notes on Topology).

Exercise 1.41. Generalize this discussion to the spheres S^m of arbitrary dimension.

1.2.7 From directional derivatives to tangent spaces

The point of view provided by the directional derivatives brings us closer to the intrinsic nature of (p,v) when talking about $(Df)_p(v)$: that of tangent vector. The key point is that $(Df)_p(v)$ depends only on the behaviour of f near p, in "the direction of v"- and how we realize that "direction" is less important. This is best seen by looking at arbitrary paths through p with the original speed v, i.e. any smooth map

$$\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$$

(with $\varepsilon > 0$) satisfying

$$\gamma(0) = p, \quad \frac{d\gamma}{dt}(0) = v. \tag{1.2.9}$$

For instance, one could take $\gamma(t) = p + tv$, but the point is that the variation of $f(p+tv)$ at $t = 0$ does not depend on this specific choice of γ.

Lemma 1.42. *If f is differentiable at p then, for any path γ satisfying (1.2.9), one has*

$$(Df)_p(v) = \frac{\partial f}{\partial v}(p) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)).$$

In particular, if f is constant along such a path γ, then $(Df)_p(v) = 0$.

This point of view becomes extremely useful when looking at more general subspaces

$$M \subset \mathbb{R}^n.$$

Definition 1.43. Let $M \subset \mathbb{R}^n$ and consider a point $p \in M$. A **smooth curve in** M is any smooth map $\gamma : I \to \mathbb{R}^n$ defined on some interval $I \subset \mathbb{R}$, which takes values in M.

A **vector tangent to** M **at** p is any vector $v \in \mathbb{R}^n$ which can be realized as the speed at $t = 0$ of a smooth curve in M that passes through p at $t = 0$ (i.e. for which $0 \in I$ and $\gamma(0) = p$):

$$v = \frac{d\gamma}{dt}(0).$$

The set of such vectors is denoted by $T_p^{\text{class}}M$; hence

$$T_p^{\text{class}}M \subset \mathbb{R}^n.$$

Although we use the name ”tangent *space*”, in general (for completely random Ms inside $\mathbb{R}^n$), $T_p^{\text{class}}M$ is just a subset of $\mathbb{R}^n$ (... but it is a vector subspace if M is ”nice”).

Exercise 1.44. Compute $T_p^{\text{class}}M$ when:

1. $M \subset \mathbb{R}^2$ is the unit circle and $p = (1,0)$.
2. $M \subset \mathbb{R}^2$ is the union of the coordinate axes and $p = (0,0)$.

Exercise 1.45. Assume that $M \subset \mathbb{R}^n$ is defined by an equation $f(x) = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth function. For $p \in \mathbb{R}^n$ we denote by $\text{Ker}_p(Df)$ the kernel (= the zero set) of the differential $(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$. Show that, in general,

$$T_p^{\text{class}}(M_f) \subset \text{Ker}_p(Df),$$

but the inclusion may be strict. Then prove this inclusion becomes and equality when

$$f(x_1, \ldots, x_n) = (x_1)^2 + \ldots + (x_n)^2 - 1.$$

Exercise 1.46. With the notations from the previous exercise show that for all $p \in M_f$ at which f is a submersion

$$T_p^{\text{class}}(M_f) = \text{Ker}_p(Df).$$

We now return to our discussion on differentials/directional derivatives, recast in terms of tangent spaces. Namely, Lemma 1.42 gives us right away:

Corollary 1.47. *Given $M \subset \mathbb{R}^n$, $p \in M$ and a function $\tilde{f} : \mathbb{R}^n \subset \mathbb{R}^k$ differentiable at p then, for any vector $v \in \mathbb{R}^n$ tangent to M at p, $(D\tilde{f})_p(v)$ depends only on $\tilde{f}|_M$.*

Of course, a similar conclusion holds slightly more generally, for any function $\tilde{f} : U \to \mathbb{R}^k$ defined on an open neighborhood $U \subset \mathbb{R}^n$ of p- the otcome being that $(D\tilde{f})_p(v)$ only depends on the values of $\tilde{f}|_M$ near p. This shows how to define the differential of a function $f : M \to \mathbb{R}^k$ which is differentiable at $p \in M$ in the sense of Definition 1.34: for $f : M \to \mathbb{R}^k$ that is differentiable at p, one has a well-defined differential

$$(Df)_p : T_p^{\text{class}}M \to \mathbb{R}^k,$$

defined using an extension $\tilde{f}$ of f near p, but independent of the extension.

Exercise 1.48. Show that if $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ and $f : M \to N$ is smooth at $p \in M$, then $(Df)_p$ takes values in $T_{f(p)}^{\text{class}}N$. Then prove the chain rule in this context and deduce that, if f is a diffeomorphism, then $(Df)_p$ is a bijection between $T_p^{\text{class}}M$ and $T_{f(p)}^{\text{class}}N$.

Finally, let us look at tangent spaces of submanifolds of $\mathbb{R}^n$.[2]

Proposition 1.49. *If $M \subset \mathbb{R}^n$ is a m-dimensional embedded submanifold then, for any $p \in M$, the tangent space of M at p is an m-dimensional vector subspace of $\mathbb{R}^n$, which can also be described as follows:*

1. *as the kernel of $(D\,eq)_p : \mathbb{R}^n \to \mathbb{R}^{L-m}$, where $eq : \widetilde{U} \to \mathbb{R}^{L-m}$ is any implicit equation defining M around p.*
2. *as the image of $(D\,par)_p : T_p\Omega \to \mathbb{R}^n$, where $par : \Omega \to M$ is any parametrization of M around p.*

Proof. Exercise.

Exercise 1.50. Compute again the tangent spaces of the spheres, but applying now the previous proposition.

1.2.8 More exercises

Exercise 1.51. Consider two smooth functions

$$U \xrightarrow{f} U' \xrightarrow{g} \mathbb{R}^p,$$

defined on opens $U \subset \mathbb{R}^n$, $U' \subset \mathbb{R}^k$. Using the interpretation of linear maps as matrices (as made precise on page 23) show that the chain rule becomes:

$$\frac{\partial g \circ f}{\partial x_i}(x) = \sum_{j=1}^{n} \frac{\partial g}{\partial y_j}(f(p)) \frac{\partial f_j}{\partial x_i}(x)$$

for all $x \in U$ and $1 \leq i \leq n$.

Exercise 1.52. Show that for any function $g : \mathbb{R} \to \mathbb{R}$, the function

$$\tilde{g} : \mathbb{R}^n \to \mathbb{R}, \quad \tilde{g}(x_1, \dots, x_n) = g((x_1)^2 + \dots + (x_n)^2)$$

is not a submersion at $x = 0$.

Exercise 1.53. Assume that $f : U_0 \to \mathbb{R}^k$ is a smooth map, $U \subset \mathbb{R}^n$ open, $p \in U$. Let

$$\chi : U \to \Omega \subset \mathbb{R}^n, \quad \chi' : U' \to \Omega' \subset \mathbb{R}^k$$

be charts, of $\mathbb{R}^n$ around p and of $\mathbb{R}^k$ around $f(p)$, respectively. What is the (maximal) domain of definition of $f_{\chi}^{\chi'}$?

Exercise 1.54. Consider

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = 3 \cdot \sqrt[3]{x^2 + 2xy + 2y^2}$$

and look around $p = (1,0)$. Find a chart χ of $\mathbb{R}^2$ around p and a chart χ' of $\mathbb{R}$ around $f(p) = 3$ such that, w.r.t. these charts,

$$f_{\chi}^{\chi'}(u,v) = u^2 + v^2.$$

Exercise 1.55. Show that

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(t) = (\cos(t), \sin(t))$$

is an immersion at each point. Then, looking around $t = 0$, find a chart χ' of $\mathbb{R}^2$ around $f(0) = (1,0)$ such that, w.r.t. this chart,

$$f_{\chi'}(t) = (t, 0).$$

[2] is this the right place?

Exercise 1.56. Show that if $f : U \to \mathbb{R}^k$, $p \in U$ satisfy the conclusion of the submersion theorem, then f must be a submersion at p. Similarly for the immersion theorem.
(Hint: try it! If it really doesn't work, then look at the next exercise).

Exercise 1.57. Assume that $f : U \to \mathbb{R}^k$ is a smooth map, $U \subset \mathbb{R}^n$ open, $p \in U$. Let χ be a chart of $\mathbb{R}^n$ around p and let χ' be a chart of $\mathbb{R}^k$ around $f(p)$. Show that f is a submersion/immersion at p if and only if $f_{\chi}^{\chi'}$ is a submersion/immersion at $\chi(p)$.

Exercise 1.58. Consider the stereographic projection w.r.t. the north pole p_N, denoted

$$\chi_N : S^2 \setminus \{p_N\} \to \mathbb{R}^2$$

and similarly the one w.r.t. the south pole, denoted χ_S. Show that

$$\chi_S \circ \chi_N^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$$

is a diffeomorphism.

Exercise 1.59. For any $\varepsilon > 0$ describe a smooth function

$$f : \mathbb{R}^n \to [0,1]$$

with the property that $f(0) > 0$ and whose support (in $\mathbb{R}^n$) is contained in the ball $\{x \in \mathbb{R}^n : ||x|| < \varepsilon\}$.

Exercise 1.60. Assume that $f : U \to U'$ and $g : U' \to U''$ are two smooth functions, with $U \subset \mathbb{R}^n$, $U' \subset \mathbb{R}^k$ and $U'' \subset \mathbb{R}^p$ opens. Show that if $g \circ f$ is a local diffeomorphism around a given point $x \in U$, then:

1. f is an immersion at x and g is a submersion at $f(x)$.
2. however, it may happen that f is not a submersion at x and g is not an immersion at $g(x)$ (describe an example!).
3. if, furthermore, f is a submersion at x or g is an immersion at $f(x)$, then both f and g are local diffeomorphisms (around x and $f(x)$, respectively).

Chapter 2
Smooth manifolds

2.1 Manifolds

2.1.1 Charts and smooth atlases

The difference between topological manifolds (see Definition 1.3) and smooth manifolds is, as the terminology suggests, that we assume smoothness for all the objects one considers (so that, on smooth manifolds, unlike for topological ones, we will be able to talk about speeds of curves, tangent vectors, differential forms, etc etc). For subspaces $M \subset \mathbb{R}^n$, making use of the ambient space $\mathbb{R}^n$, we managed to make sense of smoothness of various objects on M, such as charts- giving rise to the notion of smooth submanifold of $\mathbb{R}^n$ (as in Definition 1.36). However, in the general setting, there is no intrinsic way to make sense of smoothness just for a topological space M (not necessarily embedded into $\mathbb{R}^n$)- instead, we need extra-data on M that serves precisely that purpose. And that is the notion of smooth atlas that we start with here.

Recall from Definition 1.3 that, given a topological space M, an m-dimensional chart is a homeomorphism χ between an open U in M and an open subset $\chi(U)$ of $\mathbb{R}^m$,

$$\boxed{\chi : U \to \chi(U) \subset \mathbb{R}^m.} \qquad \text{(coordinate charts)}$$

We also say that (U, χ) is a chart for M, and we call U the domain of the chart. Given such a chart, each point $p \in U$ is determined/parametrized by its coordinates w.r.t. χ:

$$(\chi_1(p), \dots, \chi_m(p)) \in \mathbb{R}^m$$

(a more intuitive notation would be: $(x^1_\chi(p), \dots, x^m_\chi(p))$).

Given a second chart

$$\chi' : U' \to \chi'(U') \subset \mathbb{R}^m,$$

the map

$$\boxed{c^{\chi'}_{\chi} := \chi' \circ \chi^{-1} : \chi(U \cap U') \to \chi'(U \cap U')} \qquad \text{(coordinate changes)}$$

is a homeomorphism between two opens in $\mathbb{R}^m$. It will be called **the change of coordinates map** from the chart χ to the chart χ'. The terminology is motivated by the fact that, denoting

$$c^{\chi'}_{\chi} = (c_1, \dots, c_m),$$

the coordinates of a point $p \in U \cap U'$ w.r.t. χ' can be expressed in terms of those w.r.t. χ by:

$$\chi'_i(p) = c_i(\chi_1(p), \ldots, \chi_m(p)).$$

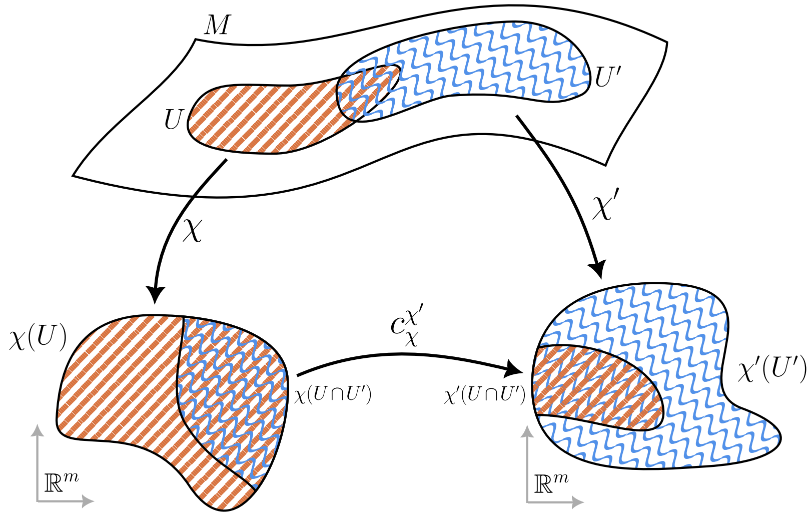


Fig. 2.1 Two charts $\chi : U \to \chi(U)$ and $\chi' : U' \to \chi'(U') \subseteq \mathbb{R}^m$ together with the change of coordinates map $c_\chi^{\chi'} : \chi(U \cap U') \to \chi'(U \cap U')$. These maps commute wherever they can be evaluated sensibly, i.e. $c_\chi^{\chi'} \circ \chi = \chi'$ holds on $U \cap U'$.

Definition 2.1. We say that two charts (U, χ) and (U', χ') are **smoothly compatible** if the change of coordinates map $c_\chi^{\chi'}$ (a map between two opens in $\mathbb{R}^m$) is a diffeomorphism.

Definition 2.2. A (m-dimensional) **smooth atlas** on a topological space M is a collection $\mathscr{A}$ of (m-dimensional) charts of M with the following properties:

1. the domains of the charts that belong to $\mathscr{A}$ cover M entirely.
2. each two charts from $\mathscr{A}$ are smoothly compatible.

Example 2.3. (Euclidean spaces) On $M = \mathbb{R}^m$ there are several interesting atlases. We mention here the extreme ones: the atlas $\mathscr{A}_{\mathbb{R}^m}$ consisting of only one chart, namely the identity chart $\mathrm{Id}_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m$, and $\mathscr{A}_{\mathbb{R}^m}^{\max}$ consisting of all smooth charts of $\mathbb{R}^m$ in the sense of Definition 1.31.

Example 2.4. (inside Euclidean spaces) For embedded m-dimensional submanifolds $M \subset \mathbb{R}^L$ (cf. Definition 1.36), there are two interesting atlases on M: the atlas $\mathscr{A}_M^{\max}$ consisting of all smooth m-dimensional charts of M in the sense of Definition 1.36, and the atlas $\mathscr{A}_M^{\text{adapt}}$ consisting of all charts that arise from smooth charts of $\mathbb{R}^L$ that are adapted to M (see the discussion following Definition 1.36).

The charts of an atlas are used to transfer notions and properties that involve smoothness from the Euclidean spaces (and opens inside) $\mathbb{R}^m$ to M; the compatibility of the charts ensures that the resulting notions (now on M) do not depend on the choice of the charts from the atlas. Hence one may say that an atlas on M allows us to

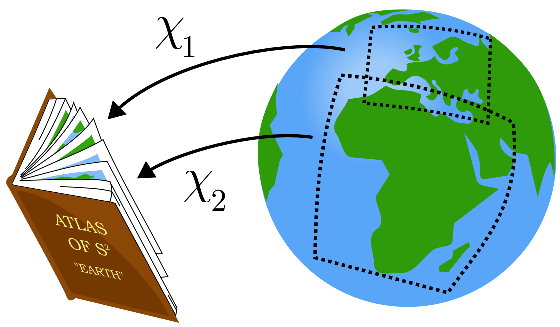


Fig. 2.2 An atlas of Earth is exactly a collection of charts that covers the surface of the globe. The same region might be included in several charts of a smooth atlas in a way that measured distances, angles and areas do not match - e.g. one chart might use the *Mercator projection* whereas another might faithfully depict areas. However, the smooth compatibility between charts does guarantee at least that a smooth path in one chart cannot develop any kinks in another.

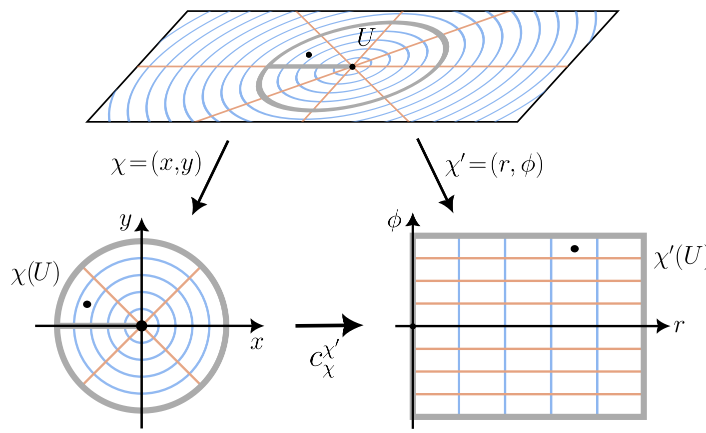


Fig. 2.3 Example of two smoothly compatible charts over the same open subset $U = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 5\} \setminus (-\infty,0] \times \{0\}$ of the plane: The identity chart $\chi = (x,y)$ as well as polar coordinates $\chi' = (r,\phi)$. The change of coordinate map going from right to left, for example, can be visualized by first collapsing the left boundary of the rectangle into the origin and then glueing the two boundary components that touch the origin together on the negative x-axis.

put a "smooth structure" on M. For instance, given a smooth m-dimensional atlas $\mathscr{A}$ on the space M, a function $f : M \to \mathbb{R}$ is called **smooth w.r.t. the atlas** $\mathscr{A}$ if for any chart (U,χ) that belongs to $\mathscr{A}$,

$$f_\chi := f \circ \chi^{-1} : \chi(U) \to \mathbb{R}$$

is smooth in the usual sense ($\chi(U)$ is an open in $\mathbb{R}^m$!). We will temporarily denote by

$$\mathscr{C}^\infty(M,\mathscr{A}) \tag{2.1.1}$$

the set of such smooth functions. However, there is a little "problem": the fact that two different atlases may give rise to the same smooth functions.

Exercise 2.5. Returning to Example 2.3, show that

$$\mathscr{C}^\infty(\mathbb{R}^m,\mathscr{A}_{\mathbb{R}^m}) = \mathscr{C}^\infty(\mathbb{R}^m,\mathscr{A}^{\max}_{\mathbb{R}^m}).$$

2.1.2 Smooth structures

As we have pointed out, there is one aspect that requires a bit of attention: the fact that two different atlases may give rise to the same "smooth structure". One way to overcome this "problem" is by using smooth atlases that are maximal:

Definition 2.6. An m-dimensional **smooth structure** on a topological space M is an m-dimensional smooth atlas $\mathscr{A}$ on M which is maximal, i.e. with the property that there is no smooth atlas strictly containing $\mathscr{A}$.

Example 2.7. ((opens in) the Euclidean spaces) On $\mathbb{R}^m$ the collection of all its smooth charts in the sense of Definition 1.31,

$$\mathscr{A}^{\max}_{\mathbb{R}^m} := \{\chi : U \to \Omega_\chi \text{ diffeomorphisms between opens } U, \Omega_\chi \subset \mathbb{R}^m\},$$

is a maximal atlas and, therefore, it defines a smooth structure on $\mathbb{R}^m$. Similarly for opens $\Omega \subset \mathbb{R}^m$ (just restrict to charts with domain $U \subset \Omega$). These will be called

the standard smooth structure on the Euclidean space $\mathbb{R}^m$/ on the open $\Omega \subset \mathbb{R}^m$.

Unless otherwise stated, from now, on the Euclidean spaces $\mathbb{R}^m$ and opens inside them will always be endowed with this smooth structure.

Example 2.8. (submanifolds of the Euclidean spaces) Similarly, for an embedded submanifold $M \subset \mathbb{R}^L$ the collection of all smooth charts of M in the sense of Definition 1.36 form a smooth maxima atlas $\mathscr{A}^{\max}_M$ and therefore defines a smooth structure on M- called **the standard smooth structure on the embedded submanifold**.

Here is a more direct characterization of the maximality condition:

Exercise 2.9. Show that, given any smooth atlas $\mathscr{A}$ on any topological space M, one has:

$$(\mathscr{A} \text{ is maximal}) \Longleftrightarrow \left(\begin{array}{c}\text{any topological chart of } M \text{ (see Def 1.3)}\\ \text{which is smoothly compatible with all the charts from } \mathscr{A}\\ \text{must belong to } \mathscr{A}\end{array}\right)$$

Actually, starting with an arbitrary (maximal or not) smooth atlas $\mathscr{A}$ on M, the collection of all charts of M that are compatible with all the charts that belong to $\mathscr{A}$,

$$\mathscr{A}^{\max} := \{\text{charts } \chi \text{ of } M : \chi \text{ is smoothly compatible with all } \chi' \in \mathscr{A}\},$$

is a new smooth atlas on M (exercise!), which is maximal (why?), and which contains $\mathscr{A}$ (why?). And the previous exercise says that $\mathscr{A}$ is maximal if and only of $\mathscr{A} = \mathscr{A}^{\max}$.

Definition 2.10. Given a smooth atlas $\mathscr{A}$ on M, **the smooth structure on M induced by the atlas $\mathscr{A}$** is the associated maximal atlas $\mathscr{A}^{\max}$.

Exercise 2.11. Show that for any smooth atlas $\mathscr{A}$ one has

$$\mathscr{C}^\infty(M, \mathscr{A}) = \mathscr{C}^\infty(M, \mathscr{A}^{\max}),$$

As we shall see a bit later (see Corollary 2.36 below), if $\mathscr{A}_1$ and $\mathscr{A}_2$ are two maximal smooth atlases, then

$$\mathscr{C}^\infty(M, \mathscr{A}_1) = \mathscr{C}^\infty(M, \mathscr{A}_2) \Longleftrightarrow \mathscr{A}_1 = \mathscr{A}_2. \tag{2.1.2}$$

One should keep in mind that very often a smooth structure is exhibited by describing a rather small atlas (ideally "the smallest possible one").

Example 2.12. (various atlases on Euclidean spaces) The standard smooth structure on $\mathbb{R}^m$ (see Example 2.7) can be induced by a very small atlas,

$$\mathscr{A}_{\mathbb{R}^m} := \{\mathrm{Id}_{\mathbb{R}^m}\},$$

i.e. the one consisting only of the identity chart

$$\chi = \mathrm{Id}_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m.$$

And a lot more is possible. E.g. here are two new smooth atlases:

$$\mathscr{A}_1 := \{\mathrm{Id}_U : U - \text{open in } \mathbb{R}^m\},$$

or $\mathscr{A}_2$ defined similarly, but using only open balls $B \subset \mathbb{R}^m$. Note that:

$$\mathscr{A}_{\mathbb{R}^m} \subset \mathscr{A}_1, \quad \mathscr{A}_{\mathbb{R}^m} \cap \mathscr{A}_2 = \{\emptyset\};$$

however, they all induce the same smooth structure on $\mathbb{R}^m$ (the standard one):

$$\mathscr{A}_1^{\max} = \mathscr{A}_2^{\max} = \mathscr{A}_{\mathbb{R}^m}^{\max}.$$

Example 2.13. Also for embedded submanifolds $M \subset \mathbb{R}^L$, there may be smaller and/or nicer atlases inducing the standard smooth structure on M described in Example 2.8. E.g., in full generality, one has the atlas $\mathscr{A}_M^{\text{adapt}}$ arising from adapted charts of $\mathbb{R}^L$ (as in Example 2.4). But probably the nicest and most convincing example is provided by the stereographic projections for the spheres- see below.

A slightly different way of understanding smooth structures is via equivalence classes of atlases where, in principle, two atlases are equivalent if they induce the same smooth functions. Let us make this more precise.

Definition 2.14. We say that two smooth atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ are **smoothly equivalent** if any chart in $\mathscr{A}_1$ is smoothly compatible with any chart in $\mathscr{A}_2$ (or, shorter: if $\mathscr{A}_1 \cup \mathscr{A}_2$ is again a smooth atlas).

This defines an equivalence relation on the collection of all smooth atlases. From this point of view, the main property of maximal atlases is that: in each equivalence class one can find one, and only one, maximal atlas (so that maximal atlases are in 1-1 correspondence with equivalence classes of smooth atlases). This is made more precise in the following simple exercise:

Exercise 2.15. Show that, for any atlas $\mathscr{A}$, $\mathscr{A}$ is (smoothly) equivalent to $\mathscr{A}^{\max}$; then show that for two atlases $\mathscr{A}_1$ and $\mathscr{A}_2$, one has

$$\mathscr{A}_1 \text{ is smoothly equivalent to } \mathscr{A}_2 \Longleftrightarrow \mathscr{A}_1^{\max} = \mathscr{A}_2^{\max}.$$

We see that one obtains a 1-1 correspondence

$$\left\{\begin{array}{c}\text{smooth}\\ \text{structures}\\ \text{on } M\end{array}\right\} = \left\{\begin{array}{c}\text{maximal}\\ \text{atlases}\\ \text{on } M\end{array}\right\} \stackrel{1-1}{\longleftrightarrow} \left\{\begin{array}{c}\text{equivalence classes of}\\ \text{smooth atlases}\\ \text{on } M\end{array}\right\},$$

and, for this reason, some text-books introduce the notion of smooth structure as an equivalence class of smooth atlases. The bottom line is that

any atlas $\mathscr{A}$ on M induces a smooth structure on M

and, depending on the point of view on smooth structure that we adopt, the smooth structure associated to an atlas $\mathscr{A}$ is interpreted either as the maximal atlas $\mathscr{A}^{\max}$ associated to $\mathscr{A}$, or as the equivalence class $[\mathscr{A}]$, respectively. The use of maximal atlases is more "down to earth"- in the sense that it avoids the use of equivalence classes. However, as we have illustrated above (e.g. in Example 2.12), one should keep in mind that very often a smooth structure is exhibited by describing a rather small atlas (ideally "the smallest possible one").

2.1.3 Manifolds

We now come to the main objects of study of this course.

Definition 2.16. A **smooth m-dimensional manifold** is a Hausdorff, second countable topological space M together with am m-dimensional smooth structure on M.

Given a smooth m-dimensional manifold M, when saying that (U,χ) is a **chart of the smooth manifold** M we mean that (U,χ) belongs to the maximal atlas $\mathscr{A}$ defining the smooth structure on M.

Example 2.17. [Euclidean spaces and its embedded submanifolds] As a conclusion of the discussions from Example 2.7 and 2.8:

- $\mathbb{R}^m$ endowed with the standard smooth becomes an m-dimensional manifold; and its charts are precisely the classical smooth charts of $\mathbb{R}^m$ in the sense of Definition 1.31.
- similarly, any embedded submanifold $M \subset \mathbb{R}^L$ as above, endowed with its standard smooth structure, becomes an m-dimensional manifold whose charts are the smooth charts from Definition 1.36. [1].

Example 2.18. (Opens) Given an m-dimensional manifold M, any non-empty open $U \subset M$ carries a natural (induced) smooth structure that makes U itself into an m-dimensional manifold: the charts of U are, by definition, the charts of M whose domain are contained in U.

In particular, any open

$$\Omega \subset \mathbb{R}^m$$

comes with a standard smooth structure making it into an m-dimensional manifold. Note that, again, this smooth structure can be induced by a very small atlas, namely

$$\mathscr{A}_\Omega := \{\mathrm{Id}_\Omega\};$$

actually, these are all the possible manifolds for which the smooth structure can be induced by an atlas consisting of one chart only- see Exercise 2.48.

Exercise 2.19. Show that the unit circle S^1 (with the topology induced from $\mathbb{R}^2$) admits an atlas made of two charts, but does not admit an atlas made of a single chart.

Exercise 2.20. Describe a smooth structure on the torus such that the underlying topology is the usual (Euclidean) one (just intuitively, on the picture for now). How many charts do you need?

Can you do the same for the Moebius band?

---

[1] actually, one can prove that that is all there is- in the sense that any manifold M can be "embedded" in some Euclidean space. This is a very interesting result, but the conclusion may be a bit misleading and it is not so important as it may seem: the "general" theory frees the embedded submanifolds from the ambient spaces and describe the geometry of the space that is independent of the ambient space- shading light even on the notions that are described, originally, using those embeddings. Think e.g. of what happens in Topology, when compactness was originally described for subsets of $\mathbb{R}^L$ requiring them to be closed and bounded, and then turned out to be a completely topological property (with important consequences). Moreover, some manifolds just do not come with "natural embeddings" into some Euclidean spaces- see e.g. the projective spaces!

Exercise 2.21. (product of manifolds) Let M and N be two manifolds of dimensions m and n, respectively and we want to make $M \times N$ into a manifold of dimension $m+n$. For that, for any smooth charts $\chi : U \to \Omega \subset \mathbb{R}^m$ of M and $\chi' : U' \to \Omega' \subset \mathbb{R}^n$ of N we would like that their product

$$\chi \times \chi' : U \times U' \to \Omega \times \Omega' \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}, \quad (p,q) \mapsto (\chi(p), \chi'(q))$$

becomes a smooth chart for $M \times N$. Show that $M \times N$ carries a unique smooth structure satisfying this property. (this is called **the product smooth structure**).

Exercise 2.22. In the definition of smooth manifolds show that the condition that M is second countable is equivalent to the fact that the smooth structure on M can be defined by an atlas that is at most countable.

Remark 2.23 (Avoiding Topology?). In an attempt to avoid any reference to topology, one may give a definition of manifolds as a structure on a set M rather than a topological space M. One problem with such an approach is that its "directness" (avoiding topology) is only apparent, and the various topological notions that inevitably show up over and over again would look even more complicated and also rather mysterious (for instance, to hide the second countability axiom we would have to require countable atlases- as indicated by the previous exercise). Another problem with such an approach (even more important than the previous one) is the fact that the various manifolds that we encounter are, before anything, topological spaces in a very natural way. Not making any reference to such natural topologies would be artificial and less intuitive.

However, here is an exercise that indicates how one could (but should not) proceed.

Exercise 2.24. Let M be a set and let $\mathscr{A}$ be a collection of bijections

$$\chi : U \to \chi(U) \subset \mathbb{R}^m$$

between subsets $U \subset M$ and opens $\chi(U) \subset \mathbb{R}^m$. We look for topologies on M with the property that each $\chi \in \mathscr{A}$ becomes a homeomorphism. We call them topologies compatible with $\mathscr{A}$.

(i) If the domains of all the $\chi \in \mathscr{A}$ cover M, show that M admits at most one topology compatible with $\mathscr{A}$.
(ii) Assume that, furthermore, for any $\chi, \chi' \in \mathscr{A}$, $\chi' \circ \chi^{-1}$ (defined on $\chi(U \cap U')$, where U is the domain of χ and U' is of χ') is a homeomorphism between opens in $\mathbb{R}^m$. Show that M admits a topology compatible with $\mathscr{A}$.

(Hint: try to define a topology basis).

2.1.4 Variations

There are several rather obvious variations on the notion of smooth manifold. For instance, keeping in mind that

$$\text{smooth} = \text{of class } \mathscr{C}^\infty,$$

one can consider a $\mathscr{C}^k$-version of the previous definitions for any $1 \leq k \leq \infty$. E.g., instead of talking about smooth compatibility of two charts, one talks about $\mathscr{C}^k$-compatibility, which means that the change of coordinates is a $\mathscr{C}^k$-diffeomorphism. One arises at the notion of **manifold of class** $\mathscr{C}^k$, or $\mathscr{C}^k$**-manifold**. For $k = \infty$ we recover smooth manifolds, while for $k = 0$ we recover topological manifolds.

Yet another possibility is to require "more than smoothness"- e.g analyticity. That gives rise to the notion of **analytic manifold**. Looking at manifolds of dimension $m = 2n$, hence modeled by $\mathbb{R}^m = \mathscr{C}^n$, one can also restrict even further- to maps that are holomorphic. That gives rise to the notion of **complex manifold**. Etc.

Another possible variation is to change the "model space" $\mathbb{R}^m$. The simplest and most standard replacement is by the upper-half planes

$$\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m \geq 0\}.$$

This gives rise to the notion of **smooth manifold with boundary**: what changes in the previous definitions is the fact that the charts are homeomorphisms into opens inside $\mathbb{H}^m$.

For instance, the closed interval $[0,1]$ and the closed disk $D^2 \subset \mathbb{R}^2$ (with their standard Euclidean topology) can be made into manifolds with boundary.

Exercise 2.25. Go back to Exercise 2.20 and do again the second part.

Exercise 2.26. If M is an m-dimensional manifold with boundary:

1. while this may be clear intuitively, give a precise definition of ”the boundary ∂M of M” (note that you cannot use the notion of boundary from Topology, as M is not part of a larger space).
2. prove that ∂M is a smooth $(m-1)$-dimensional manifold (without boundary).

Exercise 2.27. Return to products $M \times N$ of manifolds, as in Exercise 2.21. What if N is a manifold with boundary? And what if both M and N are manifolds with boundary?

2.2 Smooth maps

2.2.1 Smooth maps

Having introduced the main objects (manifolds), we now move to the maps between them. The idea will always be the same: use charts to move to Euclidean spaces, and use the standard notions there.

Definition 2.28. Let $f : M \to N$ be a map between two manifold M and N of dimensions m and n, respectively. Given charts (U, χ) and (U', χ') of M and N, respectively, the **representation of f with respect to χ and χ'** is

$$f_{\chi}^{\chi'} := \chi' \circ f \circ \chi^{-1}.$$

This map makes sense when applied to a point of type $\chi(p) \in \mathbb{R}^d$ with $p \in U$ with the property that $f(p) \in U'$, i.e. $p \in U$ and $p \in f^{-1}(U')$. Therefore, it is a map

$$f_{\chi}^{\chi'} : \mathrm{Domain}(f_{\chi}^{\chi'}) \to \mathbb{R}^n.$$

whose domain is the following open subset of $\mathbb{R}^m$:

$$\mathrm{Domain}(f_{\chi}^{\chi'}) = \chi(U \cap f^{-1}(U')) \subset \mathbb{R}^m.$$

Definition 2.29. Let M and N be two manifolds and

$$F : M \to N$$

a map between them. We say that F is **smooth** if its representation $F_{\chi}^{\chi'}$ with respect to any chart χ of M and χ' of N, is smooth (in the usual sense of Analysis).

Exercise 2.30. With the notation from the previous definitions, let $\mathscr{A}_M$ and $\mathscr{A}_N$ be two arbitrary (i.e. not necessarily maximal) atlases inducing the smooth structure on M and N, respectively. Show that, to check that F is smooth, it suffices to check that $F_{\chi}^{\chi'}$ is smooth for $\chi \in \mathscr{A}_M$ and $\chi' \in \mathscr{A}_N$.

Example 2.31. If $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ are embedded submanifolds endowed with their standard smooth structure (see Example 2.8) then the smoothness of a map $f : M \to M'$ in the sense discussed here is equivalent to the smoothness of f as a function between subsets of Euclidean spaces, i.e. in the sense of Section 1.2, Definition 1.34.

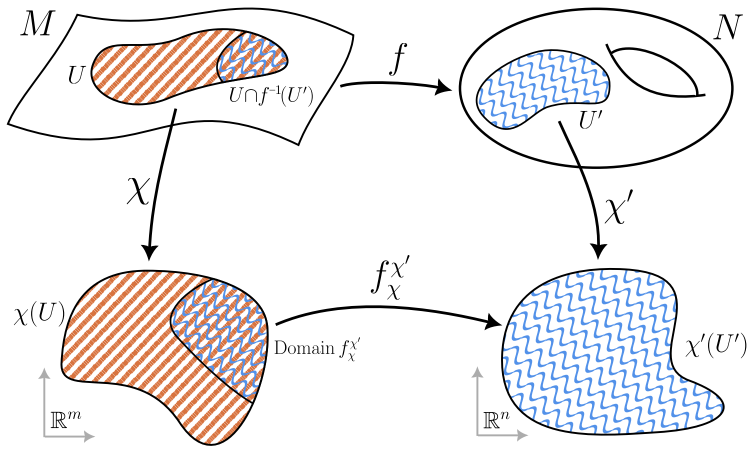


Fig. 2.4 A map $f: M \to N$ between an m- and n-dimensional manifold equipped with local charts χ and χ' defined on opens $U \subseteq M$ and $U' \subseteq N$, respectively, can locally be characterized by the representative $f_\chi^{\chi'}$: Domain $f_\chi^{\chi'} \to \mathbb{R}^n$. These maps commute wherever they can be evaluated sensibly, i.e. $\chi' \circ f = f_\chi^{\chi'} \circ \chi$ holds on $U \cap f^{-1}(U')$.

Proof. Let us call "A-smoothness" the notion from Section 1.2 of Chapter 1, and smoothness the one from this chapter. It is clear that the two coincide when M and N are open in their ambient Euclidean spaces.

Assume first that f is A-smooth. We will consider charts $\tilde{\chi}: \tilde{U} \to \Omega$ of $\mathbb{R}^{\tilde{m}}$ adapted to M (see Chapter 1, Proposition 1.37 from Section 1.2), and similarly $\tilde{\chi}': \tilde{U}' \to \Omega'$ adapted to N; they induce charts $\chi := \tilde{\chi}|_{U \cap M}$ for M, and similarly χ' for N. To prove that f is smooth around a point $p \in M$ it suffices to show that for any $\tilde{\chi}$ around p and $\tilde{\chi}'$ around $f(p)$, as above, $f_\chi^{\chi'}$ is smooth. Since f is A-smooth, we may assume that $f|_U = \tilde{f}|_U$ with $\tilde{f}: \tilde{U} \to \tilde{U}'$ is smooth; in turn, $\tilde{f}$ induces $F := \tilde{f}_{\tilde{\chi}}^{\tilde{\chi}'} : \Omega \to \Omega'$, whose restriction to $\Omega \cap (\mathbb{R}^m \times \{0\} \subset \mathbb{R}^{\tilde{m}}$ is precisely $f_\chi^{\chi'}$ (and takes values in $\Omega' \cap (\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{\tilde{n}})$. Hence we are in the situation of having a smooth map $F: \Omega \to \Omega'$ between opens in $\mathbb{R}^{\tilde{m}}$ and $\mathbb{R}^{\tilde{n}}$, taking $\Omega_0 = \Omega \cap (\mathbb{R}^m \times \{0\}$ to $\Omega_0' = \Omega' \cap (\mathbb{R}^n \times \{0\}$ and we want to show that $F|_{\Omega_0}: \Omega_0 \to \Omega_0'$ is smooth as a map between opens in $\mathbb{R}^m$ and $\mathbb{R}^n$; that should be clear.

Assume now that f is smooth. Tho check that it is A-smooth, we fix $p \in M$ and consider charts $\tilde{\chi}$ and $\tilde{\chi}'$ and proceed as above and with the same notation. This time but we do not have $\tilde{f}$, but having it is equivalent to having F extending $F_0 = f_\chi^{\chi'}$. Then we are in a similar general situation as above, when we have a smooth map $F_0: \Omega_0 \to \Omega_0'$ between opens in $\mathbb{R}^m$ and $\mathbb{R}^n$ and we want to extend it to a smooth map F between opens in $\mathbb{R}^{\tilde{m}}$ and $\mathbb{R}^{\tilde{n}}$, defined in a neighborhood of $\chi(p)$. Using the decomposition $\mathbb{R}^{\tilde{m}} = \mathbb{R}^m \times \mathbb{R}^{\tilde{m}-m}$ and similarly for $\mathbb{R}^{\tilde{n}}$, we just set $F(x,y) = (F_0(x), 0)$. 😀

Example 2.32. An extreme is when $M = I \subset \mathbb{R}$ is an open interval (which, according to our conventions, is always endowed with the standard smooth structure- see Example 2.7) and N is an arbitrary manifold. Then smooth maps

$$\gamma: I \to N$$

are called **(smooth) curves in** N. On I one can use the atlas consisting of the identity chart only, hence the smoothness of γ is checked using charts (U, χ) for M; the resulting representation of γ in the chart χ,

$$\gamma^\chi := \chi \circ \gamma : I_\chi \subset \mathbb{R}^m$$

(with domain $I_\chi = \gamma^{-1}(U) \subset I$) will then have to be smooth in the usual sense.

Example 2.33. In particular, when $N = \mathbb{R}^n$ (again with the standard smooth structure) , one can use the small atlas $\mathscr{A}_{\mathbb{R}^n}$ (Example 2.12). We see that, for a map

$$f : M \to \mathbb{R}^n,$$

the smoothness of f is checked by using charts (U, χ) for M and looking at the representation of f w.r.t. χ

$$f_\chi = f \circ \chi^{-1} : \chi(U) \to \mathbb{R}^n.$$

Exercise 2.34. Show that for any chart (U, χ) of a manifold M, $\chi : U \to \mathbb{R}^m$ is a smooth map (where U is endowed with a smooth structure as in Example 2.18.

2.2.2 Observables (or: $\mathscr{C}^\infty(M)$)

Of particular interest is the collection of real-valued smooth functions on M:

$$\mathscr{C}^\infty(M) := \{f : M \to \mathbb{R} : f - \text{smooth}\},$$

denoted $\mathscr{C}^\infty(M, \mathscr{A})$ in the previous sections (see (2.1.1). Its importance is perhaps not so predictable at this point; so let us mention that, in some sense, $\mathscr{C}^\infty(M)$ together with its algebraic structure (being able to take sums and products) encodes entirely the manifold M. On step in that direction is the claim made in (2.1.2), which we are able to prove now. Top that end, we need a lemma which is very important in various other places. Conceptually, this is what makes smoothness more flexible then analytic, holomorphic, or polynomial functions.

Lemma 2.35 (the fundamental property of smooth functions). *For any manifold M, for any $p \in M$ and any open neighborhood U of p, there exists $f \in \mathscr{C}^\infty(M)$ that is supported in U and such that $f(p) \neq 0$. Actually, one may arrange f so that $f = 1$ in a (small enough) neighborhood of p.*

Proof. Given our discussion from $\mathbb{R}^m$ (namely the proof of Theorem 1.27), there is very little that we have to do: we may assume that U is the domain of a coordinate chart $\chi : U \xrightarrow{\sim} \mathbb{R}^m$ (why?) sending p to $0 \in \mathbb{R}^m$, then choose a smooth function $f : \mathbb{R}^m \to [0,1]$ supported in the ball $B(0,1)$ and such that $f(0) \neq 0$ (as in the proof of Theorem 1.27), then move it to U via χ, i.e. consider $f \circ \chi : U \to [0,1]$; given the support property of f, it follows that if we extend f to M by declaring it to be zero outside U, we get a smooth functions $\tilde{f} : M \to [0,1]$ satisfying the desired property.

For the last part, we would need a function f on $\mathbb{R}^m$ as above, with the extra-property that $f = 1$ in a neighborhood of the origin. For that, one has to slightly improve the choice of g in the proof of Theorem 1.27: choose $g : \mathbb{R} \to \mathbb{R}$ is any smooth function that is 1 when $|t| < \frac{1}{3}$ and is 0 when $t \geq \frac{1}{2}$ and set $f(x) = g(||x||^2)$. 😀

Corollary 2.36. *Let M be a set. Assume that we have two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on M, a smooth structure $\mathscr{A}_1$ on the space $(M, \mathscr{T}_1)$ and, similarly, a smooth structure $\mathscr{A}_2$ on the space $(M, \mathscr{T}_2)$. Assume that the two smooth structures give rise to the same notions of smoothness of $\mathbb{R}$-valued functions, i.e., for arbitrary functions $f : M \to \mathbb{R}$, one has:*

$$f \text{ is smooth w.r.t. } \mathscr{A}_1 \Longleftrightarrow f \text{ is smooth w.r.t. } \mathscr{A}_2.$$

Then $\mathscr{T}_1 = \mathscr{T}_2$ and $\mathscr{A}_1 = \mathscr{A}_2$.

Proof. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two maximal smooth atlases in M such that $\mathscr{C}^\infty(M, \mathscr{A}_1) = \mathscr{C}^\infty(M, \mathscr{A}_2)$. First we show that the topologies $\mathscr{T}_i$ underlying $\mathscr{A}_i$, $i \in \{1,2\}$, must coincide. Let $U \in \mathscr{T}_1$. The previous lemma shows that, for any $p \in U$ we find $f_p \in \mathbb{C}^\infty(M, \mathscr{A}_1)$ such that

$$V_p := \{q \in M : f_p(q) \neq 0\} \subset U.$$

and q belongs to V_p. Since $\mathbb{C}^\infty(M, \mathscr{A}_1) = \mathbb{C}^\infty(M, \mathscr{A}_2)$, f_p is smooth also w.r.t. $\mathscr{A}_2$ and, therefore, $V_p \in \mathscr{T}_2$. Since $U = \cup_{p\in U} V_p$, it follows that $U \in \mathscr{T}_2$ as well. Hence $\mathscr{T}_1 \subset \mathscr{T}_2$ and, similarly, the other inclusion holds as well. Hence $\mathscr{T}_1 = \mathscr{T}_2$.

We now show that $\mathscr{A}_1 \subset \mathscr{A}_2$. Let $\chi \in \mathscr{A}_1$, $\chi : U \to \Omega \subset \mathbb{R}^m$, with $U \in \mathscr{T}_1 = \mathscr{T}_2$. Then each component χ_k of χ is smooth w.r.t. χ; however, we cannot say that it belongs to $\mathscr{C}^\infty(M, \mathscr{A}_1)$ because it is defined only on U. Therefore, for any point $p \in U$ we apply the previous lemma to find $\eta_p \in \mathscr{C}^\infty(M, \mathscr{A}_1)$ such that $\eta = 1$ in a neighborhood $V_p \subset U$ of p, and is 0 outside the closure in M of a larger neighborhood $\widetilde{V}_p \subset U$. It follows then that each $\eta_p \cdot \chi_k$, extended to the entire M by declaring to be 0 outside U, belongs to $\mathscr{C}^\infty(M, \mathscr{A}_1)$, hence also to $\mathscr{C}^\infty(M, \mathscr{A}_2)$. In particular, $\chi_k|_{V_p} = (\eta_p \cdot \chi_k)|_{V_p}$ is smooth w.r.t. $\mathscr{A}_2$. Since each $p \in U$ admits a neighborhood $V_p \subset U$ on which χ_k is smooth (w.r.t. $\mathscr{A}_2$ now), χ_k is smooth as a function on U, and so is χ (again, w.r.t. $\mathscr{A}_2$). Therefore, for any $\chi' \in \mathscr{A}_2$, $\chi_{\chi'} = (\chi \circ \chi')^{-1}$ will be smooth; the fact that $\mathscr{A}_2$ is maximal implies now that, indeed, $\chi \in \mathscr{A}_2$. This proves the inclusion $\mathscr{A}_1 \subset \mathscr{A}_2$, end then it suffices to invoke the maximality of $\mathscr{A}_2$.

And here is another interesting consequence:

Exercise 2.37. Show that, for any manifold M, $\mathscr{C}^\infty(M)$ is point separating, i.e. for any $p, q \in M$ distinct, there exists $f \in \mathscr{C}^\infty(M)$ such that $f(p) = 0$, $f(q) = 1$.

The importance of Lemma 2.35 is best understood from the perspective offered by the general discussion on subspaces of $\mathscr{C}(M)$, as discussed in the reminder on Topology, subsection 1.1.8. To apply Theorem 1.16 to $\mathscr{A} = \mathscr{C}^\infty(M)$: one has to check some simple algebraic properties which were taken care of in Exercise **??** above, and then there was the last, and more subtle, condition in that theorem; and that is precisely what the first part of the previous lemma is taking care of! Therefore we deduce:

Theorem 2.38. *On any manifold M, for any open cover $\mathscr{U}$ of M, there exists a smooth partition of unity on M subordinated to $\mathscr{U}$.*

Diving a bit more into the details of subsection 1.1.8 one see that, from the various properties discussed there, the one that is most subtle is "paracompactness". But Theorem 1.18 immediately implies that any manifold is automatically paracompact. One the other hand, when it comes to properties of subspaces $\mathscr{A} \subset C^\infty(M)$ discussed in subsection 1.1.8, the one that is more subtle is "normality"; from this perspective, the previous lemma checks the criteria for normality provided by Theorem 1.19. Therefore one also obtains:

Corollary 2.39. *On any manifold M, for any two disjoint closed subset $A, B \subset M$, there exists a smooth function $f : M \to [0,1]$ with the property that $f|_A = 0$, $f|_B = 1$.*

Remark 2.40 (For the curious students: a smooth Gelfand-Naimark ...). The "Gelfand-Naimark message" from Topology is that, for reasonable topological space X, the topological information on X can be completely recovered from the $\mathscr{C}(X)$ and its algebraic structure (the sums and products that make it into an algebra). As recalled in Remark 1.10, the way one "reconstructs" the space X from the algebra $\mathscr{C}(X)$ is by associating to any algebra A a topological space $X(A)$, called the spectrum of A, and defined as the set of all characters $\chi : A \to \mathbb{R}$ (see Remark 1.10).

Since the notion of character makes sense for any algebra, we can apply it to

$$A := \mathscr{C}^\infty(M),$$

the algebra of smooth functions on a manifold M. As before, any point $p \in M$ gives rise to a character

$$\chi_p : A \to \mathbb{R}, \quad \chi_p(f) := f(p);$$

and this gives rise to a map

$$\mathrm{GN} : M \to X(A), \quad p \mapsto \chi_p.$$

Note that the previous exercise says precisely that this map is injective! What about surjectivity? I.e., is is true that any character

$$\chi : \mathscr{C}^\infty(M) \to \mathbb{R}$$

is of type χ_p for some $p \in M$?

Theorem 2.41. *If M is a compact manifold then the map GN is 1-1.*

Proof. We are left with proving surjectivity. Hence we start with a character χ and we look for $p \in M$ such that $\chi = \chi_p$. First remark that, for the last equality to hold (given p), it suffices to require something apparently weaker condition

$$\text{if } f \in \mathscr{C}^\infty(M) \text{ is killed by } \chi \text{ (i.e. } \chi(f) = 0) \Longrightarrow f(p) = 0.$$

Indeed, for an arbitrary $f \in \mathscr{C}^\infty(M)$, since $f - \chi(f) \cdot 1$ is anyway killed by χ, the previous implication would imply $\chi(f) = f(p)$ for all f, i.e. $\chi = \chi_p$.

Therefore, it suffices to show that there exists $p \in M$ for which the previous implication holds. We proceed by contradiction. If no such p exists then we find, for each p, a function f_p with

$$\chi(f_p) = 0, \quad f_p(p) \neq 0.$$

We may actually assume that $f_p(p) > 0$ (otherwise replace f_p by its square). Since f_p is smooth (hence also continuous), for each p we find a neighborhood U_p of p s.t. $f_p > 0$ on the entire U_p. Then $\{U_p : p \in M\}$ forms an open cover of M and, using the compactness of M, we cab extract a finite subcover- corresponding to a finite number of points $p_1, \ldots, p_k$. Then the sum f of the corresponding functions f_{p_i} will have the property that

$$f \in \mathscr{C}^\infty(M), \quad \chi(f) = 0, \quad f > 0 \text{ everywhere on } M.$$

But then we can write the constant function $1 = f \cdot \frac{1}{f}$, with both f and $\frac{1}{f}$, so that

$$\chi(1) = \chi(f) \cdot \chi(\frac{1}{f}) = 0,$$

which provides us with a contradiction we were looking for. 😀

Exercise 2.42. Adapt the previous proof to show that, if M is not necessarily compact then, for any character χ on $\mathscr{C}^\infty(M)$, there exists $p \in M$ such that $\chi(f) = f(p)$ for all $f \in \mathscr{C}^\infty(M)$ that are compactly supported (i.e. vanish outside some compact).

Exercise 2.43. Given two manifolds M and N, any smooth $F : M \to N$ induces

$$F^* : \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(M),$$

which is a morphism of algebras (i.e. is linear, multiplicative and sends the unit to the unit). Prove that conversely, any morphism from the algebra $\mathscr{C}^\infty(N)$ to $\mathscr{C}^\infty(M)$ arises in this way.

Then deduce that, for any two manifolds M and N,

$$M \text{ and } N \text{ are diffeomorphic} \quad \Longleftrightarrow \quad \mathscr{C}^\infty(M) \text{ and } \mathscr{C}^\infty(N) \text{ are isomorphic as algebras}$$

2.2.3 Special maps: Diffeomorphisms, immersions, submersions

Like in the case of $\mathbb{R}^m$, there are certain types of smooth maps that deserve separate names. The first one describes the correct notion of ”isomorphisms” in Differential Geometry (analogous to linear isomorphisms in Linear Algebra, isomorphisms of groups in Group Theory, homeomorphisms in Topology).

Definition 2.44. A **diffeomorphism** between two manifolds M and N is a map $f : M \to N$ with the property that f is bijective and both f and f^{-1} are smooth.

Two manifolds M and N are said to be **diffeomorphic** if such a diffeomorphism exists.

A diffeomorphism allows one to pass from whatever differential geometric object/property on M to N and backwards; for that reason, two manifolds that are diffeomorphic are usually thought of as being ”(basically) the same as manifolds”.

Exercise 2.45. As a continuation of Exercise 2.45, show that for any chart (U, χ) of a manifold M, χ is a diffeomorphism between the domain U and its image $\Omega \subset \mathbb{R}^m$.

Exercise 2.46. In general, $\mathbb{R}^m$ has many smooth structures, though many of them (but not all!) are actually diffeomorphic. This exercise takes care of the simpler parts of these assertions. For a homeomorphism $\chi : \mathbb{R}^m \to \mathbb{R}^m$, consider

$$\mathscr{A}_\chi = \{\chi\},$$

(the atlas on $\mathbb{R}^m$ consisting of one single chart, namely χ itself).

Show that, for a general homeomorphism χ, $\mathscr{A}_\chi$ defines a smooth structure different than the standard one, but $\mathbb{R}^m$ endowed with the resulting smooth structure is diffeomorphic to the standard one.

Remark 2.47 (For the interested student: exotic smooth structures). As the previous exercise shows, the interesting question for $\mathbb{R}^m$, and as a matter of fact for any topological space M, is:

how many non-diffeomorphic smooth structures does a topological space M admit?

Even for $M = \mathbb{R}^m$ this is a highly non-trivial question, despite the fact that the answer is deceivingly simple: each of the spaces $\mathbb{R}^m$ with $m \neq 4$ admits only one such smooth structure, while $\mathbb{R}^4$ admits an infinite number of them (called "exotic" smooth structures on $\mathbb{R}^4$)!

Exercise 2.48. Show that if M is a smooth manifold with the property that its smooth structure can be induced by an atlas consisting of only one chart, then M is diffeomorphic to an open subset $\Omega \subset \mathbb{R}^m$ (endowed with the standard smooth structure- cf. Example 2.18).

With the notion of diffeomorphism at hand, and the fact that opens inside manifolds are automatically manifolds (as discussed in Example 2.18), one can now make sense of a smooth map $f : M \to N$ being a **local diffeomorphism** around a point $p \in M$: there are open neighborhoods U of p in M, and V of $f(p)$ in N, such that f restricts to a diffeomorphism $f|_U : U \to V$.

Exercise 2.49. First, convince yourself that the map

$$\exp : \mathbb{R} \to S^1, \quad t \mapsto e^{it} = (\cos t, \sin t)$$

is a local homeomorphism around any point. Put now a smooth structure on S^1 such that exp is a local diffeomorphism. Can you find more than one? Compare it with the smooth structure you found when doing Exercise 2.19.

The following shows how to reduce checking that a map is a diffeomorphism to the easier task of checking that it is a local diffeomorphism.

Lemma 2.50. *A smooth function $f : M \to N$ is a diffeomorphism if and only if it is a local diffeomorphism around any point, as well as a bijection.*

Proof. The direct implication is obvious. For the reverse one, the bijectivity assumption allows us to talk about the inverse $g : N \to M$ of f. The question is whether f and g are smooth. But smoothness is a local condition, that can be checked around points. Since f is a local diffeomorphism, it follows that both f as well as g are smooth around any point in their domain. Hence they are both smooth and, therefore, f is a diffeomorphism.

Next, we export the notion of immersion and submersion form Euclidean spaces to the general setting; the definition below is not the best one, but has the advantage that it can be given right away, before discussing tangent spaces (however, we will return to it later on- see Proposition 3.11).

Definition 2.51. Let $f : M \to N$ be a smooth map between two manifolds. We say that f is an **immersion/-submersion at** p if its local representations $f_\chi^{\chi'}$ is an immersion/submersion at $\chi(p)$ (in the usual sense from Analysis) for any chart χ of M around p and χ' of N around $f(p)$.

We say that f is an **immersion/submersion** if it is one at all $p \in M$.

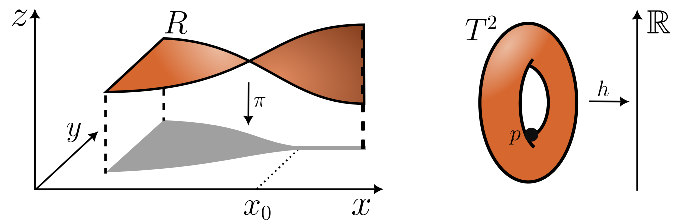


Fig. 2.5 Intuitively, an **immersion** at p is a map that locally around p does not crush the domain together, or in other words: wiggling p infinitesimally always corresponds to (first-order) movement of $f(p)$. Consider the map $\pi : R \to \mathbb{R}^3$ from the rotating ribbon R on the left that projects each point down to its shadow. It is an immersion at all points with $x < x_0$, but fails to be one as soon as the ribbon becomes exactly vertical for all $x \geq x_0$. A **submersion** at p is a map such that small wiggling away from p corresponds to movement of $f(p)$ in *all* directions in the codomain. Take the map $h : T^2 \to \mathbb{R}$ on the right that assigns to every point of the torus its height. At the marked point, it fails to be a submersion as any wiggling about p is horizontal, not changing the height to first order. Can you identify all the points where h is a submersion?

Exercise 2.52. Show that, in the previous definition, it is enough to check the required condition for (single!) one chart χ of M around p and one chart χ' of M' around $f(p)$.

Remark 2.53. As indicated above, once we will discuss that notion of tangent spaces, we will have another characterisation of the immersion and submersion conditions which does not make use of charts. That will provide another proof of, and actually extra-insight into, the previous exercise.

From the submersion and immersion theorems on Euclidean spaces, i.e. Theorem 1.32 and Theorem 1.33 from the previous chapter, we immediately deduce:

Theorem 2.54 (the submersion theorem). *If $f : M \to N$ is a smooth map between two manifolds which is a submersion at a point $p \in M$, then there exist charts χ of M around p and χ' of N around $f(p)$ such that $f_{\chi}^{\chi'} = \chi' \circ f \circ \chi^{-1}$ is given by*

$$f_{\chi}^{\chi'}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n).$$

Theorem 2.55 (the immersion theorem). *If $f : M \to N$ is a smooth map between two manifolds which is an immersion at a point $p \in M$, then there exist charts χ of M around p and χ' of N around $f(p)$ such that $f_{\chi}^{\chi'} = \chi' \circ f \circ \chi^{-1}$ is given by*

$$f_{\chi}^{\chi'}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

Moreover, the charts $\chi : U \to \Omega \subset \mathbb{R}^m$ and $\chi' : U' \to \Omega' \subset \mathbb{R}^n$ can be chosen so that

$$f(U) = \{q \in U' : \chi'_{m+1}(q) = \ldots = \chi'_n(q) = 0\}.$$

Corollary 2.56. *Consider a smooth function $f : M \to N$, $p \in M$. Then the following are equivalent:*

(1) f is a local diffeomorphism around p.
(2) f is both a submersion as well as an immersion at p.
(3) f is a submersion at p and $\dim(M) = \dim(N)$.
(4) f is an immersion at p and $\dim(M) = \dim(N)$.

Proof. Using coordinate charts one may assume $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. The equivalence between (2), (3) and (4) is immediate from linear algebra: for a linear map $A : V \to W$ between two finite dimensional vector spaces, looking at the conditions: A is injective, A is surjective, $\dim(V) = \dim(W)$, any two implies the third. Since these actually imply that A is an isomorphism, making use also of the inverse function theorem, we obtain the equivalence with (1) as well. 😀

Exercise 2.57. Consider S^1 endowed with the smooth structure previously discussed (e.g. in Exercise 2.19) and $\mathbb{R}$ endowed with it canonical smooth structure. Prove that there are no submersions or immersions $f : S^1 \to \mathbb{R}$.

Exercise 2.58. Let M and N be two smooth manifolds, consider their product (with the product smooth structure as in Exercise 2.21). Show that the two projection maps $\mathrm{pr}_M : M \times N \to M$, $\mathrm{pr}_N : M \times N \to N$ are submersions.

Exercise 2.59. Let M be a manifold and consider the product $M \times M$ (with the product smooth structure as in Exercise 2.21). Show that the diagonal inclusion $\Delta : M \longrightarrow M \times M$, $\Delta(x) = (x,x)$, is an immersion.

2.3 Immersions and submanifolds

2.3.1 Embedded submanifolds; the regular value theorem

The characterization of embedded submanifolds of Euclidean spaces in terms of adapted charts (Proposition 1.37 in subsection 1.2.6) allows us to proceed more generally and talk about embedded submanifolds M of an arbitrary manifold N. Indeed, the notion of adapted chart that appears in Proposition 1.37 has an obvious generalization to this context:

Definition 2.60. Given an n-dimensional manifold N and a subset $M \subset N$, we say that M is an embedded m-dimensional submanifold of N if for any $p \in M$, there exists a chart

$$\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^n$$

for N around p with the property that

$$\widetilde{U} \cap M = \{p \in \widetilde{U} : \widetilde{\chi}_{m+1}(p) = \ldots = \widetilde{\chi}_n(p) = 0\} \tag{2.3.1}$$

or, equivalently,

$$\widetilde{\chi}(\widetilde{U} \cap M) = \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\}).$$

A chart $(\widetilde{U}, \widetilde{\chi})$ of N satisfying this equality is called **chart of N adapted to M**.

For any such chart one can talk about **the restriction of the adapted chart to M** and denoted

$$\widetilde{\chi}|_M : U \to \Omega.$$

Its domain is $U := \widetilde{U} \cap M$, its codomain is

$$\Omega := \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\}),$$

interpreted as an (open) subset of $\mathbb{R}^m$. With these we obtain an **induced smooth structure** on M.

Exercise 2.61. Check that, indeed, the collection of all charts of type $\widetilde{\chi}|_M$ obtained from charts $\widetilde{\chi}$ of N that are adapted to M, define a smooth structure on M.

Exercise 2.62. Assume that $M \subset N$ is an embedded submanifold and let P be another manifold. Show that:

1. a function $F : P \to M$ is smooth if and only if it is smooth as an N-valued function.

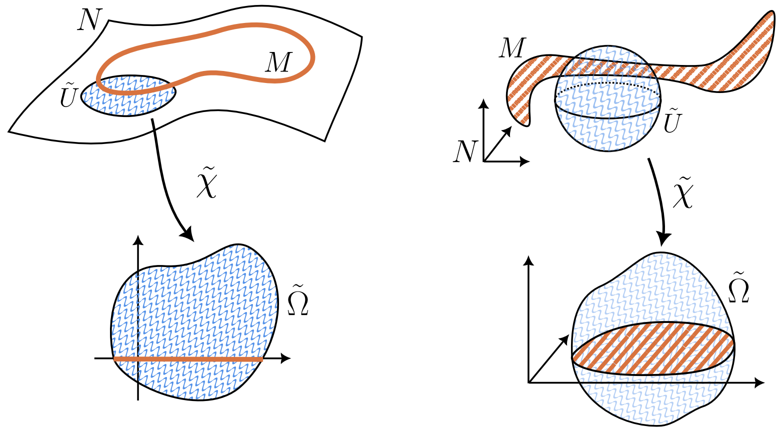


Fig. 2.6 Two examples of charts $\tilde{\chi} : \tilde{U} \to \tilde{\Omega}$ of an ambient manifold N adapted to an embedded submanifold M. The dimensions of the manifolds M, N are $(1,2)$ on the left hand side, and $(2,3)$ on the right hand side. Can you draw a picture for dimensions $(1,3)$?

2. if a function $F : M \to P$ admits a smooth extension to N, then F is smooth.
3. if M is closed in N, then the previous statement holds with "if and only if".

(Hint for the last point: use partitions of unity for N and don't forget that $N \setminus M$ is already open in N).

The notion of embedded submanifold allows us to introduce the smooth version of the notion of topological embedding. Recall that a map $F : M \to N$ between two topological spaces is a topological embedding if it is injective and, as a map from M to $F(M)$ (where the second space is now endowed with the topology induced from N), is a homeomorphism. The difference between the topological and the smooth case is that, while subspaces of a topological space inherit a natural induced topology, for smooth structures we need to restrict to embedded submanifolds.

Definition 2.63. A smooth map $F : M \to N$ between two manifolds M and N is called a **smooth embedding** if:

1. $F(M)$ is an embedded submanifold of N.
2. as a map from M to $F(M)$, F is a diffeomorphism.

Here is an useful criterion for smooth embeddings.

Theorem 2.64. *A map $F : M \to N$ between two manifolds is a smooth embedding if and only if it is both an immersion as well as a topological embedding.*

Proof. The direct implication should be clear, so we concentrate on the converse. We first show that $f(M)$ is an embedded submanifold. We check the required condition at an arbitrary point $q = F(p) \in N$, with $p \in M$. For that use the immersion theorem (Theorem 2.55) around p; we consider the resulting charts $\chi : U \to \Omega \subset \mathbb{R}^m$ for M around p and $\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^n$ for N around q; in particular,

$$F(U) = \{q \in \widetilde{U} : \widetilde{\chi}_{m+1}(q) = \ldots = \widetilde{\chi}_n(q) = 0\}.$$

Since F is a topological embedding, $F(U)$ will be open in $F(M)$, i.e. of type

$$F(U) = F(M) \cap W$$

for some open neighborhood W of $q \in N$. It should be clear now that $\widehat{U} := \widetilde{U} \cap W$ and $\widehat{\chi} := \widetilde{\chi}|_{\widehat{U}}$ defines a chart for N adapted to M.

We still have to show that, as a map $F : M \to F(M)$, F is a diffeomorphism (where $F(M)$ is with the smooth structure induced from N). It should be clear that this map continues to be an immersion. Since M and $F(M)$ have the same dimension, it follows from Corollary 2.56 that $F : M \to F(M)$ is a diffeomorphism.

And here is a very useful consequence:

Corollary 2.65. *Let $F : M \to N$ be a smooth map between two manifolds M and N, with the domain M being compact. Then F is a smooth embedding if and only if it is an injective immersion.*

Proof. Use one of the main properties of compact spaces: injective maps from compacts to Hausdorff spaces are automatically topological embeddings!

The regular value theorem presented as Theorem 2.94 can now be generalized from Euclidean spaces to more general manifolds. The setting is as follows: we are looking at a smooth map

$$F : M \to N$$

and we are interested in its fiber above a point $q \in N$:

$$F^{-1}(q), \quad (\text{with } q \in N).$$

The condition that we need here is that q is a **regular value of** F in the sense that F is a submersion at all points $p \in F^{-1}(q)$.

Theorem 2.66 (the regular value theorem). *If $q \in N$ is a regular value of a smooth map*

$$F : M \to N,$$

then the fiber above q, $F^{-1}(q)$, is an embedded submanifold of M of dimension

$$dim(F^{-1}(q)) = dim(M) - dim(N).$$

Proof. In principle, this is just another face of the submersion theorem. Let $d = m - n$, where m and n are the dimension of M and N, respectively. We check the submanifold condition around an arbitrary point $p \in M_0 := f^{-1}(q)$. For that we apply the submersion theorem to f near p to find charts $\chi : U \to \Omega$ and $\chi' : U' \to \Omega'$ of M around p and of N around $F(p)$, respectively, such that $F(U) \subset U'$ and

$$F_{\chi}^{\chi'}(x) = (x_1, \ldots, x_n) \quad \text{for all } x \in \Omega.$$

After changing χ and χ' by a translation, we may assume that $\chi(p) = 0$ and $\chi'(q) = 0$. We claim that, up to a reindexing of the coordinates, χ is a chart of M adapted to M_0. Indeed, we have

$$\chi(U \cap M_0) = \{\chi(p') : p' \in U, F(p') = q\} = \{x \in \Omega : F_{\chi}^{\chi'}(x) = 0\},$$

or, using the form of $F_{\chi}^{\chi'}$,

$$\chi(U \cap M_0) = \{x = (x_1, \ldots, x_m) \in \Omega : x_1 = \ldots = x_n = 0\},$$

proving that M_0 is an $m-n$-dimensional embedded submanifold of M.

Finally, with smooth partitions of unity at hand one can use the same arguments as in the topological case and obtain the smooth version of Theorem 1.14 from Section 1.1 of Chapter 1:

Theorem 2.67. *Any (smooth) compact manifold can be smoothly embedded into some Euclidean space.*

Proof. We use the same map as in the proof of Theorem 1.14 from Section 1.1 of Chapter 1, just that we now make use of smooth partitions of unity (which exist by Theorem 2.38). It suffices to show that the resulting topological embedding

$$i = (\eta_1, \ldots, \eta_k, \tilde{\chi}_1, \ldots, \tilde{\chi}_k) : M \to \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{k \text{ times}} \times \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{k \text{ times}} = \mathbb{R}^{k(d+1)}.$$

is also an immersion. We check this at an arbitrary point $p \in M$. Since $\sum_i \eta_i = 1$, we find an i such that $\eta_i(p) \neq 0$. We may assume that $i = 1$. In particular, p must be in U_1, and then we can choose the chart χ_1 around p and look at the representation i_{χ_1} with respect to this chart; we will check that it is an immersion at $x := \chi_1(p)$. Hence assume that the differential of i_{χ_1} at x kills some vector $v \in \mathbb{R}^d$ (and we want to prove that $v = 0$). Then the differential of all the components of i_{χ_1} (taken at x) must kill v. But looking at those components, we remark that:

- the first $\mathbb{R}$-component is $\eta := \eta_1 \circ \chi_1^{-1} : U_1 \to \mathbb{R}$.
- the first $\mathbb{R}^d$ component is the linear map $f : U_1 \to \mathbb{R}^d, f(u) = \eta(u) \cdot u$.

Hence we must have in particular:

$$(d\eta)_x(v) = 0, \quad (df)_x(v) = 0.$$

From the formula of f we see that $(df)_x(v) = (d\eta)_x(v) \cdot x + \eta(x) \cdot v$; hence, using that previous equations we find that $\eta(x) \cdot v = 0$. But $\eta(x) = \eta_1(p)$ was assumed to be non-zero, hence $v = 0$ as desired. 😀

Exercise 2.68. Show that the map from Exercise 2.105 is an embedding of $\mathbb{P}^2$ in $\mathbb{R}^4$.

Recall that, while $S^1 \times S^1$ naturally embeds in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$, the interpretation of $S^1 \times S^1$ as a torus provides an even better embedding: inside $\mathbb{R}^3$. In the following you are asked to show that this can be propagated to higher dimensional tori. Here we use products of manifolds as discussed in Exercise 2.21

Exercise 2.69. Show that the n-dimensional torus,

$$T^n := \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$$

(sitting canonically inside $\mathbb{R}^{2n}$) can actually be embedded inside $\mathbb{R}^{n+1}$.

Exercise 2.70. With the same notations as in Exercise 2.110, show that each fiber of the Hopf map $h : S^3 \to S^2$ is an embedded submanifold of S^3 which is diffeomorphic to a circle.

Exercise 2.71. Consider the height function $f : T \to \mathbb{R}$ on the torus as indicated in Fig 2.7. At which points does f fail to be a submersion?

Exercise 2.72. Let M be a smooth m-dimensional manifold and let

$$f : M \longrightarrow \mathbb{R}$$

be a smooth function. For $\lambda \in \mathbb{R}$ define

$$M_\lambda = f^{-1}(\lambda), \quad M_{\leq \lambda} := \{p \in M : f(p) \leq \lambda\}.$$

Assume that λ is a regular value of f (so that, by the regular value theorem, M_λ is an $(m-1)$-dimensional manifold. Prove that $M_{\leq \lambda}$ is an m-dimensional manifold with boundary (see subsection **??**), with

$$\partial M_{\leq \lambda} = M_\lambda.$$

How many manifolds do you obtain in this way for the height function from the previous exercise?

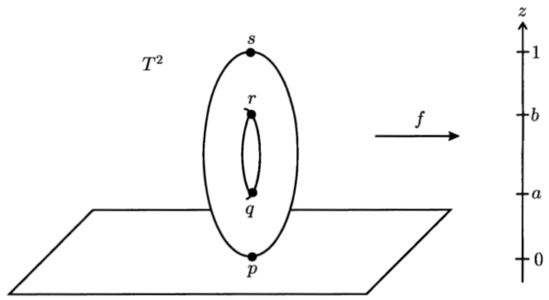


Fig. 2.7

Exercise 2.73 (part of the 2019/2020 exam). Consider

$$f : S^3 \to \mathbb{R}, \quad f(x,y,z,t) = x^2 + y^2 - z^2 - t^2.$$

(a) show that the zero-set $M_0 := f^{-1}(0)$ is an embedded submanifold of S^3.
(b) show that M_0 is diffeomorphic to the 2-torus.
(c) find all the points at which f is a submersion.

Exercise 2.74 (part of an exam from 2018; and you may want to skip (b) for now). Consider

$$M_4 := \{(x,y,z) \in \mathbb{R}^3 : x^4 + y^4 + z^4 = 1\} \subset \mathbb{R}^3,$$

and the map from M_4 to the 2-sphere S^2 given by

$$f : M_4 \to S^2, \quad f(x,y,z) = (x^2, y^2, z^2)$$

(a) Show that M_4 is a submanifold of $\mathbb{R}^3$ and f is a smooth map.
(b) Compute the tangent space of M_4 at the point $p = (\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, 0)$; more precisely, show that it is spanned by

$$\left(\frac{\partial}{\partial x}\right)_p - \left(\frac{\partial}{\partial y}\right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial z}\right)_p \in T_pM_4.$$

Similarly at the point $q = (\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$.
(c) Show that f is not an immersion at p, but it is a local diffeomorphism around q.
(d) Show that M_4 is not diffeomorphic to

$$M_3 := \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 1\} \subset \mathbb{R}^3.$$

but it is diffeomorphic to S^2.

2.3.2 General (immersed) submanifolds

In Topology (or, if you do not like that generality, just take $X = \mathbb{R}^n$) the leading principle for talking about "a subspace A of a space X" was to make sure that the inclusion $i : A \to X$ was continuous, and we chose "the best possible topology" on A doing that- and that gave rise to the induced (subspace) topology on any subset $A \subset X$. Looking for the analogous notion of "subspace" in the smooth context (i.e. a notion of "submanifold"), the first remark is that "inclusions" in the smooth context are expected to be immersions. Furthermore, starting with a subset M of a manifold N, there are several possible ways to proceed:

(1) try to make the space M (endowed with the topology induced from N) into a manifold such that the inclusion $i : M \hookrightarrow N$ is an immersion.
(2) forget about the induced topology on M and try to make the set M into a manifold such that the inclusion $i : M \hookrightarrow N$ is an immersion.
(yes, M will have a topology underlying the smooth structure, but it does not have to be the induced one).
(3) in the previous point choose "the best possible" manifold structure on M.

In (1), insisting that M has the endowed topology means that $i : M \hookrightarrow N$ is also an embedding (next to being an immersion) means that we are in the setting of Theorem 2.64. In other words, the previous section implemented precisely (1); gave rise to the notion of embedded submanifold.

The second possibility (i.e. item (2) above) gives rise to the notion of immersed submanifold.

Definition 2.75. Given a manifold N, an **immersed submanifold** of N is a subset $M \subset N$ together with a structure of smooth manifold on M, such that the inclusion $i : M \to N$ is an immersion.

We emphasize: an immersed submanifold is not just the subset $M \subset N$, but also the auxiliary data of a smooth structure on M (and, as the next exercise shows, given M, there may be several different smooth structures on M that make it into an immersed submanifold). Moreover, the required smooth structure on M induces, in particular, a topology on M; but we insist: this topology does not have to coincide with the one induced from N!

Exercise 2.76. Consider the figure eight in the plane $M = \mathbb{R}^2$. Show that it is not an embedded submanifold of $\mathbb{R}^2$, but it has at least two different smooth structures that make it into an immersed submanifold of $\mathbb{R}^2$.

Immersed submanifolds may look a bit strange/pathological, but they do arise naturally and one does have to deal with them. In general, any injective immersion

$$f : M \to N$$

gives rise to such an immersed submanifold: $f(M)$ together with the smooth structure obtained by transporting the smooth structure from M via the bijection $f : M \to f(M)$ (i.e. the charts of $f(M)$ are those of type $\chi \circ f^{-1}$ with χ a chart of M). And often immersed submanifolds do arise naturally in this way.

Example 2.77. For instance, the two immersed submanifold structures on the figure eight from the preceding exercise arise from two different injective immersions $f_1, f_2 : \mathbb{R} \to \mathbb{R}^2$ (with the same image: the figure eight), as indicated in the picture. Make pictures!

So: what happens for an embedded submanifold $M \subset N$? Endowed with "the smooth structure induced from N" (notion that only makes sense when M is embedded submanifold!) it clearly becomes an immersed submanifold. Can it be still given another smooth structure (as in the last example)? The answer is: no! Indeed, one has the following result which is a simplified version of Proposition 2.80 below.

Proposition 2.78. *If M is an embedded submanifold of the manifold N, then the set M admits precisely one smooth structure such that the inclusion into M becomes an immersion.*

To see the difference between ”immersed” and ”embedded” one just has to stare at the meaning of ”immersions”. If M (... together with a smooth structure on it) is an immersed submanifold of N, the immersion theorem tells us that there exist charts (U,χ) of N around arbitrary points $p \in M$ such that $U_M := M \cap \{q \in U : \chi_{n+1}(q) = \ldots = \chi_m(q) = 0\}$ is open in M and $\chi_M = \chi|_{U_M}$ is a chart for M. However, when saying that ”U_M is open in M”, we are making use of the topology on M induced by the smooth structure we considered on M, and not with respect to the topology induced from N (look again at the previous exercise!). This is the difference with the notion of ”embedded submanifold”.

Moving now to item (3), the best possible scenario would be when the subset $M \subset N$ has the following property: M admits a unique smooth structure that makes it into an immersed submanifold of N. When this happens we say that M has **the unique smooth structure property**. In this case there is no ambiguity what smooth structure we put on M, M becomes an immersed submanifold, and we are looking at those immersed submanifolds that are encoded just in the subset $M \subset N$ and no extra-data prescribed beforehand. There are two remarks here:

- but, again, even in this case, the underlying topological structure need not be the induced topology from N.
- on the other hand, embedded submanifolds do have this property (this is implied by the next result).

Of course, there are other examples of submanifolds with unique smooth structure besides the embedded ones: e.g. the ”dutch figure eight” (exercise); for one more see Example 2.81 below.

However, it turns out that there is a smaller and more interesting class of submanifolds with unique smooth structures (and still includes the embedded submanifolds, as well as most of the other interesting examples):

Definition 2.79. An immersed submanifold M of a manifold N is called **an initial submanifold** if the following condition holds: *for any other manifold P and any map $f : P \to M$, f is smooth if and only if it is smooth as a map with values in N*. In other words,

$$f \text{ is smooth} \Longleftrightarrow i \circ f \text{ is smoooth},$$

where $i : M \hookrightarrow N$ is the inclusion; here one may want to think on the diagram:

$$\begin{array}{ccc} P & \xrightarrow{f} & M \\ & \searrow^{i \circ f} & \downarrow i \\ & & N \end{array} .$$

What happens is that, while this condition is easier to check than the uniques smooth structure property, one has:

Proposition 2.80. *For a subset M of a manifold N one has the following:*

$$\textit{embedded submanifold} \Longrightarrow \textit{initial submanifold} \Longrightarrow \textit{the unique smooth structure property}.$$

Proof. Consider the inclusion $i : M \hookrightarrow N$. For the first implication the main point is to show that if M is embedded and $f : P \to M$ has the property that $i \circ f : P \to N$ is smooth, then f is smooth. To check that f is smooth, we we use arbitrary charts χ of P and charts χ' of N adapted to M; we also use $\chi'_M = \chi'|_M$. Note that

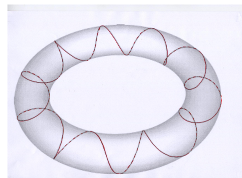
$$(i \circ f)_{\chi}^{\chi'} = f_{\chi}^{\chi'_M} : \Omega \to \mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n,$$

for some open $\Omega \subset \mathbb{R}^p$. Hence it suffices to remark that if a function $g : \Omega \to \mathbb{R}^m$ is smooth as a map with values in the larger space $\mathbb{R}^n$, then it is smooth.

For the second part assume that M is initial. In particular, it comes with a smooth structure defined by some maximal atlas $\mathscr{A}$, making the inclusion into N an immersion. To prove the uniqueness property, we assume that we have a second smooth structure with the same property, with associated maximal atlas denoted $\mathscr{A}'$. Applying the initial condition to $P = (M, \mathscr{A}')$ with f being the inclusion into N, we find that the identity map $\mathrm{id} : (M, \mathscr{A}') \to (M, \mathscr{A})$ is smooth. Applying the definition of smoothness, we see that any chart in $\mathscr{A}'$ must be compatible with any chart in $\mathscr{A}$. Therefore $\mathscr{A}' = \mathscr{A}$. 😀

Example 2.81. When N is the 2-torus and one draws a curve in N winding around, there are two interesting cases:

- the curve closes up after a few turnings. That give a subset $M_0 \subset N$.
- the curve keeps on winding around infinitely, densely inside the torus. That gives another $M_1 \subset N$.



The picture shows M_0. Draw a picture for M_1! Note that the two types of curves are not even that far apart: any one that closes up, if perturbed a little, will become of the second type. However, M_0 and M_1 are quite different:

- M_0 is embedded in the torus.
- M_1 is only immersed.

For the last point, there is a map which should be clear on the picture $f : \mathbb{R} \to N$ with image precisely M_1 and which is an immersion. Therefore making M_1 into an immersed submanifold. However, we are in the better case: M_1 is an initial submanifold (... exercise!).

Exercise 2.82. In the previous example show that M_0 is diffeomorphic to S^1 while M_1 is diffeomorphic to $\mathbb{R}$.

Using arguments similar to those from Proposition 2.80 you can try the following:

Exercise 2.83. Let M be a subset of the manifold N. Then:

(i) For any topology on M, the resulting space M can be given at most one smooth structure that makes it into an immersed submanifold of N.
(ii) If we endow M with the induced topology, then the space M can be given a smooth structure that makes it into an immersed submanifold of N if and only if M is an embedded submanifold of N; moreover, in this case the smooth structure on M is unique.

2.4 Submersions and quotients

2.4.1 Quotient smooth structures: uniqueness

Another interesting corollary of the submersion theorem is that, for submersions $f : M \to N$, smooth maps on N can be detected by looking up on M (see below for the precise statement). That is interesting because, in practice, whenever we have a set N that we would like to make it into a a manifold in a "satisfactory way", very often N is closely related to an actual manifold M, through a surjective map $f : M \to N$. In such situations, the "satisfactory way" often means: such that f is a submersion.

Corollary 2.84. *If $\pi : M \to N$ is a surjective submersion then, for functions $f : N \to \mathbb{R}$, one has:*

$$f \in C^\infty(N) \Longleftrightarrow f \circ \pi \in C^\infty(M).$$

In particular, if $\pi : M \to N$ is a surjective map from a manifold M to a set N, there is at most one way to make N into a manifold such that $\pi : M \to N$ is a submersion.

Proof. The direct implication is clear, hence we concentrate on the reverse one. We will make use of the fact that smoothness of a map is a local property: for our f, it suffices to show that for any point $y \in N$, there exists a neighborhood V_y of y such that $f|_{V_y}$ is smooth. Fixing $y \in N$, we choose the coordinate charts around x and y that follow from the submersion theorem, and we set V_y to be the domain of the chart around y. Moving by those charts to opens inside Euclidean spaces, we may assume that π is the projection onto the first n coordinates-situation in which the fact that $f \circ \pi$-smooth implies that f is smooth is clear. The (pedantic) details are left as an exercise. 😫

For the last part, one just invokes Corollary 2.36. 😀

Exercise 2.85. Show that having a submersion $\pi : M \to N$ also forces the topology on N to coincide with the so-called **quotient topology** induced by π on N, defined by:

$$V - \text{open in } N \iff \pi^{-1}(V) - \text{open in } M$$

(which arises when looking for topologies on N making π continuous, and choosing the largest one among those; this was discussed also in the introductory course in Topology).

2.4.2 The case of group actions

Next, we look at actions of groups on manifolds. You have probably seen the notion of group actions (and quotients) in the Group Theory and Introduction to Topology courses, hence it is not a surprise that they will show up in differential geometry as soon as we start looking at the first examples (next chapter). So, let us discuss group actions in the realm of differential geometry, where sets or topological spaces are replaced by manifolds.

Let M be a manifold. We denote by $\mathrm{Diff}(M)$ the set of all diffeomorphisms from M to M. Together with composition of maps, this is a group. Let Γ be another group, whose operation is denoted multiplicatively. An **action** of Γ on a manifold M is a group homomorphism

$$\phi : \Gamma \to \mathrm{Diff}(M),\ \gamma \mapsto \phi_\gamma.$$

Hence, for each $\gamma \in \Gamma$, one has a diffeomorphism ϕ_γ of M ("the action of γ on M"), so that

$$\phi_{\gamma\gamma'} = \phi_\gamma \circ \phi_{\gamma'} \ \ \forall\, \gamma, \gamma' \in M.$$

Sometimes $\phi_\gamma(x)$ is also denoted $\gamma(x)$, or simply $\gamma \cdot x$. In other words, one encodes/think of an action as a map

$$\boxed{\Gamma \times M \to M,\ (\gamma, x) \to \gamma \cdot x.}$$

Exercise 2.86. Show that the fact that each ϕ_γ is a diffeomorphism is equivalent to the fact that the map $(\gamma, x) \to \gamma \cdot x$ is smooth, where $\Gamma \times M \to M$ is endowed with the smooth structure defined by charts of type $\{\gamma\} \times U \to \mathbb{R}^m$, $(\gamma, x) \mapsto \chi(x)$, one for each $\gamma \in \Gamma$ and each chart (U, χ) of M.

Given an action of Γ on M, the **Γ-orbit through** x is defined as $(\Gamma \cdot x) := \{\gamma \cdot x : \gamma \in \Gamma\}$, and the collection of all orbits forms the so-called **orbit space**

$$M/\Gamma := \{(\Gamma \cdot x) : x \in M\}.$$

Hence a single point of M/Γ is an entire orbit. Such points arise from points in M via the **canonical projection**

$$\pi_{\mathrm{can}} : M \to M/\Gamma, \quad x \mapsto (\Gamma \cdot x),$$

just that two different points of M may give the same point in the orbit space; more precisely, one has:

$$(\Gamma \cdot x) = (\Gamma \cdot y) \iff \exists\, \gamma \in \Gamma \text{ s.t. } y = \gamma \cdot x. \tag{2.4.1}$$

In what follows, an action of a group Γ on M is said to be **free** if, for $\gamma \in \Gamma$ and $x \in M$, the equality $\gamma \cdot x = x$ can occur only when $\gamma = e$ is the identity element of Γ.

Theorem 2.87. *If Γ is a finite group acting freely on a manifold M, then the set M/Γ can be made into a smooth manifold in precisely one way such that the canonical projection $\pi_{can} : M \to M/\Gamma$ is a local diffeomorphism.*

Remark 2.88. M/Γ always carries a "natural topology", without any assumption on the group or on the action. It arises as "the best" topology on M/Γ that turns π_{can} into a continuous map, and it is precisely the quotient topology induced by π_{can} mentioned in Exercise 2.85.

In the smooth context however, M/Γ may fail to carry a similar "best smooth structure" (meaning, of course, that we require π to be a submersion). Corollary 2.84 shows that, if it exists, then it is unique, but the problem is with the existence. This problem arises already in simple situations such as $\Gamma = \mathbb{Z}_2$ (... finite!) acting on $\mathbb{R}$ by $\widehat{k} \cdot r = (-1)^k r$; we leave this as an exercise! 😰

Remark 2.89. Some of the discussion here may remind you of the introductory course on topology, where equivalence relations are discussed. If that is the case, you will notice that one can look at the entire discussion a bit differently: the very last condition in (2.4.1) defines an equivalence relation R_Γ between the points of M, the Γ-orbits are precisely R_Γ-equivalence classes and the orbit space M/Γ is precisely the quotient w.r.t. this equivalence relation. On the other hand, finding conditions on more general equivalence relations R on manifolds M to ensure that the quotient M/R can be made into a manifold into a satisfactory fashion (like the last theorem does for group actions) is more difficult.

Proof. It is interesting to point out the following consequences to the condition that Γ is finite s that the action is free, and that Γ is finite, respectively:

1. if the orbits through $x, y \in M$ do not coincide, then there exist neighborhoods U of x and V of y such that $U \cap \gamma \cdot V = \emptyset$ for all $\gamma \in \Gamma$.
2. if the action is also free then, around any $x \in M$ there exists an open neighborhood W which is Γ-small in the sense that $W \cap \gamma \cdot W = \emptyset$ for all $\gamma \in \Gamma$ distinct from the unit element e.

The first item is proven/used in Topology, when proving the quotients modulo finite groups are Hausdorff; briefly, it goes as follow: the orbits must be disjoint, hence $x \neq \gamma \cdot y$ for any γ; using that M is Hausdorff one finds neighborhoods U_γ of x and V_γ of y such that $U_\gamma \cap \gamma \cdot V_\gamma = \emptyset$; finally, one sets $U = \cap_\gamma U_\gamma$, and similarly for V, which, since Γ is finite, is a finite intersection of opens hence open. This proves the existence of the desired U and V. Notice that, while the condition on x and y is precisely that they induce two distinct points in M/Γ, the resulting U and V give rise to two disjoint subsets $\pi(U)$, $\pi(V)$. Since when endowing M/Γ with the quotient topology, $\pi_{\text{can}} : M \to M/\Gamma$ becomes an open map- i.e. sends opens to opens (why?), $\pi(U)$ and $\pi(V)$ will be disjoint opens and, therefore, M/Γ is Hausdorff.

For the second item we proceed similarly, but using that $x \neq \gamma \cdot x$ for all $\gamma \neq e$ (freeness). For each such γ we find neighborhoods W_γ and W'_γ of x such that $W_\gamma \cap \gamma \cdot W'_\gamma = \emptyset$ and consider

$$W = \left(\cap_{\gamma \neq e} W_\gamma\right) \cap \left(\cap_{\gamma \neq e} W'_\gamma\right).$$

It contains x, it is open (as a finite intersection of opens) and is Γ-small. Notice that being Γ-small is equivalent to the fact that $\pi|_W : W \to \pi(W)$ being a bijection and then, since π is continuous and open, to $\pi|_W$ being a local homeomorphism.

The second item can be used to exhibit the desired smooth structure on M/Γ. To that end, we consider charts (W, χ) on M that are Γ-small in the sense that W is. Item 2. implies that, around any point, one does find a Γ-small chart. Combining χ with the inverse of $\pi|_W$, one obtains a homeomorphism

$$\chi_0 : \pi(W) \to \chi(W) \subset \mathbb{R}^m.$$

We claim that these fit into a smooth atlas for M/Γ. So, let us fix two Γ-small charts on M, and prove that the induced charts (W, χ) and (W', χ') on M/Γ are smoothly compatible. We have to investigate the smoothness of

$$\chi_0' \circ \chi_0^{-1} = \chi' \circ (\pi|_{W'})^{-1} \circ (\pi|_W) \circ \chi,$$

on the domains where it is defined. Since χ and χ' are already smooth, we have to investigate the smoothness of

$$\tau_{W,W'} := (\pi|_{W'})^{-1} \circ (\pi|_W) : D \to D'$$

where $D \subset W$ and $D' \subset W'$ are the preimages of the overlap $\pi(W) \cap \pi(W')$ by $\pi|_W$ and $\pi|_{W'}$, respectively:

$$D = \{x \in W : \pi(x) \in \pi(W')\} \subset W, \quad D' = \{x \in W' : \pi(x) \in \pi(W)\} \subset W'.$$

We now check the smoothness of $\tau = \tau_{W,W'}$ around an arbitrary point $x_0 \in D$. Let $y_0 = \tau(x_0)$. Since x_0 and y_0 are mapped by π to the same point in M/Γ, we find γ_0 such that $y_0 = \gamma_0 \cdot x_0$. One then finds a neighborhood $D_0 \subset D$ such that $D_0' := \gamma_0 \cdot D_0$ sits inside D' and the, on D_0, τ must be given by the multiplication by γ_0 (hence smooth!).

In conclusion, we have a smooth structure on M/Γ which, by construction, makes the projection into a local diffeomorphism. Moreover, M/Γ is smooth. We leave it to the reader to prove that M/Γ is also second countable.

Remark 2.90. The proof above only used the properties mentioned in items 1. and 2. (well, 2. does imply that the action is free, but 1. and 2. may hold also when Γ is infinite):

1. if $(\Gamma \cdot x) \neq (\Gamma \cdot y)$ then there exist neighborhoods U of x and V of y such that $U \cap \gamma \cdot V = \emptyset$ for all $\gamma \in \Gamma$.
2. around any $x \in M$ there exists an open neighborhood W such that $W \cap \gamma \cdot W = \emptyset$ for all $\gamma \in \Gamma \setminus \{e\}$.

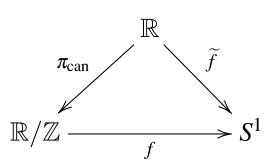
Some authors call such actions ”properly discontinuous”, but we find that a very unfortunate choice. We prefer to call them **free and proper** actions. Our choice also indicates the fact that it is not one single property, but a combination of two. More precisely, there is also the notion of **proper action**, defined without assuming any freeness: the condition is that, for any $K \subset M$ compact, the set of elements $\gamma \in \Gamma$ with the property that $\gamma \cdot K \cap K \neq \emptyset$ is finite. In particular, actions of finite groups are automatically proper. We leave it as an exercise (in Topology) to show that, indeed, an action satisfies 1. and 2. if and only if it is both free as well as proper.

Corollary 2.91. *The conclusion of the theorem holds more generally, for free and proper actions.*

Example 2.92. Giving an action of the group $(\mathbb{Z}, +)$ on a manifold M is equivalent to giving a diffeomorphism $\phi = \phi_1 : M \to M$. For instance, the translation $\mathbb{R} \to \mathbb{R}$, $r \mapsto r+1$ encodes the action

$$\mathbb{R} \times \mathbb{Z} \to \mathbb{R}, \quad (r,n) \mapsto \phi_n(r) = r+n.$$

The map $\widetilde{f} : \mathbb{R} \to S^1$, $t \mapsto (cos(2\pi t), sin(2\pi t))$ induces a bijection $f : \mathbb{R}/\mathbb{Z} \to S^1$ making the diagram below commutative



This action is both free as well as proper, hence $\mathbb{R}/\mathbb{Z}$ carries a smooth structure making π_{can} into a submersion. Actually, such a smooth structure can be obtained by transporting the smooth structure from S^1 via $\widetilde{f}$. By the uniqueness of such smooth structures, it follows that f is a diffeomorphism.

Example 2.93. Giving an action of the group $(\mathbb{Z}, +)$ on a manifold M is equivalent to giving an **involution** of M, i.e., a diffeomorphism $\phi : M \to M$ with the property that $\phi \circ \phi = \mathrm{Id}_M$. Interesting examples are provided by reflection maps $\tau(x) = -x$, defined on various subspaces of the Euclidean space. E.g.:

- on $\mathbb{R}$, where it encodes a non-free action of $\mathbb{Z}_2$ (with quotient $\mathbb{R}_{\geq 0}$);
- on the circle $S^1 \subset \mathbb{R}^2$, or even on $\mathbb{R}^2 \setminus \{0\}$, to provide free actions of $\mathbb{Z}_2$ (what are the quotient ... manifolds???);
- on the higher dimensional spheres ... but that takes us to the next chapter.

2.4.3 Quotient smooth structures: general crieteria

2.5 Examples

2.5.1 Submanifolds of $\mathbb{R}^n$ via the regular value theorem

The regular value theorem is an immediate consequence of the discussion from subsection 1.2.6 (more precisely, of Theorem 1.38), which is the main tool to check whether a given subset of an Euclidean space, $M \subset \mathbb{R}^L$, is an embedded submanifold- and, therefore, a manifold on its own. Of course, in specific examples, there is a lot more to say about those manifolds hence, when interested in such specific example, this theorem is just the first step. A more general version will be presented a bit later, in Theorem 2.66.

Theorem 2.94 (the regular value theorem in $\mathbb{R}^n$). *Assume that $M \subset \mathbb{R}^n$ can be written as the zero-set of a smooth map $f : \Omega \to \mathbb{R}^k$ defined on an open subset $\Omega \subset \mathbb{R}^n$ and assume that f is a submersion at each point $p \in M$. Then M is an $m = n-k$ dimensional embedded submanifold of $\mathbb{R}^n$.*

Here are some

Exercise 2.95. Show that the sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$

is an embedded submanifold of $\mathbb{R}^3$; and similarly for the higher dimensional spheres.

Exercise 2.96. Show that

$$M = \{(x,y,z) \in \mathbb{R}^3 : x^3+y^3+z^3-3xyz = 1\}$$

is a 2-dimensional embedded submanifold of $\mathbb{R}^3$.

Exercise 2.97. Denote by $\mathscr{M}_{m\times m}(\mathbb{R})$ the set of $m\times m$ matrices with real coefficients, and let $O(m)$ be the subset consisting of orthonormal matrices

$$O(m) = \{A \in \mathscr{M}_{m\times m}(\mathbb{R}) : A\cdot A^T = I_m.\}$$

Identifying $\mathscr{M}_{m\times m}(\mathbb{R})$ with $\mathbb{R}^{m^2}$, show that $O(m)$ is a submanifold of dimension $\frac{m(m-1)}{2}$.

2.5.2 The spheres S^m

The first example in our list are the m-dimensional spheres

$$S^m := \{(x_0,\ldots,x_m) \in \mathbb{R}^{m+1} : (x_0)^2+(x_1)^2+\ldots+(x_m)^2 = 1\}.$$

Of course, this example fits into the general discussion of embedded submanifolds of Euclidean spaces mentioned already in Example 2.17 and is a consequence of the regular value theorem that we just discussed. However, it is one of the many examples which are instructive to consider separately, even as abstract manifolds, and notice their rather special properties.

First of all, as with any subspace of a Euclidean space, we endow it with the Euclidean topology: opens are intersections of S^m with opens in $\mathbb{R}^{m+1}$. In this way, S^m is a Hausdorff, second countable space (... even compact).

Intuitively it should be clear that, locally, S^m looks like (opens inside) $\mathbb{R}^m$- and that is something that we use everyday (we do live on some sort of sphere, remember?). For instance, drawing small disks on the sphere and projecting them on planes through the origin, one can easily build charts. For instance, the upper hemisphere

$$U_0^+ = \{(x_0,\ldots,x_m) \in S^m : x_0 > 0\}$$

(not even so small!) and the projection into the horizontal plane $\{0\} \times \mathbb{R}^m \subset \mathbb{R}^{m+1}$ gives rise to

$$\chi_0^+ : U_0^+ \to \mathbb{R}^m, \quad \chi(x_0, \ldots, x_m) = (x_1, \ldots, x_m).$$

Considering the similar charts with $x_0 < 0$ or using the other coordinates, and putting them all together, we get a smooth atlas

$$(U_0^+, \chi_0^+),\ (U_0^-, \chi_0^-),\ \ldots,\ (U_m^+, \chi_m^+),\ (U_m^-, \chi_m^-) \tag{2.5.1}$$

defining a "natural" smooth structure on S^m, called **the standard smooth structure on S^m**.

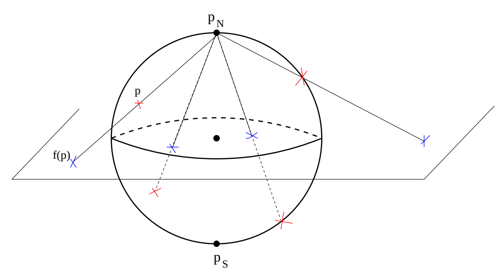
One can actually use the intuition to build similar charts and atlases; however, if you really follow your intuition, you will obtain the same smooth structure (and therefore the name "natural"). Here is an example of another smooth atlas on the sphere (... describing the same smooth structure). It is probably the most elegant one; at least it uses the least amount of charts: two. These are the so-called stereographic projections w.r.t. the north and the south poles,

$$p_N = (0, \ldots, 0, 1), \quad p_S = (0, \ldots, 0, -1) \in S^m,$$

respectively. The one w.r.t. to p_N is the map

$$\chi_N : S^m \setminus \{p_N\} \to \mathbb{R}^m$$

which associates to a point $p \in S^n$ the intersection of the line $p_N p$ with the horizontal hyperplane (see Figure 2.5.2).



The stereographic projection (sending the red points to the blue ones)

Fig. 2.8

Computing the intersections, we find the precise formula:

$$\chi_N : S^m \setminus \{p_N\} \to \mathbb{R}^n, \quad \chi_N(x_0, x_1, \ldots, x_m) = \left(\frac{x_0}{1 - x_m}, \ldots, \frac{x_{m-1}}{1 - x_m} \right)$$

(while for χ_S we find a similar formula, but with $+$s instead of the $-$s). Reversing the process (i.e. computing its inverse), we find

$$\chi_N^{-1} : \mathbb{R}^m \to S^m \setminus \{p_N\}, \quad \chi_N^{-1}(u_1, \ldots, u_m) = \left(\frac{2u_1}{|u|^2 + 1}, \ldots, \frac{2u_m}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1} \right)$$

and we deduce that χ_N is a homeomorphism. And similarly for χ_S. Computing the change of coordinates between the two charts we find

$$\chi_S \circ \chi_N^{-1} : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}, \quad \chi_S \circ \chi_N^{-1}(u) = \frac{u}{|u|^2}.$$

This is clearly smooth, composed with itself is the identity, therefore it is a diffeomorphism (... therefore also solving Exercise 1.58 from Chapter 1). We deduce that the two charts

$$(S^m \setminus \{p_N\}, \chi_N), \quad (S^m \setminus \{p_S\}, \chi_S) \tag{2.5.2}$$

define a smooth m-dimensional atlas on S^m.

Exercise 2.98. Show that the stereographic projections give rise to the standard smooth structure on S^m. Or, more precisely, show that the smooth atlas (2.5.2) induces the same smooth structure as (2.5.1).

Exercise 2.99. Consider the height function

$$f : S^2 \to \mathbb{R}, \quad f(x,y,z) = z.$$

At which points in the sphere does f fail to be a submersion?

Remark 2.100 (For the curious students: exotic spheres). As for the Euclidean spaces, it becomes very interesting (and exciting!) when we ask about the existence of other smooth structures on S^m (endowed with the Euclidian topology). Of course, for the reasons we explained, we only look at non-diffeomorphic smooth structures. Again, the answers are deceivingly simple (and the proofs highly non-trivial):

- except for the exceptional case $m = 4$ (see below), for $m \leq 6$ the standard smooth structure on S^m is the only one we can find (up to diffeomorphisms).
- S^7 admits precisely 28 (!?!) non-diffeomorphic smooth structures.
- S^8 admits precisely 2.
- ...
- and S^{11} admits 992, while S^{12} only one!
- and S^{31} more than 16 million, while S^{61} only one (and actually, next to the case S^m with $m \leq 6$, S^{61} is the only odd-dimensional sphere that admits only one smooth structure).
- ...

Moreover, some of the exotic (i.e. non-diffeomorphic to the standard) spheres can be described rather simply. For instance, fixing $\varepsilon > 0$ small enough, inside the small sphere of radius ε, $S^9_\varepsilon \subset \mathbb{C}^5$,

$$W_k := \{(z_1,z_2,z_3,z_4,z_5) \in S^9_\varepsilon : z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0\} \subset \mathbb{C}^5 \cong \mathbb{R}^{10}$$

(one for each integer $k \geq 1$) are all homeomorphic to S^7, they are all endowed with smooth structures induced from the standard (!) one on $\mathbb{R}^{10}$, but the first 28 of them are each two non-diffeomorphic (and after that they start repeating). In this way one obtains a rather explicit description of all the 28 smooth structures on S^7.

While the number of smooth structures on S^m is well understood for $m \neq 4$, the case of S^4 remains a mystery: is there just one smooth structure? Is there a finite number of them (and how many?)? Or there is actually an infinite number of them? The smooth Poincare conjecture says that there is only one; however, nowadays it is believed that the conjecture is false.

2.5.3 The projective spaces $\mathbb{P}^m$

Probably the simplest example of a smooth manifold that does not sit *naturally* inside a Euclidean space (therefore for which the abstract notion of manifold is even more appropriate) is the m-dimensional projective space $\mathbb{P}^m$. Recall that it consists of all lines through the origin in $\mathbb{R}^{m+1}$:

$$\mathbb{P}^m = \{l \subset \mathbb{R}^{m+1} : l - \text{one dimensional vector subspace}\}.$$

Each point $x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}$ gives rise to the line l_x through the origin and x, hence $l_x := \mathbb{R} \cdot x \subset \mathbb{R}$, so that we can write

$$\mathbb{P}^m = \left\{l_x : x \in \mathbb{R}^{m+1} \setminus \{0\}\right\}, \quad \text{where } (l_x = l_y) \Longleftrightarrow (y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}^*)$$

$\mathbb{P}^m$ is best understood by relating it back to $\mathbb{R}^{m+1} \setminus \{0\}$, via the map

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{P}^m, \quad x \mapsto l_x.$$

For instance, this gives rise to a natural topology on $\mathbb{P}^m$: the quotient topology w.r.t. π (already mentioned before, see e.g. Exercise 2.85)- that is, the largest one that makes π continuous, explicitly described as follows:

$$U \subset \mathbb{P}^m \text{ is open} \quad \Longleftrightarrow \quad \pi^{-1}(U) \text{ is open in } \mathbb{R}^{m+1}.$$

All these define $\mathbb{P}^n$ as a set, and as a topological space. For the smooth structure, it is handy to change the notation for the lines l_x to

$$[x_0 : x_1 : \ldots : x_m] := l_x = \mathbb{R} \cdot x, \quad \text{for } x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}.$$

The use of the symbol : in the notation should suggest "division" and is motivated by

$$[x_0 : x_1 : \ldots : x_m] = [y_0 : y_1 : \ldots : y_m] \Longleftrightarrow y_k = \lambda \cdot x_k \text{ for some } \lambda \in \mathbb{R}^* \text{ and all } k\text{s}. \tag{2.5.3}$$

The coordinate notation is also more appropriate when one is searching for coordinate charts. The natural smooth structure on $\mathbb{P}^m$ is obtained starting from a simple observation: the last equality above allows us in principle to make the first coordinate equal to 1:

$$[x_0 : x_1 : \ldots : x_m] = \left[1 : \frac{x_1}{x_0} : \ldots : \frac{x_m}{x_0}\right],$$

i.e. to use just coordinates from $\mathbb{R}^m$; with one little problem- when $x_0 = 0$ (this "little problem" is what forces us to use more than one chart). We arrive at a very natural chart for $\mathbb{P}^m$: with domain

$$U_0 = \{[x_0 : x_1 : \ldots : x_m] \in \mathbb{P}^m : x_0 \neq 0\}$$

and defined as

$$\chi^0 : U_0 \to \mathbb{R}^m,\ \chi^0([x_0 : x_1 : \ldots : x_m]) = \left(\frac{x_1}{x_0}, \ldots, \frac{x_m}{x_0}\right).$$

And similarly when trying to make the other coordinates equal to 1:

$$\chi^i : U_i \to \mathbb{R}^m,\ \chi^0([x_0 : x_1 : \ldots : x_m]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_m}{x_i}\right),$$

defined on U_i defined by $x_i \neq 0$.

Exercise 2.101. Show that, indeed,

$$(U_0, \chi^0), (U_1, \chi^1), \ldots, (U_m, \chi^m),$$

is, indeed, a smooth atlas (hence it gives rise to a smooth structure on $\mathbb{P}^m$).

Remark 2.102 (gluing the antipodal points of the sphere). By rescaling elements $x \in \mathbb{R}^{m+1} \setminus \{0\}$ to force them to be of norm one or, more geometrically, by intersecting the lines through the origin with the unit sphere, one can replace $\mathbb{R}^{m+1} \setminus \{0\}$ by S^m in the discussion above. Hence

$$\mathbb{P}^m = \{l_x : x \in S^m\} \quad \text{where, for } x, y \in S^m\text{:} \ (l_x = l_y) \Longleftrightarrow (y = x \text{ or } y = -x) \tag{2.5.4}$$

and it is not very difficult to show that the quotient topology induced by π coincides with the one induced by its restriction

$$\pi|_{S^m} : S^m \to \mathbb{P}^m, \quad x \mapsto l_x.$$

These allow us to think of $\mathbb{P}^m$ as being obtained from S^m by gluing any point $x \in S^m$ to its antipodal $-x$. More elegantly, one may say that we deal with an action of $\mathbb{Z}_2$ on S^m, and

$$\mathbb{P}^m = S^m/\mathbb{Z}_2,$$

at least as topological spaces. Notice that one of the gains of using S^m is that it follows right away that $\mathbb{P}^m$ is compact (as the quotient of a compact space). On the other hand, one could also invoke Theorem 2.87 to make $\mathbb{P}^m$ into a smooth manifold. Using the uniqueness part of the theorem, the following should be a rather easy exercise:

Exercise 2.103. Show that the smooth structure induced by the charts (U_k, χ_k) coincides with the one obtained by applying Theorem 2.87 to the action of $\mathbb{Z}_2$ on S^n given by the reflection $x \mapsto -x$.

The following exercise show that, when $m = 1$, $\mathbb{P}^1$ is not really new: it is diffeomorphic to the circle $S^1 \subset \mathbb{R}^2$.

Exercise 2.104. Consider the function

$$f : \mathbb{P}^1 \to S^1, \quad f([x : y]) = \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{2xy}{x^2 + y^2} \right)$$

and do the following:

1. show that f is well-defined and smooth.
2. show hat f is actually a diffeomorphism.
3. after identifying $\mathbb{P}^1$ with S^1 using f, what does the map $H : S^1 \to \mathbb{P}^1$ become?
4. explain f with a picture.

Exercise 2.105. Let $f : \mathbb{P}^2 \to \mathbb{R}^4$ be the function given by

$$f([x : y : z]) = (xy,\ yz,\ zx,\ y^2 - z^2) \quad \text{for } (x, y, z) \in S^2.$$

Please do the following:

1. check that f is well-defined and write the general formula for f (not only for (x, y, z) in the sphere).
2. compute the representation of f with respect to the charts χ^i of $\mathbb{P}^2$ (and the identity chart for $\mathbb{R}^4$).
3. show that f is an immersion.
4. we also compose f with the projection $\mathbb{R}^4 \to \mathbb{R}^3$ on the first three coordinates,

$$g := \mathrm{pr} \circ f : \mathbb{P}^2 \to \mathbb{R}^3;$$

   Show that its image is the following explicit subspace of $\mathbb{R}^3$:

$$R := \left\{ (X, Y, Z) \in \mathbb{R}^3 : |X|, |Y|, |Z| \leq \frac{1}{2},\ (XY)^2 + (YZ)^2 + (ZX)^2 = XYZ \right\} \ldots$$

5. ... but this is not an embedded submanifold of $\mathbb{R}^3$.
6. show that, however, $g : \mathbb{P}^2 \to \mathbb{R}^3$ is an immersion everywhere except for six points $p_1, \ldots, p_6$ ...
7. ... but even after removing those six points, $g(\mathbb{P}^2 \setminus \{p_1, \ldots, p_6\})$ is not an embedded submanifold of $\mathbb{R}^3$.
8. show that, on the other hand,

$$R_0 = \{(X, Y, Z) \in R : XYZ \neq 0\}$$

   is an open dense subset of R which is an embedded submanifold of $\mathbb{R}^3$, and there is open dense subset $\mathbb{P}^2_0 \subset \mathbb{P}^2$ such that $g|_{\mathbb{P}^2_0} : \mathbb{P}^2_0 \to R_0$ is a diffeomorphism.

The image R of g is know as "the Roman surface", or the "Steiner surface" (discovered by Steiner in Rome in 1844, according to Wikipedia). One can actually show that $\mathbb{P}^2$ cannot be embedded in $\mathbb{R}^3$. The g and R above can be seen as an attempt to find an immersion of $\mathbb{P}^2$ in $\mathbb{R}^3$ (non-injective, of course). You may be surprised to hear that such immersions actually exist. Finding an explicit one is quite a bit more difficult but also very interesting, and gives rise to Boy's surface in $\mathbb{R}^3$ ... but I let you google this one ...

Exercise 2.106. Show that the smooth structure on $\mathbb{P}^m$ discussed here makes the canonical map

$$H : S^m \to \mathbb{P}^m, \quad H(x_0, \ldots, x_m) = [x_0 : \ldots : x_m]$$

into a submersion.

Exercise 2.107. As a continuation of the previous exercise: show that it is the unique smooth structure on $\mathbb{P}^m$ for which H is a submersion.

Remark 2.108 (For the interested students: minimal number of charts). Note that this smooth structure on $\mathbb{P}^m$ is also a first example in which, at least intuitively, we need many more than just two charts to form an atlas. The question, valid for any manifold M, asking what is the minimal number of charts one needs to obtain an atlas of M, is a very interesting one. In general one can show that one can always find an atlas consisting of no more than $m+1$ charts where m is the dimension of M (not very deep, but not trivial either!). Therefore, denoting by $N_0(M)$ the minimal number of charts that we can find, one always has $N_0(M) \leq \dim(M)+1$. The atlas described above confirms this inequality for $\mathbb{P}^m$. However, in concrete examples, we can always do better. E.g. we have seen that $N_0(S^m) = 2$ (well, why can't it be 1?). The same holds for all compact orientable surfaces. Thinking a bit (but not too long) about $\mathbb{P}^m$ one may expect that $N_0(\mathbb{P}^m) = m+1$; however, that is not the case. The precise computation was carried out by M. Hopkins, except for the cases $n = 31$ and $n = 47$. For those cases we know that $N_0(\mathbb{P}^{31})$ is either 3 or 4, while $N_0(\mathbb{P}^{46})$ is either 5 or 6. For all the other cases, writing $m = 2^k a - 1$ with a odd, one has

$$N_0(\mathbb{P}^m) = \begin{cases} \max\{2, a\} & \text{if } k \in \{1,2,3\} \\ \text{the least integer} \geq \frac{m+1}{2(k+1)} & \text{otherwise} \end{cases} \quad \square$$

2.5.4 The complex projective spaces $\mathbb{CP}^m$

Returning to the basics, note that there is a complex analogue of $\mathbb{P}^m$; for that reason $\mathbb{P}^m$ is sometimes denoted $\mathbb{RP}^m$ and called **the real projective space**. The m-dimensional **complex projective space** $\mathbb{CP}^m$ is defined completely analogously but using complex lines $l_z \subset \mathbb{C}^{m+1}$, i.e. 1-dimensional complex subspaces of $\mathbb{C}^{m+1}$ (and 1-dimensional is in the complex sense). Again, one can write

$$\mathbb{CP}^m = \left\{[z_0 : z_1 : \ldots : z_m] : (z_0, z_1, \ldots, z_m) \in \mathbb{C}^{m+1} \setminus \{0\}\right\}$$

where $[z_0 : z_1 : \ldots : z_m]$ is just a notation for the line through the origin and the point $z = (z_0, \ldots, z_m) \in \mathbb{C}^{m+1}$. Hence

$$[z_0 : z_1 : \ldots : z_m] = [\lambda \cdot z_0 : \lambda \cdot z_1 : \ldots : \lambda \cdot z_m] \quad \text{for } \lambda \in \mathbb{C}^*.$$

Analogously to the real case, one can realize

$$\mathbb{CP}^m = \left(\mathbb{C}^{m+1} \setminus \{0\}\right)/\mathbb{C}^*.$$

Using $\mathbb{R}^{2m} = \mathbb{C}^m$ we can also represent the $(2m+1)$-dimensional sphere as:

$$S^{2m+1} = \{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} : |z_0|^2 + \ldots + |z_m|^2 = 1\}$$

and there is an obvious map

$$H : S^{2m+1} \to \mathbb{CP}^m, \quad H(z_0, \ldots, z_m) = [z_0 : \ldots : z_m].$$

Intersecting the (complex) lines with this sphere we can realize each line as as $[z]$ with $z \in S^{2m+1}$; two like this, $[z_1]$ and $[z_2]$, are equivalent if and only if $z_2 = \lambda \cdot z_1$, where this time $\lambda \in S^1$; i.e. the group $\mathbb{Z}_2$ from the real case is replaced by the group S^1 of complex numbers of norm 1 (endowed with the usual multiplication). We obtain

$$\mathbb{CP}^m = S^{2m+1}/S^1.$$

As for the smooth structure one proceeds completely analogously, keeping in mind the identification $\mathbb{C}^m = \mathbb{R}^{2m}$: we get a (smooth) atlas made of $m+1$ charts.

$$\mathscr{A} = \{(U_0, \chi^0), \ldots, (U_m, \chi^m)\}$$

given by

$$U_i = \{[z_0 : z_1 : \ldots : z_m] \in \mathbb{CP}^m : z_i \neq 0\},$$
$$\chi^i : U_i \to \mathbb{C}^m = \mathbb{R}^{2m},$$
$$\chi^i([z_0 : z_1 : \ldots : z_m]) = (\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_m}{z_i}).$$

Remark 2.109 (For the interested students: minimal number of charts for $\mathbb{CP}^m$). Note that the change of coordinates is actually given by holomorphic maps- therefore $\mathbb{CP}^m$ is also a complex manifold (see Section 2.1.4). Note that m is the complex dimension of $\mathbb{CP}^m$; as a manifold, it is $2m$-dimensional. As a curiosity: for the minimal number of charts needed to cover $\mathbb{CP}^m$, the answer is much simpler in the complex case:

$$N_0(\mathbb{CP}^m) = \begin{cases} m+1 & \text{if } m \text{ is even} \\ \frac{m+1}{2} & \text{if } m \text{ is odd} \end{cases}. \quad \square$$

Similar to Exercise 2.104, the following exercise show that, when $m = 1$, $\mathbb{CP}^1$ is not really new: it is diffeomorphic to the 2-sphere. This will be discussed together with the map

$$H : S^3 \to \mathbb{CP}^1, \quad H(z_0, z_1) = [z_0 : z_1].$$

The notation H (and h for the map in the next exercise) are related to the name of Hopf (... fibration).

Exercise 2.110. Consider the map

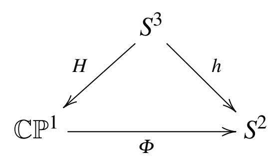
$$h : S^3 \to S^2, \quad h(z_0, z_1) := (|z_0|^2 - |z_1|^2, 2i \cdot \overline{z_0} \cdot z_1)$$

or, using real coordinates, h sends $(x, y, z, t) \equiv (x + i \cdot y, z + i \cdot t)$ to

$$h(x, y, z, t) = (x^2 + y^2 - z^2 - t^2, 2(yz - xt), 2(xz + yt)).$$

Show that:

1. h is well defined and it is a smooth submersion.
2. H is a smooth submersion.
3. There exists and is unique a map $\Phi : \mathbb{CP}^1 \to S^2$ such that $h = \Phi \circ H$ i.e. a commutative diagram:



4. Φ is smooth.
5. Φ is actually a diffeomorphism.

Exercise 2.111. Returning to arbitrary dimensions and the canonical map $H : S^{2m+1} \to \mathbb{CP}^m$ then, as in Exercise 2.106, show that the smooth structure on $\mathbb{CP}^m$ discussed here is uniquely determined by the condition that the projection H becomes a submersion. Try to further generalize, and eventually deduce something about smooth structures on quotients of smooth manifolds.

2.5.5 The torus T^2

We now concentrate on 2-dimensional manifolds (surfaces). Some of them (e.g. the 2-sphere and the 2-torus) sit rather canonically inside Euclidean spaces, but others (e.g. the projective plane or the Klein bottle) do not. The point is that each one of them is an interesting manifold on its own (completely independent on how it may sit inside an Euclidean space), each one of them can be "embedded" in an Euclidean space (providing "concrete models"), but while some of those embeddings are very natural (e.g. the concrete realization of the sphere in $\mathbb{R}^2$), other are not (e.g. any model of the Klein bottle). Therefore, it is interesting to free our mind and to think of them as abstract manifolds.

We start with ”the torus”- which is the standard name for subspaces of $\mathbb{R}^3$ which look like the surface of a doughnut; we use it generically for any manifold that is diffeomorphic to such a doughnut surface (or just homeomorphic, if you are doing Topology ...). There are various ways to produce explicit models and, for each one of them, the mathematics to make it precise.

Knutsel model: All the surfaces can be obtained from a planar figure, usually a square, after gluing some of its sides. For the torus, one simply glues each pair of opposite sides of a square, as shown in Figure 2.9. The outcome is a real-life torus.

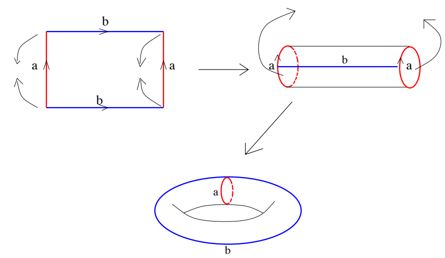


Fig. 2.9

The mathematical tool to make this precise is the notion of equivalence relation: given a set/space/manifold M, an equivalence relation R on M can be thought of as encoding the ”gluing information” (so that we will be gluing precisely the equivalent points of M), and the the quotient space M/R can be thought of as the result of the gluing. Recall that

$$M/R = \{R(x) : x \in M\},$$

where $R(x)$ is the R-orbit through $x \in M$ (also called the equivalence class of x), $R(x) = \{y \in M : (x,y) \in R\}$. Hence an element/point of M/R is an entire equivalence class, and the intuition that “we glue precisely the equivalent points” takes the precise form:

$$R(x) = R(y) \Longleftrightarrow (x,y) \in R.$$

The fact that “the points of M/R are obtained from those of M” is made precise by/encoded in the surjective map

$$\pi_{\text{can}} : M \to M/R$$

that sends x to $R(x)$, called canonical projection. This projection is very important since it allows one to understand/study the quotient M/R by relating it to M For instance, if M is a topological space, then it is natural to look for topologies on M/R that make π_{can} continuous and, among those, pick up “the most interesting one”. That is what is known in Topology as the **quotient topology** on M/R, defined by:

$$V - \text{open in } M/R \iff \pi_{\text{can}}^{-1}(V) - \text{open in } M.$$

When M is a manifold, the question of whether M/R admits a “canonical smooth structure” is more subtle; we will return to it later on.

Back to the torus, the relevant space is $M = [0,1] \times [0,1]$, two points $x, y \in M$ are (declared) equivalent if they get glued to each other, i.e. if

R1: $x = y$, or
R2: $x = (t,0)$ and $y = (t,1)$ with $t \in [0,1]$, or the other way around, or
R3: $x = (0,s)$ and $t = (1,s)$ with $s \in [0,1]$, or the other way around.

This defines an equivalence relation R_{torus} and the conclusion is that $[0,1] \times [0,1]/R_{\text{torus}}$ is a precise (though quite abstract) version of the torus, defined now as a topological space.

One way to exhibit a smooths structure on the abstract model without making use of the concrete ones (to be discussed below) is by noticing that the equivalence relation R_{torus} comes from a group actions on the entire plane $\mathbb{R}^2$. The group is $(\mathbb{Z}^2,+)$ acting on $\mathbb{R}^2$ by

$$(n,m)\cdot(x,y) = (x+n, y+m).$$

Notice now that the corresponding equivalence relation $R_{\mathbb{Z}^2}$ on $\mathbb{R}^2$ has the following properties:

- each point in $\mathbb{R}^2$ is equivalent with at least one point in the square $[0,1]\times[0,1]$;
- the restriction from $\mathbb{R}^2$ to the square is precisely the equivalence relation R_{torus} from the knutsel model.

It follow that

$$[0,1]\times[0,1]/R_{\text{torus}} \equiv R^2/\mathbb{Z}^2.$$

Finally, $R^2/\mathbb{Z}^2$ has a smooth structure (unique, by Corollary 2.84) with the property that the projection $\pi : \mathbb{R}^2 \to R^2/\mathbb{Z}^2$ is a submersion. This can be seen directly, or one could now invoke Corollary 2.91.

Remark 2.112 ("knutsel models" for S^2 and $\mathbb{P}^2$). For the surfaces that arise in the last two sections, the sphere S^2 and the projective plane $\mathbb{P}^2$, one can describe similar "knutsel models". For $S2$, the procedure is pretty clear and is described in Figure 2.10.

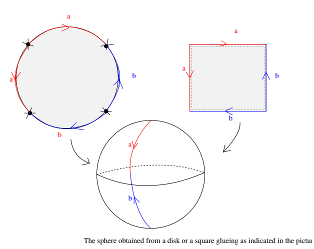


Fig. 2.10

For $\mathbb{P}^2$, since it cannot be realised inside $\mathbb{R}^3$, it is a bit harder to use the intuition; actually, one may even claim that a "knutsel model" helps in picturing how the projective plane actually looks like.

The model is shown in Figure 2.11 but it still requires a little explanation: the disk in the picture is the unit disk $D^2 \subset \mathbb{R}^2$. The reason that it appears is that it can be used to "parametrise" $\mathbb{P}^2$. More precisely, any $y \in D^2$ gives rise to a line $l(y) \subset \mathbb{R}^3$ as follows: the perpendicular on the horizontal plane $x_2 = 0$ going through y intersects the northern hemisphere $S^2_+ = \{x \in S^2 : x_2 \geq 0\}$ into a point $\widetilde{y}$, and then $l(y)$ is the line through the origin and $\widetilde{y}$. All lines arise in this way, just that some different ys may give the same line. Actually, one remarks right away that

$$l(y) = l(y') \Longleftrightarrow y = y' \text{ or } y \in \partial D^2 (= S1) \text{ and } y = -y'.$$

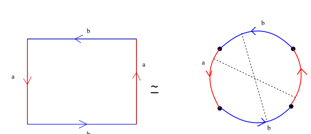


Fig. 2.11

Therefore, $\mathbb{P}^2$ can be thought of as obtained from D^2 by gluing any point in the boundary with its opposite. This is precisely what the disk in the picture describes, while the passing to a square model should be clear. ✓

Beweging model: For the next model of the torus one has to move a bit as follows: place yourself perpendicularly on the XOY plane with your chest at the origin O, take a circle of some radius r (... smaller than the length $\widetilde{r}$ of your hand), hold it by its center (you find a way to do that ...) and then rotate yourself a full 360 degrees. The locus spanned by the circle will be a torus. This description translates mathematically into a parametrised version of the torus. To that end, notice that the points in the resulting tous are determined by the angles a and b as shown in the picture.

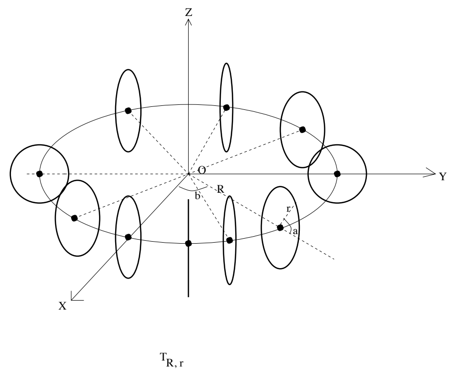


Fig. 2.12

Computing the resulting coordinates, one obtains the parametric description:

$$T^2 = T^2_{\widetilde{r},r} = \{((\widetilde{r} + r \cdot \cos a) cos b,\ (\widetilde{r} + r \cdot \cos a) sin b,\ r \cdot \sin a) :\ a, b \in [0, 2\pi]\} \subset \mathbb{R}^3. \tag{2.5.5}$$

Denoting the coordinates by x, y, z and eliminating a and b using $\sin^2 + \cos^2 = 1$ one finds the implicit description

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 : \left(\sqrt{x^2 + y^2} - \widetilde{r}\right)^2 + z^2 = r^2\}. \tag{2.5.6}$$

Exercise 2.113. Use the regular value theorem in $\mathbb{R}^3$ (see Theorem 2.94) to conclude that $T^2_{\widetilde{r},r}$ is a manifold.

This concrete model of the torus is related to quotient one as follows. First of all, it is related to the square via

$$\widetilde{f} : [0,1] \times [0,1] \to T^2, \quad (t, s) \mapsto ((\widetilde{r} + r \cdot \cos 2\pi t) cos 2\pi s,\ (\widetilde{r} + r \cdot \cos 2\pi t) sin 2\pi s,\ r \cdot \sin 2\pi t)\,.$$

This map has the property that $f(x) = f(y)$ holds precisely in one of the cases (R0)-(R2) described above, hence it induces a bijection between $[0,1] \times [0,1]/R$ and $T^2_{\widetilde{r},r}$ (and one can also check that this is actually a homeomorphism).

The geography model: But probably the shortest description of the torus is simply $S^1 \times S^1$. The fact that such a product, a priori sitting inside $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$, can be realised inside $\mathbb{R}^3$, is interesting on its own. On the other hand, the fact that the standard models inside $\mathbb{R}^3$ can be interpreted as such a product amounts to realising that the torus carries a longitude $\times$ latitude grid. This is indicated in Figure 2.13, where the red/blue circles stand for "longitude/latitude lines"; looking at a general point in the torus one sees that, through it, there passes precisely one blue and one red circle. Fixing some reference point on the torus (which will play the role of "null Island point") one obtains a 0-longitude (red) circle and a 0-latitude (blue) circle, which will play the role of "coordinate axes": any point on the torus will have a "longitudinal coordinate" obtained by intersecting the red circle through it with the 0-latitude (blue) circle and, similarly, a "latitudinal coordinate". We see that the coordinates are now elements of the circle and, therefore, one gets a pair (z_1, z_2) with $z_1, z_2 \in S^1 \times S^1$ as coordinates.

In the explicit description (2.5.5), the coordinates of a point in the torus will be simply $z_1 = e^{i \cdot a}$ and $z_2 = e^{i \cdot b}$. To emphasise that everything works out smoothly we formulate the next exercise, but we should also mention that the most elegant way to do it can be carried out only later on, after we discuss tangent spaces (Exercise 2.114 to come).

Exercise 2.114. Prove that

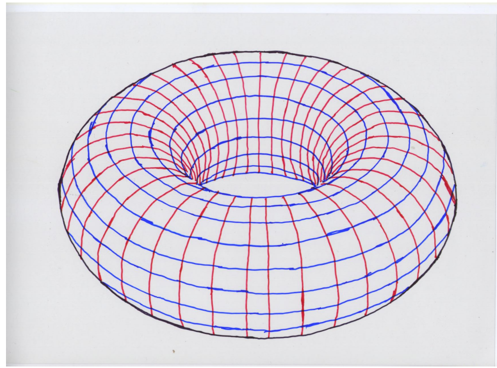


Fig. 2.13

$$f_0 : S^1 \times S^1 \to T^2, \quad (e^{i\cdot a}, e^{i\cdot b}) \mapsto ((\widetilde{r} + r\cdot \cos a)cos b,\ (\widetilde{r} + r\cdot \cos a)sin b,\ r\cdot \sin a)$$

is a diffeomorphism, where $T^2 = T^2_{\widetilde{r},r}$.

All together, one obtains a sequence of diffeomorphisms

$$\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1 \cong T^2_{\widetilde{r},r}, \quad (t,s) + \mathbb{Z}^2 \mapsto \left(e^{2i\pi t}, e^{2i\pi s}\right) \mapsto (\widetilde{r} + r\cdot \cos 2\pi t) sin 2\pi s,\ r\cdot \sin 2\pi t).$$

Notice however that that the description of the torus as $\mathbb{R}^2/\mathbb{Z}^2$ is closest to the one provided by the knutsel model. Indeed, the square $[0,1]\times[0,1] \subset \mathbb{R}^2$ interacts with equivalence relation R_Γ ($\Gamma = \mathbb{Z}_2$) on $\mathbb{R}^2$ as follows:

- each point in $\mathbb{R}^2$ is equivalent with at least one point in the square $[0,1]\times[0,1]$;
- the restriction of the equivalence relation from $\mathbb{R}^2$ to the square is precisely the equivalence relation R_{torus} from the knutsel model.

The full torus: The torus also comes with a "full-version". Playing the game from Fig 2.12, one would now need to grab and rotate a full 2-dimensional disk of radius r,

$$D^2_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le \widetilde{r}^2\}.$$

The outcome will be a subspace

$$T^2_{\widetilde{r},r,\text{solid}} \subset \mathbb{R}^3.$$

Extending the computation we did for the torus to the full torus, one obtains a parametrization

$$T^2_{\widetilde{r},r,\text{solid}} = \left\{(\widetilde{r} + ru)\cos a, (\widetilde{r} + ru)\sin a, v) : a \in [0,2\pi], (u,v) \in D^2\right\},$$

where D^2 is the unit disk; this gives rise to the following analogue of (2.5.6):

$$T^2_{\widetilde{r},r,\text{solid}} = \{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2} - \widetilde{r}\right)^2 + z^2 \le r^2\}.$$

as well as to a diffeomorphism

$$f : S^1 \times D^2 \to T^2_{\widetilde{r},r,\text{solid}}, \quad (e^{ia}, (u,v)) \mapsto (\widetilde{r} + ru)\cos a, (\widetilde{r} + ru)\sin a, rv). \tag{2.5.7}$$

Exercise 2.115. Show that T^2_{solid} is a manifold with boundary, and that it is diffeomorphic to $S^1 \times D^2$.

Of course, the diffeomorphism from the exercise, when restricted to the boundary, will become the diffeomorphism between T^2 and $S^1 \times S^1$ discussed above.

And here is a very nice property of the solid torus, that one sometimes states simply as: the 3-sphere can be obtained by gluing together two solid tori along their boundary. The following exercise explains this process and its interaction with the Hopf map from Exercise 2.110.

Exercise 2.116. Consider again the Hopf map in the explicit form

$$h : S^3 \to S^2, \quad h(x,y,z,t) = (x^2+y^2-z^2-t^2, 2(yz-xt), 2(xz+yt)).$$

Consider also the decomposition of S^2 into the open upper and lower hemisperes,

$$S^2_+ = \{(u,v,w) \in S^2 : u \geq 0\}, \quad S^2_- = \{(u,v,w) \in S^2 : u \leq 0\}$$

with the common intersection the circle identified with

$$S^1 = S^2_+ \cap S^2_+ = \{(u,v,w) \in S^2 : u = 0\}.$$

Show that:

1. each of the hemispheres are manifolds with boundary diffeomorphic to the unit disk D^2.
2. the pre-image of the common circle S^1 via h is homeomorphic to the torus.
3. the pre-image of the upper/lower hemisphere via h is homeomorphic to the solid torus.

Therefore, the pre-image via h of the decomposition $S^2 = S^2_+ \cup S^2_-$ becomes a decomposition of S^3 into two copies of the solid torus.

Note that we do not quite have yet the theoretical foundation to make this exercise into a "smooth one"; but, with the right concepts, all the homeomorphisms will become smooth (diffeomorphisms).

2.5.6 The Moebius band

The most popular manifold with boundary is the Moebius band. According to wikipedia, it "is a surface that can be formed by attaching the ends of a strip of paper together with a half-twist. As a mathematical object, it was discovered by Johann Benedict Listing and August Ferdinand Moebius in 1858, but it had already appeared in Roman mosaics from the third century CE." Here we look at it following the same lines of the discussion of the torus.

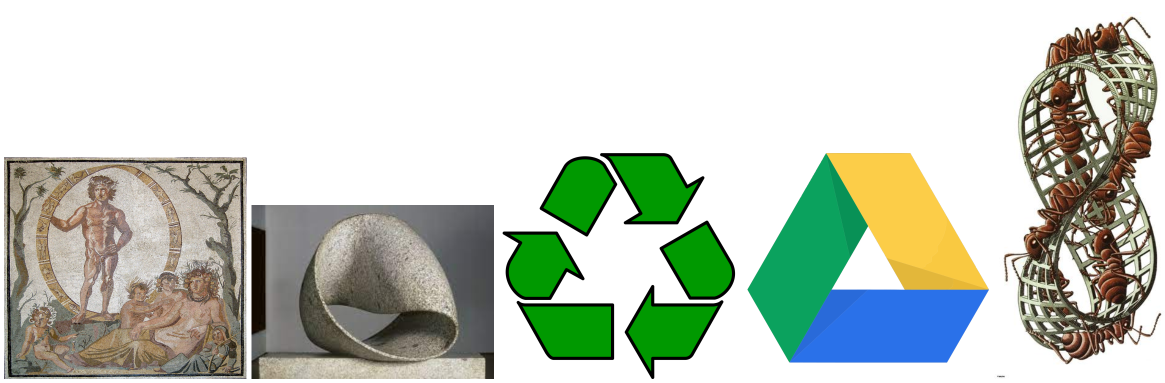


Fig. 2.14

Knutsel model: Most probably, the way that the Moebius was first shown to you was by using a long strip of paper and gluing two of its edges not in the most obvious way (that would give you a cylinder), but first twisting the strip creating a ribbon. Of course, "long strip" is so as to have a nice model made of paper (without rupturing the paper); in principle, this is again about the unit square $[0,1]\times[0,1]$, where we now glue two of its opposite edges while changing their orientation, as shown in Figure 2.15.

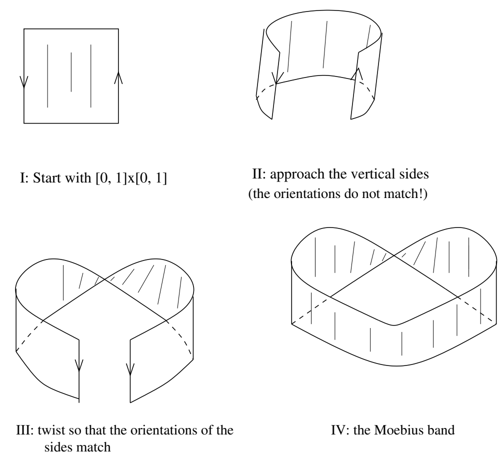


Fig. 2.15

More precisely (from a mathematical viewpoint), we deal with the quotient space $[0,1]\times[0,1]/R_{\text{Moe}}$ where R_{Moe} is the equivalence relation described as follows: two points $x=(t,s), x'=(t',s')\in[0,1]\times[0,1]$ are declared equivalent if:

R1: $(t,s)=(t',s')$, or
R2: $t=0, t'=1$ and $s'=1-s$, or
R3: $t=1, t'=0$ and $s'=1-s$.

This makes sense of the Moebius band as a topological space, living on its own (independent on how we choose to embed it inside an Euclidean space).

Beweging model: To produce a model inside $\mathbb{R}^3$ that can be described in formulas, we proceed like we did for the torus. We place ourselves in the same position (perpendicular on the XOY plane, etc), but now holding a stick by its middle, ands while rotating ourselves a full 360 degrees, we rotate the stick by 180 degrees (turning it upside down). Of course, we assume the rotations to be uniform (constant spead). See Figure 2.16. To write down explicit formulas then, again, we denote by $\widetilde{r}$ the length of the arm, and by r half of the length of the stick. The starting position is the segment A_0B_0 perpendicular on XOY with middle point $P_0=(\widetilde{r},0,0)$. At any moment, the segment stays in the plane through the origin and its middle point, which is perpendicular on the XOY plane.

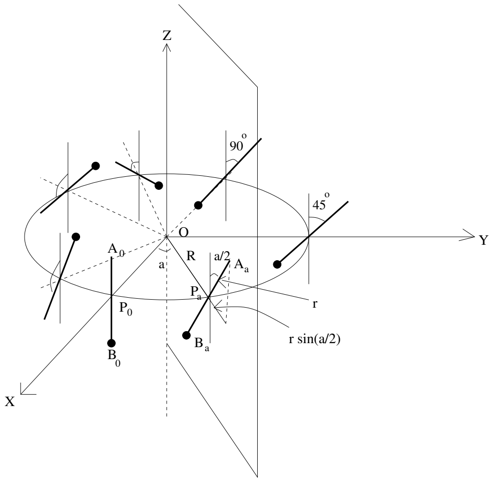
We parametrise the movement by the angle a shown in the picture. At any moment a, the segment stays in the plane perpendicular on the XOY plane that goes through the origin and the middle point

$$P_a=(\widetilde{r}\cos(a),\widetilde{r}\sin(a),0).$$

At this moment, the precise position of the segment, denoted A_aB_a, is determined by the angle that it makes with the perpendicular on the plane XOY through P_a; call it b. This angle depends on a. Due to the assumptions (namely that while a goes from 0 to 2π, b only goes from 0 to π, and that the rotations are uniform), we have $b=a/2$ (see 2.16). We deduce

$$A_a=\left(\left(\widetilde{r}+r\sin\frac{a}{2}\right)\cos(a),\left(\widetilde{r}+r\sin\frac{a}{2}\right)\sin(a),\, r\cos\frac{a}{2}\right).$$

and a similar formula for B_a (obtained by replacing r by $-r$). One obtains the following explicit model for the Moebius band:



Explicit realization of the Moebius band in $|R^3$: $M_{R,r}$

Fig. 2.16

$$M_{\widetilde{r},r} = \left\{ \left(\left(\widetilde{r} + tr\sin\frac{a}{2}\right)\cos(a), \left(\widetilde{r} + tr\sin\frac{a}{2}\right)\sin(a), tr\cos\frac{a}{2}\right) : a \in [0,2\pi], t \in [-1,1] \right\} \quad (2.5.8)$$
$$= \{((\widetilde{r} + tr\sin b)\cos 2b, (\widetilde{r} + tr\sin b)\sin 2b, tr\cos b) : b \in [0,\pi], t \in [-1,1]\}$$

Exercise 2.117. Show that $M_{\widetilde{r},r} \subset \mathbb{R}^3$ is, indeed, a manifold with boundary, and a different choice of r and $\widetilde{r}$ produces diffeomorphic manifolds.

To fix one concrete model, one usually takes $\widetilde{r} = 2$ and $r = 1$.

Exercise 2.118. Consider $f : [0,1] \times [0,1] \to \mathbb{R}^3$ given by

$$f(t,s) = ((2 + (2s-1)\sin(\pi t))\cos(2\pi t), (2 + (2s-1)\sin(\pi t))\sin(2\pi t), (2s-1)\cos(\pi t)).$$

Check that $f(t,s) = f(t',s')$ holds only in one of the situations R1-R3 above. Deduce that the abstract Moebius band $[0,1] \times [0,1]/R_{\text{Moe}}$ is homeomorphic to the $M_{2,1} \subset \mathbb{R}^3$.

The model inside the tautological bundle over $\mathbb{P}^1$: The explicit model $M_{\widetilde{r},r}$ clearly lives inside the solid torus $T_{\widetilde{r},r}$ and, via the identification of the solid torus with $S^1 \times D^2$, see (2.5.7), it is clear that $M_{\widetilde{r},r}$ will be identified to the following subspace

$$\left\{ \left(e^{ia}, te^{\frac{ia}{2}}\right) : e^{ia} \in S^1, t \in [-1,1] \right\} \subset S^1 \times D^2.$$

Now, the key remark is that the appearance of e^{ia} and $e^{\frac{ia}{2}}$ together is at the very heart of the diffeomorphism between $\mathbb{P}^1$ and S^1 mentioned in Exercise 2.104; indeed, that diffeomorphism takes the line through $e^{\frac{ia}{2}}$ precisely to the point $e^{ia} \in S^1$. Therefore, pairs $\left(e^{ia}, te^{\frac{ia}{2}}\right)$ as above may be interpreted as a pairs (l_z, tz) with $z \in S^1$ and $t \in [-1,1]$. We discover the so-called tautological line bundle over $\mathbb{P}^1$,

$$E = \{(l,v) \in \mathbb{P}^1 \times \mathbb{R}^2 : v \in l\} \subset \mathbb{P}^1 \times \mathbb{R}^2.$$

Inside it there are various subspaces obtained by posing conditions on $|v|$, such as

$$E_{\leq 1} = \{(l,v) \in E : |v| \leq 1\}, \quad E_{<1} = \{(l,v) \in E : |v| < 1\}, \text{ etc},$$

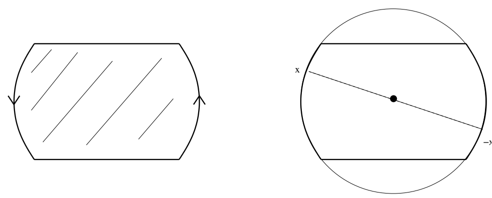
and the conclusion is that we have a diffeomorphism

$$E_{\leq 1} \to M_{\widetilde{r},r}, \quad (l,z) \mapsto ((\widetilde{r} + tr\sin b)\cos 2b, (\widetilde{r} + tr\sin b)\sin 2b, tr\cos b),$$

where $b \in [0,\pi]$ and $t \in [-1,1]$ are chosen so that l is the line through the origin and e^{ib}, and $z = te^{ib}$. Of course, $E_{<1}$ corresponds to the open Moebius band, while the smaller $E_{\leq \varepsilon} \to E_1$ correspond to thiner Moebius bands.
The model inside $\mathbb{P}^2$: Another copy of the Moebius band can be detected inside the projective plane $\mathbb{P}^2$. To visualise it, we use the model for the projective plane that was mentioned in Remark 2.112: as the space obtained from D^2 by gluing the antipodal points on its boundary. Consider inside D^2 the "band"

$$B = \{(x,y) \in D^2 : -\frac{1}{2} \leq y \leq \frac{1}{2}\}. \subset D^2.$$

The gluing process that produces $\mathbb{P}^2$ affects B in the following way: it glues the "opposite curved sides" of B as in the picture (Figure 2.17), and gives us the Moebius band.



The Moebius band inside the projective plane

Fig. 2.17

Paying attention to what happens to $D^2 - \text{int}(B)$ in the gluing process, you see that it contributes with a copy of the closed disk.

Exercise 2.119. Fill in the details and deduce that $\mathbb{P}^2$ can be obtained by starting from a Moebius band and a disk, and gluing them along the boundary circle.

As a quotient modulo a group: To continue the analogy with the discussion from the torus, let us point out that the Moebius band can also be realised as a quotient modulo a group action. Again, the group is $\mathbb{Z}^2$ acting on the space $\mathbb{R}^2$, but with the action:

2.5.7 The Klein bottle

A good friend of the torus is the so called Klein bottle, call it K.

Knutsel model: As for the torus, it can be obtained from a square by gluing each pair of opposite edges, as shown in Figure 2.18, i.e., changing the orientation in one of the pairs.
Beweging model: We leave it to the imagination of the reader to find motions that produce the Klein bottle. One ends up with an explicit model of K as a subspace of $\mathbb{R}^4$:

$$K = \{((2+cos(a))cos(b), (2+cos(a))sin(b), sin(a)cos(b/2), sin(a)sin(b/2)) : a, b \in [0,2\pi]\}. \tag{2.5.9}$$

Exercise 2.120. Explain why this subspace of $\mathbb{R}^4$ can be interpreted as the result of the gluing from Fig. 2.18.

Crafting the torus: Next, we point out that the Klein bottle is to the torus what the projective space is to the sphere: a quotient which is obtained by gluing "antipodal points". To see this, we go back to the square used to obtain the Klein bottle. Since the gluing is so problematic, let us mirror the square as in the picture; to get the Klein bottle we would first have to fold the longer square back (a gluing process itself) and perform the original gluing. However, in the longer square, the gluing that of the opposite side (which was problematic at the beginning) disappears: we

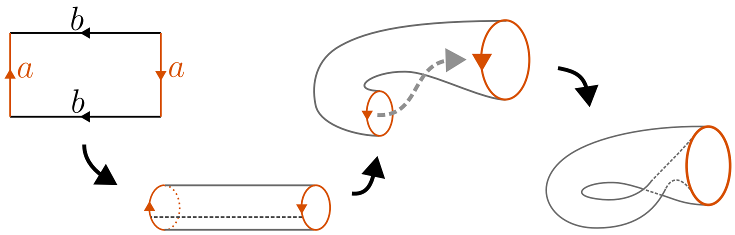


Fig. 2.18 Construction of the Klein bottle by gluing: Take a rectangle and glue the sides b together to get a cylinder. Then twist one end around so that the remaining a sides can be glued together in opposite orientation from what we would do to get a torus. Note that when visualizing this in $\mathbb{R}^3$, we are forced to create a self intersection when pulling the a sides close, but this does not affect the abstract topological gluing procedure.

can perform it and we get precisely the torus! However, to get the Klein bottle we see we have to keep on going and finish gluing the rest. Inspecting the picture, it will eventually be clear that what we still have to do is to glue points in the torus which are "antipodal" (reflections of each other with respect to the origin).

This can be further interpreted as a quotient of the torus modulo a $\mathbb{Z}_2$-action, similar to $\mathbb{P}^2 = S^2/\mathbb{Z}_2$:

$$K = T/\mathbb{Z}_2.$$

Realising the torus as $T = S^1 \times S^1$, the action of $\mathbb{Z}_2$ is simply given by $(z_1, z_2) \mapsto (-z_1, -z_2)$. While part of this discussion is purely topological and "hand waving", the last description we achieved is precise and makes sense in the realm of smooth manifolds. Actually, one can now apply Theorem 2.87 right away and make K into a smooth manifold.

Gluing two Moebius bands: In general, given two manifolds with boundary (subsection 2.1.4) N_1 and N_2, if they have the same boundary, they can be glued along their boundary producing a new manifold,

$$N_1 \cup_\partial N_2,$$

(now a manifold without boundary!). Making this precise is not completely trivial, but the statement should be clear intuitively. For instance, if N_1 is the unit disk, and $N_2 \subset \mathbb{R}^2$ consists of vectors of norm ≥ 1, then $N_1 \cup_\partial N_2$ is precisely $\mathbb{R}^2$.

This construction can be applied in particular to $N_1 = N_2 = M$ the Moebius band. Using the knutsel model for M, Figure 2.19 indicates that the outcome is precisely the Klein bottle.

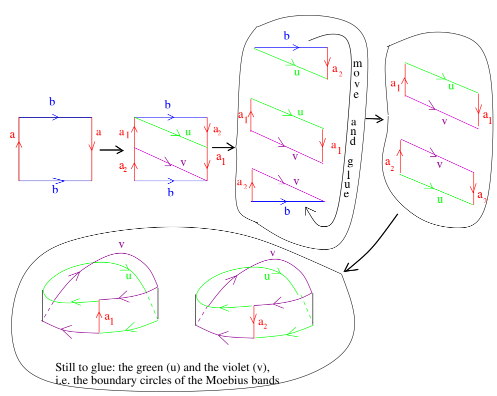


Fig. 2.19

Exercise 2.121. Explain how the picture describes the Klein bottle as the result of gluing two Moebius bands.

2.5.8 ... and the rest

Of course, the spaces that we have mentioned here: the torus, the Klein bottle, the 2-sphere S^2 and the projective plane $\mathbb{P}^2$, they are all examples of surfaces or, in our language, of 2-dimensional manifolds. One may remember from other courses (Meetkunde een Topologie?) that, at least topologically, surfaces are classified into:

- orientable ones: the sphere S^2, the torus T and then, for each g, the torus with g-wholes T_g (g-times connected sum of T with itself).
- non-orientable ones: the projective plane $\mathbb{P}^2$ and, similar to T_g, the spaces P_h obtained as the connected sum of h copies of $\mathbb{P}^2$.

The same classification result holds also in the smooth context. But, before that, one has to make sense of all these spaces as smooth manifolds. And that can be done using some basic very general operations with manifolds:

- Manifolds with boundary (subsection 2.1.4), N_1 and N_2, if they have the same boundary, they can be glued along their boundary producing a new manifold,
$$N_1 \cup_\partial N_2,$$
now a manifold without boundary!
- Given an m-dimensional manifold without boundary M, after removing a small open ball (in a coordinate chart) one obtains a manifold with boundary; let us denote it M° and call it "cut-M". It boundary is just S^{m-1}. Hence, if we have two m-dimensional manifolds M_1 and M_2 and we consider their cuts, we can just apply the gluing from the previous point. Hence one gets a new m-dimensional manifold,
$$M_1 \sharp M_2 := M_1^\circ \cup_{S^{m-1}} M_2^\circ,$$
called the connected sum of M_1 and M_2.

To do all of this in detail and properly (e.g. to see that the connected sum does not depend on how we remove the balls) requires a bit of work which goes beyond the scope of this course. But the intuition should be clear. And it can be applied right away: in this way all the surfaces

$$T_g = \underbrace{T\sharp \dots \sharp T}_{g \text{ times}}$$

and

$$P_h = \underbrace{\mathbb{P}^2\sharp \dots \sharp \mathbb{P}^2}_{h \text{ times}}$$

become smooth manifolds. Furthermore, various constructions that you may have seen already in Topology, makes sense in the context of smooth manifolds. For instance the fact that the projective space $\mathbb{P}^2$ can be described as obtained by gluing a disk D^2 to the Moebius band:

$$\mathbb{P}^2 = \text{Moebius} \cup_\partial D^2.$$

In turn, this property can be read a bit differently: the cut projective space (i.e. after removing a ball) is the Moebius band. Therefore, for any 2-dimensional manifold M, the operation of taking the connected sum with $\mathbb{P}^2$,

$$\mathbb{P}^2 \sharp M$$

means: remove a ball from M and, along the boundary circle, glue back a Moebius band.

Exercise 2.122. You should convince yourself now that the Klein bottle can also be described as

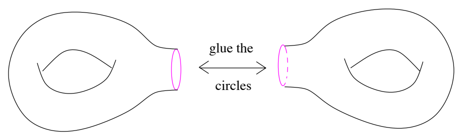


Fig. 2.20

$$K = \mathbb{P}^2 \sharp \mathbb{P}^2.$$

2.5.9 Grassmanians

Similar to considering lines through the origin (i.e. 1-dimensional vector subspaces) in $\mathbb{R}^{n+1}$ like we did for $\mathbb{P}^n$, Grassmannians are obtains when considering higher dimensional subspaces. More precisely, one defines

$$G_k(\mathbb{R}^{n+k}) := \{V \subset \mathbb{R}^{n+k} : V \text{ is a } k\text{-dimensional vector subspace of } \mathbb{R}^{n+k}\}.$$

To introduce the smooth structure, we fix an point $V \in G_k(\mathbb{R}^{n+k})$ and look at "nearby points". Using the standard inner product on $\mathbb{R}^{n+k}$, we consider the orthogonal $V^\perp \subset \mathbb{R}^{n+k}$, so that we have a direct sum decomposition

$$\mathbb{R}^{n+k} = V \oplus V^\perp.$$

The key idea is based on the intuition that, if V is thought of as an "OX" axis (and $V^\perp$ as OY), k-dimensional sub-spaces that are "close to V" look like graphs of function (from our OX to our OY). This takes the precise form of associating to any linear map $L : V \to V^\perp$ the k-dimensional subspace of $\mathbb{R}^{n+k}$:

$$V_L := \{v + L(v) : v \in V\} \subset \mathbb{R}^{n+k}.$$

Of course, V corresponds to $L = 0$. We now define

$$\mathrm{Op}(V) := \left\{V_L : L \in \mathrm{Lin}\left(V, V^\perp\right)\right\}.$$

2.5.10 More examples: classical groups and ... Lie groups

One very interesting (and special) class of manifolds are the so-called Lie groups: they are both groups, as well as manifolds, and the two structures are compatible:

Definition 2.123. A **Lie group** G is a group which is also a manifold, in a compatible fashion, i.e., such that the multiplication $m : G \times G \to G$, $m(g,h) = gh$, and the inversion $\iota : G \to G$, $\iota(g) = g^{-1}$, are smooth maps.

A **Lie group homomorphism** between two Lie groups G and H is any group homomorphism $f : G \to H$ which is also smooth. When f is also a diffeomorphism, we say that f is an isomorphism of Lie groups.

Example 2.124. The unit circle S^1, identified with the space of complex numbers of norm one,

$$S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

is a Lie group with respect to the usual multiplication of complex numbers. Actually, we see that

$$S^1 = SU(1).$$

Example 2.125. Similarly, the 3-sphere S^3 can be made into a (non-commutative, this time) Lie group. For that we replace $\mathbb{C}$ by the space of quaternions:

$$\mathbb{H} = \{x + iy + jz + kt : x, y, z, t \in \mathbb{R}\}$$

where we recall that the product in $\mathbb{H}$ is uniquely determined by the fact that it is $\mathbb{R}$-bilinear and $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. Recall also that for

$$u = x + iy + jz + kt \in \mathbb{H}$$

one defines

$$u^* = x - iy - jz - kt \in \mathbb{H}, \ \ |u| = \sqrt{uu^*} = \sqrt{x^2 + y^2 + z^2 + t^2} \in \mathbb{R}.$$

Then, the basic property $|u \cdot v| = |u| \cdot |v|$ still holds and we see that, identifying S^3 with the space of quaternionic numbers of norm 1, S^3 becomes a Lie group.

Remark 2.126 (For the interested students: spheres, Lie groups, etc). One may wonder: which spheres can be made into Lie groups? Well, it turns out that S^0, S^1 and S^3 are the only ones!

On the other hand, it is interesting to understand what happens with the arguments we used for S^1 and S^3 (and which were very similar to each other) in higher dimensions. The main point there was the multiplication on $\mathbb{C}$ and $\mathbb{H}$ and the presence of a norm such that

$$|x \cdot y| = |x| \cdot |y| \tag{2.5.10}$$

(so that, for two elements in the sphere, i.e. of norm one, their product is again in the sphere). So, to handle S^n similarly, we would need a ”normed division algebra” structure on $\mathbb{R}^{n+1}$, by which we mean a multiplication ”$\cdot$” on $\mathbb{R}^{n+1}$ that is bilinear and a norm satisfying the previous condition. Again, it is only on $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^4$ that such a multiplication exists. If we do not insist on the associativity of the multiplication, there is one more possibility: $\mathbb{R}^8$ (the so called octonions). But nothing else! And this was known since the 19th century! But why did people care about such operations in the 19th century? Well ... it was number theory and the question of which numbers (integers) can be written as a sum of two, three, etc squares. For sum of two squares the central formula which shows that a product of two numbers that can be written as a sum of two squares can itself be written as a sum of two squares is:

$$(x^2 + y^2)(a^2 + b^2) = (xa - yb)^2 + (xb + ya)^2.$$

Or, in terms of the complex numbers $z_1 = x + iy, z_2 = a_i b$, the norm equation (2.5.10). The search for similar “magic formulas” for sum of three squares never worked, but it did for four:

$$(x^2 + y^2 + z^2 + t^2)(a^2 + b^2 + c^2 + d^2) =$$

$$(xa + yb + zc + td)^2 + (xb - ya - zd + tc)^2 +$$
$$+ (xc + yd - za - tb)^2 + (xd - yc + zb - ta)^2.$$

This is governed by the quaternions and its norm equation (2.5.10).

Example 2.127 (GL_n). Probably the most important example is the **general linear group** $GL_n(\mathbb{R})$, consisting of $n \times n$ invertible matrices with real entries:

$$GL_n(\mathbb{R}) = \{A \in \mathscr{M}_{n \times n}(\mathbb{R}) : \det(A) \neq 0\}$$

They sit inside the space of all $n \times n$ matrices (for some n natural number)- which is itself a Euclidean space (just that the variables are arranged in a table rather than in a row):

$$\mathscr{M}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}.$$

Furthermore, since the determinant is continuous as a map

$$\det : \mathscr{M}_{n\times n}(\mathbb{R}) \to \mathbb{R},$$

it follows that $GL_n(\mathbb{R})$ is open inside $\mathscr{M}_n(\mathbb{R})$. Therefore $GL_n(\mathbb{R})$ inherits a natural smooth structure: the one induced by the atlas consisting of one single chart (with image open in $\mathbb{R}^{n^2}$):

$$GL_n(\mathbb{R}) \ni A = (A^i_j)_{ij} \quad \overset{\chi}{\longmapsto} \quad (A^1_1, \ldots, A^1_n, A^2_1, \ldots, A^2_n, \ldots, A^n_1, \ldots, A^n_n) \in \mathbb{R}^{n^2}.$$

Hence, indeed, $GL_n(\mathbb{R})$ is a Lie group of dimension n^2. A similar discussion applies to matrices with complex coefficients, giving rise to a similar Lie group $GL_n(\mathbb{C})$; identifying $\mathbb{C}$ with $\mathbb{R}^2$, one has:

$$\mathscr{M}_{\times n}(\mathbb{C}) \cong \mathbb{R}^{(2n)^2},$$

so that $GL_n(\mathbb{C})$ will be a Lie group of dimension $4n^2$.

Example 2.128 (subgroups of GL_n). Inside the GL_ns one can find several interesting Lie groups such as

- **the orthogonal group**: $O(n) = \{A \in GL_n(\mathbb{R}) : A \cdot A^T = I\}$ (where A^T denotes the transpose of A).
- **the special orthogonal group**: $SO(n) = \{A \in O(n) : \det(A) = 1\}$.
- **the special linear group**: $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}$.
- **the unitary group**: $U(n) = \{A \in GL_n(\mathbb{C}) : A \cdot A^* = I\}$ (where A^* is the conjugate transpose).
- **the special unitary group**: $SU(n) = \{A \in GL_n(\mathbb{C}) : A \cdot A^* = I, \det(A) = 1\}$.
- **the symplectic group**: $Sp_n(\mathbb{R}) := \{A \in GL_{2n}(\mathbb{R}) : A^T JA = J\}$, where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in GL_{2n}(\mathbb{R}). \tag{2.5.11}$$

Each one of these caries a natural smooth structure; actually, they are all embedded submanifolds of the corresponding (Euclidean) spaces of matrices (see below). Since the group operations are restrictions of the ones of the GL_ns, they will continue to be smooth- so that all these groups will be Lie groups.

For the claim that they are embedded submanifolds of $GL_n \subset \mathscr{M}_{n\times n}$, since they are all given by (algebraic) equations, there is a natural to proceed: use the regular value theorem (Theorem 2.66). As an illustration, we provide the details for the case of $O(n)$.

Example 2.129 (the regular value theorem approach in the case of $O(n)$). The equation defining $O(n)$ suggest we should be looking at

$$f : GL_n(\mathbb{R}) \to \mathscr{M}_{n\times n}(\mathbb{R}), \quad f(A) = A \cdot A^T.$$

However, it is important to note that the values of this function takes values in a smaller Euclidean space. Indeed, since in general, $(A \cdot B)^T = B^T \cdot A^T$, the matrices of type $A \cdot A^T$ are always symmetric. Therefore, denoting by $\mathscr{S}_n$ the space of symmetric $n \times n$ matrices (again a Euclidean space, of dimension $\frac{n(n+1)}{2}$), we will deal with f as a map

$$f : GL_n(\mathbb{R}) \to \mathscr{S}_n.$$

(If we didn't remark that f was taking values in $\mathscr{S}_n$, we would not have been able to apply the regular value theorem; however, the problem that we would have encountered would clearly indicate that we have to return and make use of $\mathscr{S}_n$ from the beginning; so, after all, there is no mystery here).

Now, f is a map from an open in a Euclidean space (and we could even use f defined on the entire $\mathscr{M}_{n\times n}(\mathbb{R})$), with values in another Euclidean space; and it is clearly smooth since it is given by polynomial expressions. So, at each point $A \in GL_n(\mathbb{R})$, the differential of f at A,

$$(df)_A : \mathscr{M}_{n\times n}(\mathbb{R}) \to \mathscr{S}_n$$

can be computed by the usual formula:

$$(df)_A(X) = \frac{d}{dt}\Big|_{t=0} f(A+tX) = \frac{d}{dt}\Big|_{t=0} \left(A\cdot A^T + t(A\cdot X^T + A^T\cdot X) + t^2 X\cdot X^T\right) = A\cdot X^T + X\cdot A^T.$$

Since $O(n) = f^{-1}(\{I\})$, we have to show that $(df)_A$ is surjective for each $A \in O(n)$. I.e., for $Y \in \mathscr{S}_n$, show that the equation $A\cdot X^T + X\cdot A^T = Y$ has a solution $X \in \mathscr{M}_{n\times n}(\mathbb{R})$. Well, it does, namely: $X = \frac{1}{2}YA$ (how did we find it?). Therefore $O(n)$ is a smooth submanifold of GL_n of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Exercise 2.130. Do the same for the other groups in the list. At least for one more.

On the other hand, instead of doing a case-by-case analysis, one can also invoke the following general result:

Theorem 2.131. *Any closed subgroup of $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ is automatically an embedded submanifold and, therefore, becomes a Lie group.*

Of course, this theorem is very useful and one should at least be aware of it. A proof will be given at the end of this section.

Example 2.132 (low dimensions). For n small, the classical subgroups of GL_n have can be further recognized as manifolds/groups that we have already looked at. Here are a couple of examples which we leave as exercises:

Exercise 2.133. Show that

$$f : S^1 \to SO(2),\ \ f(x,y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

is an isomorphism of Lie groups.

Exercise 2.134. Show that

$$F : S^3 \to SU(2),\ \ F(\alpha,\beta) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

where we interpret S^3 as $\{(\alpha,\beta) \in \mathbb{C}^2 : |\alpha|^2 + |\beta|^2 = 1\}$, is an isomorphism of Lie groups.

By analogy with the last example, one may expect that S^3 is isomorphic also to $SO(3)$ (at least the dimensions match!). However, the relation between these two is more subtle:

Exercise 2.135. Recall that we view S^3 inside $\mathbb{H}$. We also identify $\mathbb{R}^3$ with the space of pure quaternions

$$\mathbb{R}^3 \xrightarrow{\sim} \{v \in \mathbb{H} : v + v^* = 0\},\ (a,b,c) \mapsto ai + bj + ck.$$

For each $u \in S^3$, show that $A_u(v) := u^* v u$ defines a linear map $A_u : \mathbb{R}^3 \to \mathbb{R}^3$ which, as a matrix, gives an element $A_u \in SO(3)$. Then show that the resulting map

$$\phi : S^3 \to SO(3),\ u \mapsto A_u$$

is smooth, is a group homomorphism, is a surjective local diffeomorphism, but each fiber has two elements (it is a 2-1 cover). Finally, deduce that $SO(3)$ is diffeomorphic to the real projective space $\mathbb{P}^3$.

Exercise 2.136. The aim of this exercise is to show how $GL_n(\mathbb{C})$ sits inside $GL_{2n}(\mathbb{R})$, and similarly for the other complex groups. Well, there is an obvious map

$$j : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R}), \quad A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Show that:

1. j is an embedding, with image $\{X \in GL_{2n}(\mathbb{R}) : J \cdot X = X \cdot J\}$, where J is the matrix (2.5.11).
2. j takes $U(n)$ into $O(2n)$.
3. actually j takes $U(n)$ into $SO(2n)$.

(Hint for (3): prove that $\det(j(Z)) = |\det(Z)|^2$ for all $Z \in GL_n(\mathbb{C})$. For this: show that for B invertible, there exists a matrix $X \in GL_n(\mathbb{R})$ such that

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & X^{-1} \\ -I & X \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

Proof (of Theorem 2.131 ... for the interested students). We will be using the exponential map of matrices ($\exp(A) = \sum_n \frac{A^n}{n!}$), which can be seen as a smooth map

$$\exp : \mathscr{M}_{n\times n} \to GL_n \subset \mathscr{M}_{n\times n}$$

Since its differential at the zero-matrix is

$$(d\exp)_0 : \mathscr{M}_{n\times n} \to \mathscr{M}_{n\times n}, \quad X \mapsto \frac{d}{dt}\Big|_{t=0} \exp(tX) = X,$$

the inverse function theorem implies that exp restricts to a diffeomorphism between some open neighborhood of $0 \in \mathscr{M}_{n\times n}$ to an open neighborhood of $I \in \mathrm{GL}_n$. The resulting inverse, that could be called "logarithm", can then be interpreted as chart for GL_n.

For closed subgroups $G \subset GL_n$ there is a rather simple idea to exhibit a smooth structure on G: looking for matrices whose exponential lands in G one introduces

$$\mathfrak{g} := \{X \in \mathscr{M}_{n\times n} : \exp(tX) \in G \;\forall\, t \in \mathbb{R}\} \subset \mathscr{M}_{n\times n}(\mathbb{R}), \tag{2.5.12}$$

and the idea is to use $\exp|_{\mathfrak{g}}$ to produce a chart for G around $\exp(0) = I$ (the identity matrix); for charts around other points $A \in \mathrm{GL}_n$, one uses the group structure to move from I (and around) to A. Of course, exhibiting charts on G to make G into a manifold would not be necessary if we manage to prove (as stated) that G is an embedded submanifold. The point of the simple idea above is that it can actually be adjusted to provide not only charts for $\mathfrak{g}$ but also adapted charts- therefore proving the stronger statement that G is an embedded submanifold. Time to start the proof!

Therefore, assume that $G \subset GL_n$; for simplicity, assume we work over $\mathbb{R}$. We introduce $\mathfrak{g}$ given by (2.5.12). It is handy to realize what G being closed implies about $\mathfrak{g}$: to ensure that $\exp(tX) \in G$ holds *for all* $t \in \mathbb{R}$, it suffices that it holds for a sequence $t_i \to 0$ of nonzero real numbers. The proof actually reveals something slightly stronger:

Lemma 2.137. *Given a matrix X, if there exists a sequence of nonzero real numbers with $t_i \to 0$, and a sequence of matrices $X_i \to X$ such that $\exp(t_iX_i) \in G$ for all i, then $\exp(tX) \in G$ for all t (i.e., $X \in \mathfrak{g}$).*

Proof. Since $\exp(-Y) = \exp(Y)^{-1}$ for all Y, after eventually changing the signs of some t_is and t, we may assume that $t_i > 0$ for all i and that we are looking at $\exp(tX)$ with $t > 0$. Fix t. For each i, the positive number t/t_i sits in an interval $[k_i, k_i + 1]$ for some integer k_i. One finds that $t - t_i \leq k_it_i \leq t$, i.e. we managed to write our t as a limit of integer multiples of the t_is- namely $t = \lim_{i\to\infty} k_it_i$. In turn, this allows us to write $\exp(tX)$ as a limit of elements of G:

$$\exp(tX) = \lim_{i\to\infty} \exp(k_it_iX_i) = \lim_{i\to\infty} \exp(t_iX_i)^{k_i}.$$

Since G is closed, we obtain $\exp(tX) \in G$.

With this lemma at hand one can now show that the sum of two matrices $X, Y \in \mathfrak{g}$ is again in $\mathfrak{g}$. Indeed, writing

$$e^{tX}e^{tY} = e^{f(t)}, \quad \text{with} \quad \lim_{t\to 0} \frac{f(t)}{t} = X + Y,$$

take t_n to be any sequence of real numbers converging to 0 and $X_n = \frac{f(t_n)}{t_n}$. It follows that $\mathfrak{g}$ is a vector subspace of $\mathscr{M}_{n\times n}$. In turn, this allows us to choose a complement $\mathfrak{g}'$ of $\mathfrak{g}$ in $\mathscr{M}_{n\times n}(\mathbb{R})$- which is needed to adapt the original idea we mentioned above to proving that G is an embedded submanifold. More precisely, we aim to show that one can find an open neighborhood V of the origin in $\mathfrak{g}$, and a similar one V' in $\mathfrak{g}'$, so that

$$\phi : V \times V' \to GL_n, \phi(X, X') = e^X \cdot e^{X'}$$

is a local diffeomorphism onto an open neighborhood of the identity matrix and so that

$$G \cap \phi(V \times V') = \{\phi(v, 0) : v \in V\}. \tag{2.5.13}$$

This means that the submanifold condition is verified around the identity matrix and then, using left translations, it will hold at all points of G. Note also that the differential of ϕ at 0 (with ϕ viewed as a map defined on the entire $\mathfrak{g} \times \mathfrak{g}'$) is the identity map:

$$(d\phi)_0 : \mathfrak{g} \times \mathfrak{g}' \to T_I GL_n = \mathscr{M}_{n\times n}(\mathbb{R}), (X, X') \mapsto \frac{d}{dt}_{t=0} e^{tX} e^{tX'} = X + X'.$$

In particular, ϕ is indeed a local diffeomorphism around the origin. Hence the main condition that we have to take care of is (2.5.13), i.e. that if $X \in V$, $X' \in V'$ satisfy $e^X \cdot e^{X'} \in G$, then $X' = 0$. Hence it suffices to show that one can find V' so that

$$X' \in V', e^{X'} \in G \Longrightarrow X' = 0$$

(then just choose any V small enough such that ϕ is a diffeomorphism from $V \times V'$ into an open neighborhood of the identity in GL_n). For that we proceed by contradiction. If such V' would not exist, we would find a sequence $Y_n \to 0$ in $\mathfrak{g}'$ such that $e^{Y_n} \in G$. Since Y_n converges to 0, we find a sequence of integers $k_n \to \infty$ such that $X_n := k_n Y_n$ stay in a closed bounded region of $\mathfrak{g}'$ not containing the origin (e.g. on $\{X \in \mathfrak{g}' : 1 \leq ||X|| \leq 2\}$, for some norm on $\mathfrak{g}'$); then, after eventually passing to a subsequence, we may then assume that $X_n \to X \in \mathfrak{g}'$ for non-zero X. On the other hand, setting $t_n = \frac{1}{k_n}$, we see that $e^{t_n X_n} = e^{Y_n} \in G$ hence Lemma 2.137 would imply $X \in \mathfrak{g}$, providing the desired contradiction. 😀

For closed subgroups $G \subset GL_n$, the spaces $\mathfrak{g}$ used in the proof above (given by (2.5.12)) are not just a tool but, as we shall see above, are of fundamental importance: they are are the linear counterpart of Lie groups and, together with the relevant algebraic structure, they contain almost all the information on G. The relevant structure is, besides that of vector space, the one provided by the commutator of matrices:

$$[X,Y] = XY - YX.$$

Using an argument similar to the one we used to prove that $\mathfrak{g}$ is a vector subspace of $\mathscr{M}_{n\times n}$, you can now try to show the following:

Exercise 2.138. Given a closed subgroup $G \subset GL_n$, show that $\mathfrak{g}$ is closed under the commutator bracket of matrices:

$$X, Y \in \mathfrak{g} \Longrightarrow [X,Y] \in \mathfrak{g}.$$

(Hint: use an argument similar to the one we used to prove that $\mathfrak{g}$ is a vector subspace of $\mathscr{M}_{n\times n}$, making use of Lemma 2.137).

Chapter 3
Tangent spaces

3.1 Intro

So far, the fact that we dealt with general manifolds M and not just embedded submanifolds of Euclidean spaces did not pose any serious problem: we could make sense of everything right away, by moving (via charts) to opens inside Euclidean spaces.

However, when trying to make sense of ”tangent vectors”, the situation is quite different. Indeed, already intuitively, when trying to draw a ”tangent space” to a manifold M, we are tempted to look at M as part of a bigger (Euclidean) space and draw planes that are ”tangent” (in the intuitive sense) to M. The fact that *the tangent spaces of M are intrinsic to M and not to the way it sits inside a bigger ambient space* is remarkable and may seem counterintuitive at first. And it shows that the notion of tangent vector is indeed of a geometric, intrinsic nature. Given our preconception (due to our intuition), the actual general definition may look a bit abstract. For that reason, we describe several different (but equivalent) approaches to tangent spaces. Each one of them starts with one intuitive perception of tangent vectors on $\mathbb{R}^m$, and then proceeds by concentrating on the main properties (to be taken as axioms) of the intuitive perception.

However, before proceeding, it is useful to clearly state what we expect- i.e. a list of ”wishes” for whatever we are looking for:

- **TW1. Tangent spaces**: for M-manifold, $p \in M$, have a vector space T_pM.
- **TW2. Differentials**: for smooths map $F : M \to N$ (between manifolds), $p \in M$, have an induced linear map

$$(dF)_p : T_pM \to T_{F(p)}N,$$

  called the differential of F at p. For the differential we expect, like inside $\mathbb{R}^m$:
  - $F = \mathrm{Id}_M : M \to M$ is the identity map, $(dF)_p$ should be the identity map of T_pM.
  - **The chain rule**: for $M \xrightarrow{F} N \xrightarrow{G} P$ smooth maps between manifold and $p \in M$,

$$(d\,G \circ F)_p = (dG)_{F(p)} \circ (dF)_p.$$

- **TW3. Nothing new in** $\mathbb{R}^m$: Of course, for embedded submanifolds $M \subset \mathbb{R}^{\tilde{m}}$, these tangent spaces should be (canonically) isomorphic to the previously defined tangent spaces T_pM and dF should become the usual differential.

We also expect that embeddings $N \hookrightarrow M$ induce injective maps $T_pN \hookrightarrow T_pM$, but this will turn out to be the case once one imposes the milder condition that, for any open $U \subset M$, the resulting maps $T_pU \to T_pM$ are isomorphisms.

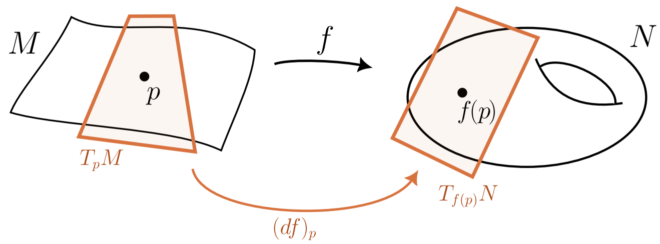


Fig. 3.1 Tangent space at p of a manifold M as well as at the image $f(p)$ under a smooth map $f: M \to N$. We want the differential $(df)_p$ of f at p to be a linear map between these.

Exercise 3.1. Show that, whatever construction of the tangent spaces T_pM we give so that it satisfies the previous properties, for any diffeomorphism $F: M \to N$ and any $p \in M$, the differential of F at p,

$$(dF)_p : T_pM \to T_{F(p)}N,$$

is a linear isomorphism. Deduce that for any m-dimensional manifold M, T_pM is an m-dimensional vector space.

However, even before these properties, one should keep in mind that the way we should think intuitively about tangent spaces is:

> *T_pM is made of speeds (at $t = 0$) of curves γ in M passing through p at $t = 0$.*

This is clear and precise in the case when $M \subset \mathbb{R}^{\tilde{m}}$, where the embedding was used to make sense of "speeds". For conciseness, we introduce the notation

$$\mathbf{Curves}_p(M) := \{\gamma : (-\varepsilon, \varepsilon) \to M \ \textbf{smooth, with} \ \varepsilon > 0, \gamma(0) = p\}$$

(valid for all manifold M and $p \in M$). Therefore, one would like to add to the previous list of wishes the following:

- **TW4. Speeds of curves (leading principle!):** for any curve γ in M that starts at p, "the speed $\frac{d\gamma}{dt}(0)$" should make sense as an element of T_pM, and

$$T_pM = \left\{ \text{"}\frac{d\gamma}{dt}(0)\text{"} : \gamma \in \mathrm{Curves}_p(M) \right\}, \tag{3.1.1}$$

  so that we can think of tangent vectors at p as "speeds of curves that start at p".

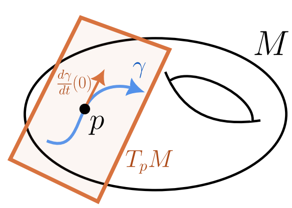


Fig. 3.2 We want to construct the tangent space in a point $p \in M$ in such a way that vectors correspond to the speeds $\frac{d\gamma}{dt}(0)$ of curves $\gamma : (-\varepsilon, \varepsilon) \to M$ that pass through p at time $t = 0$. Note that the image of the curve determines the direction of the vector, but not it's length: A reparametrized curve that passes through the same image twice as fast will correspond to a vector of twice the size.

Remark 3.2 (Extra-for the interested student: tangent spaces very abstractly: via equivalence relations). One way to introduce tangent spaces is by realizing the "leading principle" mentioned above by "brute force", starting from the following remark: although "the speed of γ" (at $t = 0$) is not defined yet, we can make sense right away of the property that two such curves $\gamma_1, \gamma_2 \in \mathrm{Curves}_p(M)$ have the same speed at $t = 0$: if their representations γ_1^χ and γ_2^χ with respect to any chart χ around p have this property:

$$\frac{d\gamma_1^\chi}{dt}(0) = \frac{d\gamma_2^\chi}{dt}(0).$$

(recall that, for $\chi : U \to \Omega \subset \mathbb{R}^m$, $\gamma_i^\chi = \chi \circ \gamma_i$ is a curve in $\Omega \subset \mathbb{R}^m$). It is not difficult to see (exercise!) that it suffices to check this condition for one chart χ. When this happens, we write

$$\gamma_1 \sim_p \gamma_2,$$

and another simple exercise is to check that $\sim_p$ is an equivalence relation. The "brute force" approach to tangent spaces would then be to define:

- T_pM as the quotient of $\mathrm{Curves}_p(M)$ modulo the equivalence relation $\sim_p$.
- the speed $\frac{d\gamma}{dt}(0)$ as the $\sim_p$ equivalence class of γ.

Then equality (3.1.1) that corresponds to our intuition is forced in tautologically. Although this does work, and it is sometimes taken as definition in some text-books/lecture notes, we find it too abstract for such an intuitive concept. Instead, we will discuss the tangent space via charts, as well as via derivations. Nevertheless, one should always keep in mind the intuition via speeds of paths (and make it precise in whatever model we use). □

3.2 Tangent vectors via charts

The brief philosophy of this approach is:

while a chart χ for M allows one to represent points p by coordinates,
it will allow us to represent tangent vectors $v \in T_pM$ by vectors in $\mathbb{R}^m$

3.2.1 The general notion

Concentrating on the tangent vectors rather than on the charts, the previous "slogan" reads:

a tangent vector to M at p can be seen
through each chart, as a vector in the Euclidean space $\mathbb{R}^m$

Remark 3.3 (bla, bla, bla- leading to the actual motivation/first definition of tangent spaces; i.e., while this is hopefully insightful, in principle you may just skip this and jump right to the definition). The actual definition can be discovered as the outcome of "wishful thinking": we want to extend the definition of T_pM from the case of embedded submanifolds of Euclidean spaces to arbitrary manifolds M, satisfying the "wish list" TW1 and TW2 above. We see that, for a general manifold M and $p \in M$, choosing a chart (U, χ) for M around p and viewing χ as a smooth map from (an open inside) M to (an open inside) $\mathbb{R}^m$, the wishful thinking tells us that we expect an induced map (even an isomorphism)

$$(d\chi)_p : T_pM \to \mathbb{R}^m.$$

Therefore, an element $v \in T_pM$ can be represented w.r.t. to the chart χ by a vector in the standard Euclidean space:

$$v^\chi := (d\chi)_p(v) \in \mathbb{R}^m.$$

What happens if we change the chart χ by another chart χ' around p? Then $v^{\chi'}$ should be changed in a way that is dictated by the change of coordinates

$$c = c_\chi^{\chi'} = \chi' \circ \chi^{-1}.$$

Indeed, since $\chi' = c \circ \chi$ and we want the chain rule still to hold, we would have:

$$v^{\chi'} = (d\chi')_p(v) = (d\,c)_{\chi(p)}\left((d\chi)_p(v)\right),$$

i.e.

$$v^{\chi'} = (d\,c)_{\chi(p)}(v^\chi). \tag{3.2.1}$$

Therefore, whatever the meaning of T_pM is, we expect that the elements $v \in T_pM$ can be represented w.r.t local charts χ (around p) by vectors $v^\chi \in \mathbb{R}^m$ which, when we change the chart, transform according to (3.2.1). Well, if that is what we want from T_pM, let us just define it that way! □

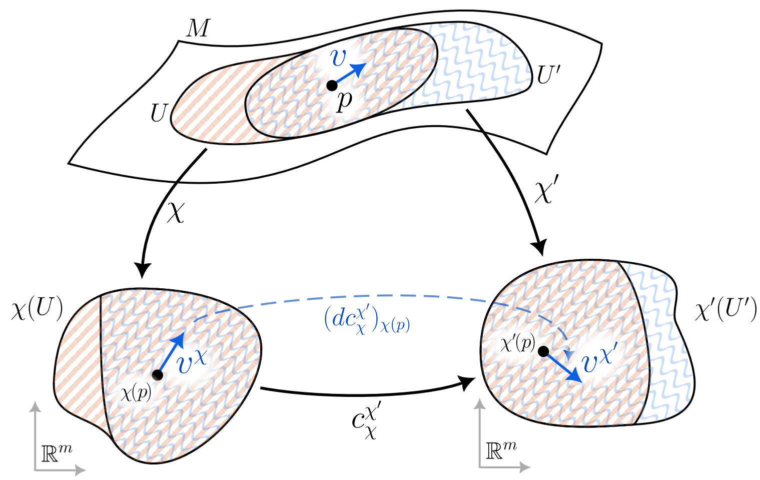


Fig. 3.3 A tangent vector v at a point $p \in M$ assigns to every chart $\chi : U \to \chi(U)$ around p a representation v^χ. When changing coordinates to a new chart $\chi' : U' \to \chi'(U')$ along $c_\chi^{\chi'}$, this representation transforms into $v^{\chi'}$ by applying the differential $(dc_\chi^{\chi'})_{\chi(p)}$. Note that for a map between Euclidean spaces such as $c_\chi^{\chi'}$, the differential is just the total derivative in the usual sense.

Definition 3.4. Given an m-dimensional manifold M and $p \in M$. We denote by

$$\mathrm{Chart}_p(M)$$

the collection of all charts χ of M whose domain contains the point p. A **tangent vector of** M **at** p is a function v

$$v : \mathrm{Chart}_p(M) \to \mathbb{R}^m, \quad \chi \mapsto v^\chi,$$

with the property that, for any two charts $\chi, \chi' \in \mathrm{Chart}_p(M)$ one has

$$v^{\chi'} = (d\,c)_{\chi(p)}(v^\chi), \tag{3.2.2}$$

where $c = c_\chi^{\chi'}$ is the change of coordinates from χ to χ'.

We denote by T_pM the vector space of all such tangent vectors of M at p- made into a vector space using the vector space structure on $\mathbb{R}^m$ (i.e. $(v+w)^\chi := v^\chi + w^\chi$, etc.)

Some comments are in order here. First of all, v^χ is the notation that we use for the value of v on the chart χ; the vector $v^\chi \in \mathbb{R}^m$ is also called **the representation of the tangent vector v with respect to the chart χ**, or simply v **on the chart χ**. Note also that, as above, $c = \chi' \circ \chi^{-1}$ is a smooth map from an open in $\mathbb{R}^m$ containing $\chi(p)$, to another open in $\mathbb{R}^m$), so that $(d\,c)_{\chi(p)}$ appearing in the condition (3.2.2) is the usual differential defined as a linear map

$$(d\,c)_{\chi(p)} : \mathbb{R}^m \to \mathbb{R}^m.$$

In more detail, the vector space structure on T_pM is defined as follows: for $v, w \in T_pM$, and a scalar $\lambda \in \mathbb{R}$, the sum $v+w$ and the multiplication $\lambda \cdot v$,

$$v+w, \quad \lambda \cdot v \in T_pM,$$

are defined using the similar operations $+$ and $\cdot$ from $\mathbb{R}^m$:

$$(v+w)^\chi := v^\chi + w^\chi, \quad (\lambda \cdot v)^\chi := \lambda \cdot v^\chi.$$

Staring for a minute at the condition (3.2.2) imposed on tangent vectors $v \in T_pM$, we see that once we know v^χ for one single chart $\chi \in \mathrm{Chart}_p(M)$, we completely know v. Actually, we obtain:

Lemma 3.5. *If we fix a chart $\chi_0 \in \mathrm{Chart}_p(M)$, then*

$$T_pM \to \mathbb{R}^m, \quad v \mapsto v^{\chi_0}$$

is an isomorphism of vector spaces.

Proof. As we just pointed out, once we fixed χ_0, and given $v_0 \in \mathbb{R}^m$, the element $v \in T_pM$ such that $v^{\chi_0} = v_0$ is unique (if it exists) and it must be given by the formula

$$v^\chi = (d\,c^\chi_{\chi_0})_{\chi_0(p)}(v_0),$$

for any other chart χ around p. Strictly speaking we still prove (3.2.2) but that follows from the chain rule and the remark that $c^{\chi'}_{\chi_0} = c^{\chi'}_{\chi} \circ c^{\chi}_{\chi_0}$. 😀

We deduce in particular that any chart $\chi \in \mathrm{Chart}_p(M)$ gives rise to a basis

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p \in T_pM \tag{3.2.3}$$

of the tangent space T_pM, which corresponds to the canonical basis $e_1, \ldots, e_m$ of $\mathbb{R}^m$ via the isomorphism from the previous lemma. In other words,

$$\left(\frac{\partial}{\partial \chi_i}\right)_p \in T_pM \text{ is the unique vector which, on the chart } \chi, \text{ is } \left(\frac{\partial}{\partial \chi_i}\right)_p^\chi = e_i.$$

The basis (3.2.3) is called **the canonical basis of T_pM with respect to χ**.

Exercise 3.6. If χ and χ' are two charts of M around p, $c = c^{\chi'}_{\chi}$, show that

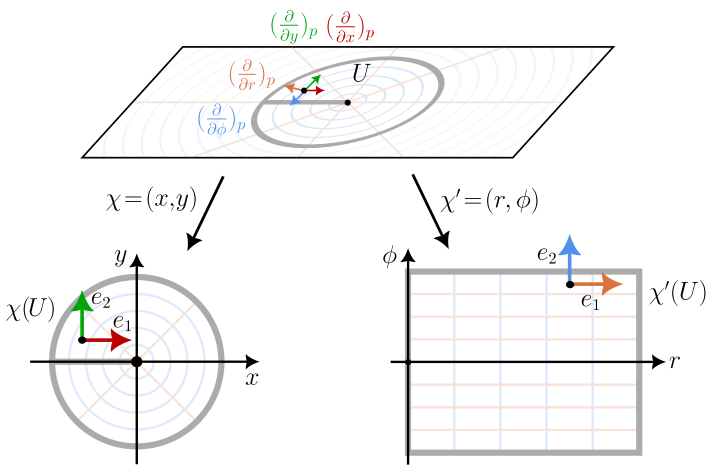


Fig. 3.4 In the situation of Fig. 2.3, we fix a point $p \in U$ and can consider the vectors based at p that are represented by unit vectors within the identity and polar coordinates. They always point in the direction in which the corresponding coordinate increases.

$$\left(\frac{\partial}{\partial \chi_i}\right)_p = \sum_j \frac{\partial c_j}{\partial x_i}(\chi(p)) \left(\frac{\partial}{\partial \chi'_j}\right)_p. \tag{3.2.4}$$

3.2.2 *The case of* $\mathbb{R}^m$ *(the standard identifications)*

For $M = \mathbb{R}^m$ (or an open in $\mathbb{R}^m$) and $p \in M$, one can use the previous lemma combined with the fact that $\mathbb{R}^m$ comes with a standard chart (around any point!): the identity chart Id. One obtains

- an isomorphism
$$\text{standard}_p : T_p\mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^m, \quad v \mapsto v^{\text{Id}} \tag{3.2.5}$$
- a canonical basis, denoted
$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p\mathbb{R}^m$$

These will be called **the standard identification** of the abstract tangent space $T_p\mathbb{R}^m$ with $\mathbb{R}^m$, and **standard basis** of $T_p\mathbb{R}^m$. Of course, the standard identification just picks up the coefficients with respect to the standard basis:

$$\sum_i v_i \left(\frac{\partial}{\partial x_i}\right)_p \mapsto (v_1, \ldots, v_m).$$

The inverse map translates into the fact that any vector $w \in \mathbb{R}^m$ gives rise to a tangent vector

$$\widetilde{w} \in T_p\mathbb{R}^m,$$

uniquely determined the condition that, with respect to the identity chart, $\widetilde{w}^{\text{Id}} = w$. Explicitly, on an arbitrary chart χ of $\mathbb{R}^m$ around p (now just a smooth map from an open in $\mathbb{R}^m$ containing p, to $\mathbb{R}^m$),

$$\widetilde{w}^{\chi} := (d\chi)_p(w) \in \mathbb{R}^m.$$

As pointed out at the begining, the same discussion applies to any open inside Euclidean spaces (why open?), giving rise to standard identifications

$$\mathbf{standard}_p^{\Omega} : T_p\Omega \xrightarrow{\sim} \mathbb{R}^m, \quad v \mapsto v^{\mathbf{Id}} \quad (\textbf{for } \Omega \subset \mathbb{R}^m \textbf{ open}). \tag{3.2.6}$$

3.2.3 Speeds of curves

Let us show that, indeed, the definition of the tangent spaces from this section accommodates the leading principle: we can make sense of

$$\frac{d\gamma}{dt}(0) \in T_pM$$

for $\gamma \in \mathrm{Curves}_p(M)$. The actual definition should be clear: the representation of this vector with respect to a chart χ should be the speed (at $t = 0$) of the representation γ^χ of γ with respect to χ. In more detail: if $\chi : U \to \Omega$ is a chart of M around p, consider the resulting curve $\gamma^\chi = \chi \circ \gamma$ in $\Omega \subset \mathbb{R}^m$, and take its derivatives at 0- which is a vector in $\mathbb{R}^m$. This gives an function

$$\mathrm{Chart}_p(M) \to \mathbb{R}^m, \quad \chi \mapsto \frac{d\gamma^\chi}{dt}(0) = \frac{d}{dt}\bigg|_{t=0} \gamma^\chi(t),$$

which is clearly linear and satisfies the Leibniz identity; therefore we obtain a tangent vector denoted

$$\frac{d\gamma}{dt}(0) \in T_pM.$$

In a short formula:

$$\left(\frac{d\gamma}{dt}(0)\right)^\chi := \frac{d\gamma^\chi}{dt}(0).$$

Of course, there is nothing special for $t = 0$, and in the same way one talks about

$$\frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M.$$

for all t in the domain of γ.

Exercise 3.7. For $M = \mathbb{R}^m$ show that the abstract derivative $\frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^m$ defined above corresponds, via the standard identification (3.2.6), to the usual derivative at 0 of γ.

Then check that for any $w \in \mathbb{R}^m$, the corresponding tangent vector at p (i.e. $\widetilde{w} \in T_p\mathbb{R}^m$ corresponding to w via the standard identification (3.2.6)) coincides with the speed (in the sense we have just defined) of the curve $t \mapsto p + t \cdot w$:

$$\tilde{w} = \frac{d}{dt}\bigg|_{t=0} (p + t \cdot w).$$

3.2.4 Differentials

Finally, let us show show that, as desired, smooth maps induce differentials at the level of tangent spaces.

Lemma 3.8. *Given a smooth map $F : M \to N$ between manifolds, and given $p \in M$, there exists a linear map*

$$(dF)_p : T_pM \to T_{F(p)}N, \quad v \mapsto (dF)_p(v)$$

unique determined by the property that, for any $\chi \in \mathrm{Chart}_p(M)$ *and* $\chi' \in \mathrm{Chart}_{F(p)}(N)$*, one has:*

$$((dF)_p(v))^{\chi'} = (dF_{\chi}^{\chi'})_{\chi(p)}(v^{\chi})$$

for all $v \in T_pM$*.*

Proof. Fixing v and χ, the required condition forces the definition of $(dF)_p(v)$; to see that one ends up with a tangent vector to N at $F(p)$ (i.e. the condition (3.2.2) is satisfied in this context) one uses again the chain rule. One still has to check that the resulting map does not depend on the choice of χ- but that follows again from the chain rule. 😀

Definition 3.9. For a smooth map $F : M \to N$ and $p \in M$, the resulting linear map $(dF)_p : T_pM \to T_{F(p)}N$ is called **the differential of** F **at the point** p.

Exercise 3.10. Show that the differential of smooth functions has the following properties:

1. the chain rule continues to hold: for $M \xrightarrow{F} N \xrightarrow{G} P$, $p \in M$,

$$(dG \circ F)_p = (dG)_{F(p)} \circ (dF)_p.$$

2. if F is a (local) diffeomorphism than $(dF)_p$ is an isomorphism.
3. when $F : M \to N$ with M and N opens in Euclidean spaces then, via the standard identifications (3.2.6) for M and N, dF is identified with the standard differential DF.

$$\begin{array}{ccc} T_pM & \xrightarrow{(dF)_p} & T_{F(p)}N \\ {\scriptstyle \mathrm{standard}_p^M}\downarrow\sim & & \sim\downarrow{\scriptstyle \mathrm{standard}_{F(p)}^N} \\ \mathbb{R}^m & \xrightarrow[(DF)_p]{} & \mathbb{R}^n \end{array}$$

4. $(dF)_p$ is uniquely characterized also by the condition that, for any $\gamma \in \mathrm{Curves}_p(M)$, one has:

$$(dF)_p\left(\frac{d\gamma}{dt}(0)\right) = \frac{dF \circ \gamma}{dt}(0).$$

Note that, with the general notion of tangent spaces and differentials at hand, one has a neater characterization of immersions and submersions (which is actually taken as definition in some text-books- with the disadvantage that one has to wait until those concepts are introduced).

Proposition 3.11. *A smooth map* $F : M \to N$ *between two manifolds is an immersion (or submersion) at a point* $p \in M$ *if and only if* $(dF)_p$ *is injective (or surjective, respectively).*

Proof. Choose coordinate charts (U, χ) for M around p (U', χ') for N around $q := F(p)$ such that $F(U) \subset U'$, so that we can consider the restriction

$$F|U : U \to U'.$$

It suffices to prove the statement for this restricted map (why?). On the other hand, since $\chi : U \to \Omega$ and $\chi' : U' \to \Omega'$ are diffeomorphism, $(d\chi)_p$ and $(d\chi')_q$ are isomorphisms.

$$\begin{array}{ccc} T_pU & \xrightarrow{(dF)_p} & T_qU' \\ {\scriptstyle (d\chi)_p}\downarrow & & \downarrow{\scriptstyle (d\chi')_q} \\ T_{\chi(p)}\Omega & \xrightarrow[(dF_{\chi}^{\chi'})_{\chi(p)}]{} & T_{\chi'(q)}\Omega' \end{array}$$

We see that we can pass further to Euclidean spaces- i.e. it suffices to prove the statement for

$$F_{\chi}^{\chi'} : \Omega \to \Omega'$$

(around the point $\chi(p)$). However, the differential of this function is identified, via the standard identifications, to the usual differential (cf. the previous exercise) therefore, what we have to check becomes the actual definition of immersions/submersions. 😀

3.2.5 Submanifolds $M \subset N$

We apply the previous discussion to the inclusion of embedded submanifolds: whenever $M \subset N$ is an embedded submanifold of a manifold N, the inclusion $i : M \hookrightarrow N$ is a smooth map and, therefore, at each $p \in M$, it induces an injective map

$$(di)_p : T_pM \to T_pN.$$

This allows one to promote a vector $v \in T_pM$ tangent to a submanifold M to a tangent vector for the larger manifold N. Actually, while for $p \in M$, $i(p) \in N$ is still (denoted by) p, a tangent vector $v \in T_pM$ is still denoted by v when interpreted as a vector tangent to N (i.e. $(di)_p(v)$ is still denoted by v); in particular, we think of T_pM as a subspace of T_pN:

$$T_pM \subset T_pN.$$

This makes the notation somehow simpler and agrees with the intuition. Note that, thinking of tangent spaces as consisting of speeds of curves, the previous inclusion is rather tautological as the curves in M are curves also in N:

$$\left\{ \frac{d\gamma}{dt}(0) : \gamma \in \text{Curves}_p(M) \right\} \subset \left\{ \frac{d\gamma}{dt}(0) : \gamma \in \text{Curves}_p(N) \right\}.$$

Exercise 3.12. With the previous identifications and notations in mind show that, for $M \subset N$ embedded submanifold, $p \in M$, $v \in T_pM$, v interpreted as a vector tangent to N has the following property that determines it uniquely: on any chart χ of N adapted to M, it is precisely v on the induced chart $\chi|_M$ for M.

The previous discussion can be applied in particular to embedded submanifolds of Euclidean spaces

$$M \subset \mathbb{R}^n.$$

The resulting inclusion of $T_pM \subset T_p\mathbb{R}^n$ combined with the standard identification of $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ from (3.2.5) gives rise to:

$$\text{standard}_p^M := \text{standard}_p|_{T_pM} : T_pM \to \mathbb{R}^n. \tag{3.2.7}$$

Lemma 3.13. *For any embedded submanifold $M \subset \mathbb{R}^n$, the map (3.2.7) is injective and its image is precisely the classical tangent space $T_p^{class}M$ from Chapter 1 (Definition 1.43). In other words, it becomes an isomorphism*

$$standard_p^M : T_pM \xrightarrow{\sim} T_p^{class}M.$$

Proof. Defined as a composition of an injective map with an isomorphism, standard$_p^M$ is injective. By the very definition of the map we see that, choosing a chart (U, χ) for $\mathbb{R}^n$ around p, adapted to M, with induced chart on M denoted $\chi|_M$, 😀

Exercise 3.14. Please revisit some of the previous exercises on immersions/submersions and find now more elegant/simpler arguments:

1. The height function on S^2 (Exercise 2.99) and on the torus (Exercise 2.71),
2. The explicit Hopf map h from Exercise 2.110,
3. Part (c) of Exercise 2.73,
4. And then also parts (b) and (c) of Exercise 2.74.

3.3 Tangent vectors as directional derivatives

Here is another approach to the tangent spaces; the brief philosophy of this approach is:

> *tangent vectors at p allow us to take directional derivatives (at p) of smooth functions*

3.3.1 The general notion

To implement the previous "slogan" we return to the differential

$$(df)_p(v)$$

of a smooth function $f : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^m$ open) at a point $p \in \Omega$, applied to a <u>(tangent) vector</u> $v \in \mathbb{R}^m$, that we reinterpreted it as a v-derivative at p

$$(df)_p(v) = \frac{\partial f}{\partial v}(p) = \partial_v(f)(p).$$

Hence, if we want to understand what the (tangent) vector v really does (at p), one may say that it defines a function

$$\partial_v = \frac{\partial}{\partial v} : \mathscr{C}^\infty(\Omega) \to \mathbb{R};$$

What are the main properties of this function? Well, it is clearly linear and, moreover, it acts on the product of functions according to the Leibniz rule at p:

$$\partial_v(fg) = f(p) \cdot \partial_v(g) + g(p) \cdot \partial_v(g) \tag{3.3.1}$$

for all $f, g \in \mathscr{C}^\infty(\Omega)$. Note that here we are making use of the following algebraic structures on $C^\infty(\Omega)$: the vector space structure (for linearity) and the product of functions (to make sense of the Leibniz identity at p); all together, we are making use of the "algebra structure" on $C^\infty(\Omega)$. The main point is that, conversely, any map

$$\partial : \mathscr{C}^\infty(\Omega) \to \mathbb{R}$$

which is linear and satisfies the Leibniz identity at p is of type ∂_v for a unique vector $v \in \mathbb{R}^m$ (this will follow from the discussion below). This gives another perspective/possible approach to tangent spaces.

Definition 3.15. Given an m-dimensional manifold M and $p \in M$, a **derivation of M at p** is any linear map

$$\partial : \mathscr{C}^{\infty}(M) \to \mathbb{R}$$

which is linear and which satisfies the Leibniz identity (3.3.1) at p.

We denote by $T_p^{\mathbf{deriv}}M$ the vector space of all such derivations M at p- made into a vector space using the usual addition and multiplication by scalars from $\mathbb{R}$.

In more detail, the vector space structure on $T_p^{\text{deriv}}M$ is defined as follows: for $\partial, \partial' \in T_p^{\text{deriv}}M$, and a scalar $\lambda \in \mathbb{R}$, the sum $\partial + \partial' \in T_p^{\text{deriv}}M$ and the multiplication $\lambda \cdot \partial \in T_p^{\text{deriv}}M$, are defined using the similar operations $+$ and $\cdot$ from $\mathbb{R}$:

$$(\partial + \partial')(f) := \partial(f) + \partial'(f), \quad (\lambda \cdot \partial)(f) := \lambda \cdot \partial(f).$$

3.3.2 The case of Euclidean spaces

When $M = \Omega$ is an open in $\mathbb{R}^m$ (endowed with the canonical smooth structure), $p \in \Omega$, then the operation of taking the usual partial derivatives at p,

$$\left(\frac{\partial}{\partial x_i}\right)_p : \mathscr{C}^{\infty}(U) \to \mathbb{R}, \quad f \mapsto \frac{\partial f}{\partial x_i}(p),$$

are derivations at p- hence they can be interpreted as vectors

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p^{\text{deriv}}\Omega.$$

As it will follow from the discussion below, they form a basis of $T_p^{\text{deriv}}\Omega$.

Similarly, for any manifold M, a chart $\chi : U \to \Omega \subset \mathbb{R}^m$ of M around p induces **partial derivatives at p w.r.t. the chart χ**,

$$\frac{\partial f}{\partial \chi_i}(p) := \frac{\partial f_{\chi}}{\partial x_i}(\chi(p)),$$

where $f_{\chi} = f \circ \chi^{-1} : \Omega \to \mathbb{R}$. And then vectors

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p \in T_p^{\text{deriv}}M \tag{3.3.2}$$

which form a basis of $T_p^{\text{deriv}}M$- but that is still to be proven; here we take care of the case $M = \mathbb{R}^m$:

Proposition 3.16. *For any $p \in \mathbb{R}^m$, the vectors (3.3.2) form a basis of $T_p^{deriv}\mathbb{R}^m$.*

Proof. For notational simplicity we assume that $p = 0$. For the linear independence we assume that $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ are scalars with the property that

$$\lambda_1 \cdot \left(\frac{\partial}{\partial x_1}\right)_0 + \ldots + \lambda_m \cdot \left(\frac{\partial}{\partial x_m}\right)_0 = 0$$

and we have to prove that each λ_i vanishes. For that one applies the left hand side of the previous equality (a derivation at p!) to the coordinate function $x \mapsto x_i$ and we obtain that, indeed, $\lambda_i = 0$.

We still have to check that any derivation at 0,

$$\partial : \mathscr{C}^\infty(\mathbb{R}^m) \to \mathbb{R},$$

is a linear combination of the standard partial derivatives at 0:

$$\partial = \sum_i \lambda_i \left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x_i} \right)_0 \quad \text{with } \lambda_i \in \mathbb{R}.$$

We will show that $\lambda_i := \partial(x_i) \in \mathbb{R}$ do the job. To show this, we use the fact that any $f \in \mathscr{C}^\infty(\mathbb{R}^m)$ can be written as

$$f(x) = f(0) + \sum_i x_i g_i(x), \quad \text{with } g_i \in \mathscr{C}^\infty(\mathbb{R}^m) \tag{3.3.3}$$

(see below). Using the Leibniz identity (which also implies that ∂ is 0 on constant functions) we obtain

$$\partial(f) = \sum_i \lambda_i g_i(0),$$

while $\frac{\partial f}{\partial x_i}(0) = g_i(0)$. Hence we are left with proving that any f can be written in the form (3.3.3). For this we use the identity

$$h(1) - h(0) = \int_0^1 \frac{dh}{dt}(t)dt$$

for all smooth functions $h : [0,1] \to \mathbb{R}$; of course, we want to choose h such that $h(1) = f(x)$ and $h(0) = f(0)$. Choose then $h(t) = f(tx)$ and we obtain the desired identity (3.3.3) with

$$g_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt.$$

😀

3.3.3 Speeds of curves

Let us point out, right away, that also this model for tangent spaces serves the original purpose: we can make sense of speeds

$$\frac{\boldsymbol{d\gamma}}{\boldsymbol{dt}}(0) = \partial_{\frac{d\gamma}{dt}(0)} \in T_p^{\text{deriv}} M \tag{3.3.4}$$

for $\gamma \in \text{Curves}_p(M)$. The bold colour is here to distinguish this notion of speed from the one from section 3.2.3. It is true that the two will be identified but, even then, we sometimes still prefer a different notation for the speed when we want to interpret it as a derivation. Therefore, from now on, for (3.3.4), we will be using the notation $\partial_{\frac{d\gamma}{dt}(0)}$. The actual definition should be clear: just consider the variation of functions along γ (at $t = 0$):

$$\partial_{\frac{d\gamma}{dt}(0)} : \mathscr{C}^\infty(M) \to \mathbb{R}, \quad \partial_{\frac{d\gamma}{dt}(0)}(f) := \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)).$$

Of course, one can use the derivative at any t on the domain of γ, giving rise to

$$\partial_{\frac{d\gamma}{dt}(t)} \in T_{\gamma(t)}^{\text{deriv}} M.$$

3.3.4 Differentials

Let us also explain show how smooth maps induce differentials at the level of tangent spaces as in TW2 above, denoted $(\boldsymbol{d}F)_p : T_p^{\text{deriv}}M \to T_{F(p)}^{\text{deriv}}N$, where we use a slightly different green notation to distinguish it from the differentials from the previous section (soon to be identified with each other). Although the discussion is a bit tedious, especially if you never looked at duals of duals, or functions on a set of functions, everything is based on a simple remark: any smooth map $F : M \to N$ between two manifolds induces a map at the level of smooth functions,

$$F^* : \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(M), \quad g \mapsto F^*(g) := g \circ F,$$

that is compatible with the algebraic structures. It follows that if

$$\partial : \mathscr{C}^\infty(M) \to \mathbb{R}$$

is a derivation of M at a point $p \in M$ then

$$\partial \circ F^* : \mathscr{C}^\infty(N) \longrightarrow \mathbb{R}, \quad g \mapsto \partial(F^*(g)) = \partial(g \circ F)$$

is a derivation of N at $F(p)$. Denoting this derivation by

$$(\boldsymbol{d}F)_p(\partial) \in T_{F(p)}^{\text{deriv}}N,$$

one obtains the desired differential (of F at p)

$$(\boldsymbol{d}F)_p : T_p^{\text{deriv}}M \to T_{F(p)}^{\text{deriv}}N, \quad \partial \mapsto (\boldsymbol{d}F)_p(\partial) = \partial \circ F^*.$$

Exercise 3.17. Fill in the missing details and prove that this construction has the desired properties from TW2 above (including the chain rule).

3.3.5 Extra-for the interested student: the algebraic nature of the construction

It is clear that the model for the tangent space provided by $T_p^{\text{deriv}}M$ is more algebraic. Let us make this a bit more explicit. This discussion fits in very well with the previous discussions on the algebra of smooth functions (Remark 2.40), going back all the way to the "Gelfand-Naimark philosophy" (see also Remark 1.10).

Looking at the definition of $T_p^{\text{deriv}}M$ and remembering that evaluation at p can be seen as a character χ_p of $\mathscr{C}^\infty(M)$, there is a clear way one can proceed in general: for any algebra A and any character χ on A we introduce the notion of χ-derivation on A, by which we mean a linear map $\chi : A \to \mathbb{R}$ satisfying the derivation identity:

$$\partial(ab) = \chi(a)\partial(b) + \partial(a)\chi(b)$$

for all $a, b \in A$. We denote by $\text{Der}_\chi(A)$ the (vector) space of all such derivations. Intuitively, this may be interpreted as "the tangent space of $X(A)$ at χ".

When applied to $A = \mathscr{C}^\infty(M)$, while we know (when M is compact) that any character on A is necessarily of type χ_p for some p (unique), the corresponding derivations give

$$\text{Der}_{\chi_p}(\mathscr{C}^\infty(M)) = T_p^{\text{deriv}}M.$$

For this reason, for a general algebra A, one may think of its characters as "points", while for a character χ one may think of $\text{Der}_\chi(A)$ as "the tangent space of $X(A)$ at χ".

Exercise 3.18. When $A = \mathbb{R}[X_1, \ldots, X_m]$ is the algebra of polynomials in m variables show that any character is the evaluation χ_x at some point $x \in \mathbb{R}^m$, and then show that $\text{Der}_{\chi_x}(A)$ is a finite dimensional vector space and exhibit a basis.

Then consider another A: the algebra of polynomial functions on the circle S^1. And, again, consider the characters χ_p for $p \in S^1$ and compute $\text{Der}_{\chi_p}(A)$.

One warning however: while when looking at characters/points it does not make a difference whether we use the algebra $\mathscr{C}^\infty(M)$ or the one of continuous functions, the situation is dramatically different when it comes to tangent vectors:

Exercise 3.19. Show that $\mathrm{Der}_{\chi_p}(\mathscr{C}(M)) = 0$. (Hint: the proof is much shorter than the bla-bla preceeding the exercise)

3.3.6 The local nature of derivations

We now discuss another aspect that $T_p^{\mathrm{deriv}}M$ reveals: the local nature of derivations.

Lemma 3.20. *Any derivation of M at p, $\partial \in T_p^{deriv}M$, has the following local property: for $f_1, f_2 \in \mathscr{C}^\infty(M)$ one has:*

$$f_1 = f_2 \text{ in a neighborhood of } p \Longrightarrow \partial(f_1) = \partial(f_2).$$

Proof. Let $f = f_1 - f_2$. We know that $f = 0$ in a neighborhood U of p and we want to prove that $\partial(f) = 0$. We may assume that U is chosen so that it is diffeomorphic to a ball $B(0,\varepsilon) \subset \mathbb{R}^m$, by a diffeomorphism that takes p to 0. As we have already noticed, one can find a smooth function on $\mathbb{R}^m$ that is supported inside $B(0,\varepsilon)$ and is non-zero at 0 (e.g. take $x \mapsto g(\frac{1}{\varepsilon^2}||x||^2)$, where $g : \mathbb{R} \to \mathbb{R}$ is the function from Exercise 1.28). Moving from $B(0,\varepsilon)$ to U, and extending by zero outside U, we find a function $\eta \in C^\infty(M)$ which is supported inside U and $\eta(p) \neq 0$. Then $\eta f = 0$, and applying the Leibniz identity for ηf we find $\eta(p)\partial(f) = 0$ hence, since $\eta(p) \neq 0$, one must have $\partial(f) = 0$.

Assume that $U \subset M$ is an open containing p. Then any derivation $\partial \in T_p^{\mathrm{deriv}}U$ gives a derivation $\tilde{\partial} \in T_p^{\mathrm{deriv}}M$ simply by

$$\tilde{\partial}(f) := \partial(f|_U).$$

Of course, the resulting map

$$T_p^{\mathrm{deriv}}U \to T_p^{\mathrm{deriv}}M, \quad \partial \mapsto \tilde{\partial}$$

is just the map induced (cf. Exercise 3.17 above) by the inclusion $U \hookrightarrow M$.

Lemma 3.21. *Given $p \in U \subset M$ with U open, the canonical map $T_p^{deriv}U \to T_p^{deriv}M$ is a linear isomorphism.*

Proof. For injectivity, let $\partial : \mathscr{C}^\infty(U) \to \mathbb{R}$ be a derivation at p such that $\partial(\tilde{f}|_U) = 0$ for all $\tilde{f} \in \mathscr{C}^\infty(M)$; we must show that $\partial = 0$. This follows immediately from the following:

- $\partial(f)$ depends only on f in an arbitrarily small neighborhood of p. This is the content of the previous lemma.
- for any $f \in \mathscr{C}^\infty(U)$, there exists $\tilde{f} \in \mathscr{C}^\infty(M)$ s.t. $f = \tilde{f}$ in some neighborhood of p.

To prove the last item, we take $\tilde{f} := \eta \cdot f$ where $\eta \in \mathscr{C}^\infty(M)$ is supported inside U (so that $\tilde{f}$ is defined and smooth on the entire M) and $\eta = 1$ in some (smaller) neighborhood of p. The existence of η is a local problem- therefore it suffice to build a smooth function η on $\mathbb{R}^m$ that is supported in the ball $B(0,1)$ and is 1 in a neighborhood of 0 (say on $B(0,\frac{1}{3})$); for that one takes again η of type $\eta(x) = g(||x||^2)$ where $g : \mathbb{R} \to \mathbb{R}$ is any smooth function that is 1 when $|t| < \frac{1}{3}$ and is 0 when $t \geq \frac{1}{2}$.

For the surjectivity, we start with $\tilde{\partial} : \mathscr{C}^\infty(M) \to \mathbb{R}$ and we want to build ∂. By now the definition should be clear: or $f \in \mathscr{C}^\infty(U)$, choose any $\tilde{f} \in \mathscr{C}^\infty(M)$ that coincides with f near p (possible by the first part) and set $\partial(f) := \tilde{\partial}(\tilde{f})$. The previous lemma shows that the definition of $\partial(f)$ does not depend on the choice of $\tilde{f}$. And this can used also to check that ∂ is still a derivation at p. E.g., for the Leibniz identity, choosing extensions $\tilde{f}$ or f and $\tilde{g}$ or g, then use the extension $\tilde{f}\tilde{g}$ or fg and use the Leibniz identity for $\tilde{\partial}(\tilde{f}\tilde{g})$.

Remark 3.22 (Extra- for the interested student). Lemma 3.20 can be packed into a more conceptual conclusion: any derivations ∂ at p descends to the quotient space

$$\mathscr{C}_p^\infty(M) := \mathscr{C}^\infty(M)/I_p = C^\infty(M)/\sim_p.$$

where I_p is the space of functions that vanish around p and $\sim_p$ is the associated equivalence relation:

$$f_1 \sim_p f_2 \Longleftrightarrow f_1 = f_2 \text{ in a neighborhood of } p.$$

For $f \in \mathscr{C}^\infty(M)$, its equivalence class is denoted

$$\mathrm{germ}_p(f) \in \mathscr{C}^\infty_p(M)$$

and is called **the germ of** f **at** p. The space of germs at p, i.e. our quotient $\mathscr{C}^\infty_p(M)$, is still an algebra (since I_p is an ideal of $\mathscr{C}^\infty(M)$); explicitly, the operations are induced from $\mathscr{C}^\infty(M)$:

$$\mathrm{germ}_p(f+g) = \mathrm{germ}_p(f) + \mathrm{germ}_p(g), \quad etc.$$

The locality property from the previous lemma can now be interpreted as an isomorphism

$$T_p^{\mathrm{deriv}} M \cong \mathrm{Der}_p(\mathscr{C}^\infty_p(M))$$

between the tangent space and the space of all derivations of $\mathscr{C}^\infty_p(M)$ with respect to the evaluation at p; note that the main difference with the similar discussion using $\mathscr{C}^\infty(M)$ is that, this time, $\mathscr{C}^\infty_p(M)$ has a unique character (the evaluation at p). □

3.3.7 Comparing with the previous definition of tangent spaces

There is also a very natural interaction (which turns out to be an isomorphism) with the tangent space from the previous section. The starting remark is that any $v \in T_pM$ gives rise to a derivation at p, ∂_v. To define $\partial_v(f)$, we fix a chart $\chi : U \to \Omega \subset \mathbb{R}^m$ of M around p; then we can use the representation of f with respect to χ, $f_\chi : \Omega \to \mathbb{R}$, as well as the one of v, $v^\chi \in \mathbb{R}^m$ and set:

$$\partial_v(f) := \frac{\partial f_\chi}{\partial v^\chi}(\chi(p)) = (df_\chi)_{\chi(p)}(v^\chi).$$

Similar to previous arguments, using the chain rule one checks that this quantity actually does not depend on the choice of χ. Therefore we obtain

$$\partial_v : \mathscr{C}^\infty(M) \to \mathbb{R};$$

it should be clear that it is a derivation of M at p, hence

$$\partial_v \in T_p^{\mathrm{deriv}} M.$$

This defines a linear map

$$I_p^M : T_pM \to T_p^{\mathrm{deriv}} M, \quad v \mapsto \partial_v.$$

Remark 3.23 (getting some more insight into I_p^M). To get a better feeling about this map, note that:

- It sends the speeds of curves in the sense of the previous section (i.e. $v = \frac{d\gamma}{dt}(0)$) to the speeds of curves as discussed in this section (subsection 3.3.3):

$$I_p^M\left(\frac{d\gamma}{dt}(0)\right) = \frac{\boldsymbol{d}\gamma}{\boldsymbol{d}t}(0)\left(= \partial_{\frac{d\gamma}{dt}(0)}\right).$$

- For a chart χ around p, I_p^M sends the induced basis induced by χ as discussed in the previous section, i.e. (3.2.3), to the similar vectors (3.3.2) discussed in this section.
- This construction is also compatible with the differentials: if $F : M \to N$ is a smooth map and we consider the differential of F defined on T_pM as in the previous section (subsection 3.2.4) and the one defined on $T_p^{\mathrm{deriv}}M$ as in this scetion (subsection 3.3.4) one has

$$\begin{array}{ccc} T_pM & \xrightarrow{(dF)_p} & T_{F(p)}N \\ \downarrow{\scriptstyle I_p^M} & & \downarrow{\scriptstyle I_{F(p)}^N} \\ T_p^{\mathrm{deriv}}M & \xrightarrow[(\boldsymbol{d}F)_p]{} & T_{F(p)}^{\mathrm{deriv}}N \end{array} \quad , \quad (\boldsymbol{d}F)_p \circ I_p = I_{F(p)} \circ (dF)_p \quad .$$

Exercise 3.24. Check/prove these assertions.

Theorem 3.25. *The map $I_p^M : T_pM \to T_p^{deriv}M$ is a linear isomorphism.*

Note that we could have discussed this theorem earlier on and then transfer, via I, all the properties/constructions that we knew for T_pM to $T_p^{\mathrm{deriv}}M$. Giving the proof of this theorem earlier would not have been completely trivial; however, the main reason for not doing that is that it was instructive to look at $T_p^{\mathrm{deriv}}M$ independently and check that it does satisfy the properties that we want. In particular, that revealed the advantages of $T_p^{\mathrm{deriv}}M$ as well as the further nature of tangent spaces.

Proof (Proof of Theorem 3.25). Fix a chart $\chi : U \to \mathbb{R}^m$ around p. The argument lives on the following diagram

$$\begin{array}{ccc}
T_pM & \xrightarrow{\;I_p^M\;} & T_p^{\mathrm{deriv}}M \\
{\scriptstyle (di)_p}\big\uparrow & & \big\uparrow{\scriptstyle (\boldsymbol{d}i)_p} \\
T_pM & \xrightarrow{\;I_p^U\;} & T_p^{\mathrm{deriv}}U \\
{\scriptstyle (d\chi)_p}\big\downarrow & & \big\downarrow{\scriptstyle (\boldsymbol{d}\chi)_p} \\
T_0\Omega & \xrightarrow{\;I_0^{\Omega}\;} & T_0^{\mathrm{deriv}}\Omega \\
{\scriptstyle (dj)_0}\big\downarrow & & \big\downarrow{\scriptstyle (\boldsymbol{d}j)_0} \\
T_0\mathbb{R}^m & \xrightarrow{\;I_0^{\mathbb{R}^m}\;} & T_0^{\mathrm{deriv}}\mathbb{R}^m
\end{array}$$

where $i : U \to M$ and $j : \Omega \to \mathbb{R}^m$ are the inclusions. The vertical maps on the left hand side are all isomorphism because of the properties of the tangent spaces and differential (inclusions of opens induce iso in tangent spaces, and the differential of a diffeomorphism is an isomorphism- as a consequence of TW2). Similarly for the vertical maps on the right hand side: $(\boldsymbol{d}\chi)_p$ for exactly the same reasons as for $(d\chi)_p$, while $(\boldsymbol{d}i)_p$ and $(\boldsymbol{d}j)_0$ was proven in Lemma 3.21 (and is certainly more involved than $(di)_p$ and $(dj)_0$). By the properties of I discussed above, all the squares are commutative. Notice that, in a commutative square, if three of the maps are isomorphisms, then so is the fourth. Therefore, we see that by moving up on the diagram, to prove that I_p^M is an isomorphism, it suffices to prove that the bottom horizontal map is. But, since that map sends the basis $\left(\frac{\partial}{\partial x_i}\right)_p$ to the similar collection (3.3.2) in T^{deriv}, Proposition 3.16 implies that the map is indeed an isomorphism. 😀

3.4 Tangent spaces: conclusions

Let us put together the main conclusions on tangent spaces.

3.4.1 Via charts, or as derivations

For any m-dimensional manifold M, $p \in M$, we have two ways of looking at the tangent space T_pM:

- **T1:** *its elements $v \in T_pM$ can be represented w.r.t. any chart χ around p by vectors $v^\chi \in \mathbb{R}^m$; the passing from one chart to the other was given by*

$$v^{\chi'} = (dc)_{\chi(p)}(v^\chi), \quad \textit{where } c = c_\chi^{\chi'} = \chi' \circ \chi^{-1}.$$

- **T2:** *its elements $v \in T_pM$ can be interpreted as derivations on $\mathscr{C}^\infty(M)$ at p.*

The equivalence between the two is given by the isomorphism

$$I_p : T_pM \to T_p^{\mathrm{deriv}}M$$

of Theorem 3.25. From now on we will identify the two (via I_p) and use only the notation T_pM. The upshot is that, when describing a vector $v \in T_pM$ we have the choice to describe it w.r.t. a chart, or as a derivation at p.

3.4.2 Speeds

Most importantly, any curve $\gamma \in \mathrm{Curves}_p(M)$ has a speed

$$\frac{d\gamma}{dt}(0) \in T_pM$$

Explicitly:

- T1-description: w.r.t. a chart χ, it is $\frac{d\gamma^\chi}{dt}(0) \in \mathbb{R}^m$.
- T2-description: as a derivation at p it sends a smooth function f to $\frac{df\circ\gamma}{dt}(0)$.

(and, with the same descriptions, one has $\frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M$ for any t in the domain of γ).

And this gives the best way to think (and even to work with) tangent spaces:

- **T0:** $T_pM = \{\frac{d\gamma}{dt}(0) : \gamma \in Curves_p(M)\}$*, where we know that two such curves have the same speed at $t = 0$ if and only if that happens for their representations w.r.t. a/any chart.*

3.4.3 Basis with respect to a chart

The tangent space is an m-dimensional vector space- hence isomorphic to $\mathbb{R}^m$. Each chart χ around p gives rise to a basis of T_pM (hence to a specific isomorphism with $\mathbb{R}^m$):

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p \in T_pM,$$

called the canonical basis w.r.t. χ:

- T1-description: w.r.t. χ, $\left(\frac{\partial}{\partial \chi_i}\right)_p$ corresponds to $e_i \in \mathbb{R}^m$.
- T2-description: as derivations, they are given by the partial derivatives w.r.t. χ:

$$\frac{\partial f}{\partial \chi_i}(p) := \frac{\partial f_\chi}{\partial x_i}(\chi(p)). \quad (*)$$

- T0-description: as speeds, $\left(\frac{\partial}{\partial \chi_i}\right)_p$ are induced by the paths

$$t \mapsto \chi^{-1}(\chi(p) + te_i).$$

The way that these vectors change when we change the chart is easiest read off from the point of view of [T1], which gives the formula (3.2.4). Or, using the partial derivatives given by the point of view of [T2], we have the more compact formula:

$$\left(\frac{\partial}{\partial \chi_i}\right)_p = \sum_{j=1}^m \frac{\partial c_j}{\partial \chi_i}(\chi(p)) \left(\frac{\partial}{\partial \chi'_j}\right)_p = \sum_{j=1}^m \frac{\partial \chi'_j}{\partial \chi_i}(p) \left(\frac{\partial}{\partial \chi'_j}\right)_p \quad (\text{where } c = \chi' \circ \chi^{-1}).$$

Or, if one denotes by x and y the charts χ and χ', respectively, and c by y but interpreted as a function of x, one come across the more compact (but a bit sloppy) formula:

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}.$$

3.4.4 Differentials

One can talk about the differential

$$(dF)_p : T_pM \to T_{F(p)}N \quad \text{for any smooth map } f : M \to N.$$

Again, this can described from the three points of view:

- T0-description: It sends the speed (at $t = 0$) of $\gamma \in \text{Curves}_p(M)$ to the one of $F \circ \gamma \in \text{Curves}_{F(p)}(N)$.
- T1-description: For $v \in T_pM$ represented w.r.t. χ by v^χ, $(dF)_p(v)$ is represented w.r.t. a/any chart χ' around $f(p)$ by the image of v^χ by $(dF_\chi^{\chi'})_{\chi(p)}$.
- T2-description: $(dF)_p(v)$, as a derivation, acts on a function $g \in \mathscr{C}^\infty(M)$ by acting with v on $g \circ F$.

Or, if we want to write down $(dF)_p$ w.r.t. bases induced by charts χ around p and χ' around $F(p)$:

$$\left(\frac{\partial}{\partial \chi_i}\right)_p \xmapsto{(dF)_p} \sum_{j=1}^m \frac{dF_\chi^{\chi',j}}{dx_i}(\chi(p)) \left(\frac{\partial}{\partial \chi'_j}\right)_{F(p)},$$

where we use the representation $F_{\chi}^{\chi'} = \chi' \circ F \circ \chi^{-1}$ of F w.r.t. the two bases. Or, in other words, the matrix corresponding to $(dF)_p$ w.r.t. the two bases is precisely the matrix of the partial derivatives of $F_{\chi}^{\chi'}$ at $\chi(p)$.

And we should also keep in mind the characterization of immersions/submersions from Proposition 3.11:

$$F \text{ is an immersion/submersion at } p \Longleftrightarrow (dF)_p \text{ is injective/surjective},$$

and this should be the way to think about immersions/submersions from now on!

3.4.5 Submanifolds

Let us restrict for simplicity to embedded submanifolds. Given such a submanifold $N \subset M$ of a manifold M, the inclusion $i : N \hookrightarrow M$ is an immersion, therefore, for each $p \in N$,

$$(di)_p : T_pN \to T_{i(p)}M$$

is an injection. In the same way that we do not write $i(p)$ but p, this injection will be viewed as an inclusion

$$T_pN \subset T_pM.$$

This inclusion can be described using any of the three viewpoints:

- T0-description: this is completely clear/tautological: since any curve in N is also one in M, the speed $\frac{d\gamma}{dt}(0) \in T_pN$ is now identified with $\frac{d\gamma}{dt}(0) \in T_pM$ (we do not even have a notation to distinguish the two!).
- T1-description: for $v \in T_pN$, when viewed in T_pM it is the unique tangent vector with the property that, w.r.t. any chart χ of M adapted to N, the representation of v w.r.t. χ is the representation of $v \in T_pN$ w.r.t. $\chi|_N$.
- T2-description: as derivations, if $v \in T_pN$, it acts on $f \in \mathscr{C}^\infty(M)$ by acting with v on $f|_N$.

In this way tangent spaces give us (linear) information on the way that N sits inside M (near p).

Finally, here is the full version of the regular value theorem proven in Chapter 2 (Theorem 2.66 there), this time with exra-information on the tangent spaces. Recall that a regular value of $f : M \to N$ is any q with the property that f is a submersion at all points $p \in f^{-1}(q)$.

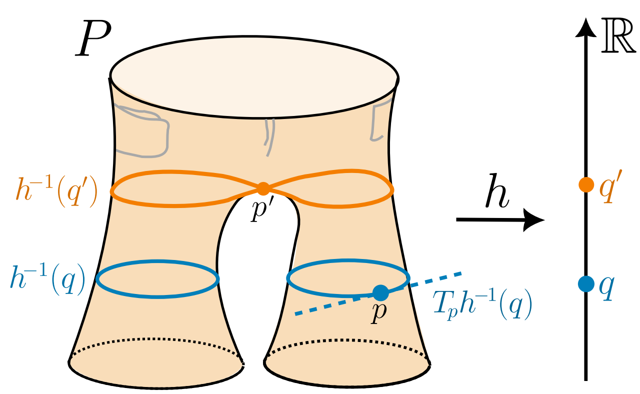


Fig. 3.5 The regular value theorem visualized for the case of the *pair of pants* manifold P, together with the map $h : P \to \mathbb{R}$ that assigns to each point its height over the ground. For the regular value q, the *level set* $h^{-1}(q)$ is an embedded submanifold of codimension one. The tangent space at $p \in h^{-1}(q)$ consists exactly of $\mathrm{Ker}\,(dh)_p$, i.e. those directions in P that get smashed together by h. After all, small movement in these directions does not change the value of h, so we remain in $h^{-1}(q)$. The set $h^{-1}(q')$ fails to be a manifold since h is not a submersion at p' (so q' is not a regular value).

Theorem 3.26 (the regular value theorem- with tangent spaces). *If $q \in N$ is a regular value of a smooth map*

$$F : M \to N,$$

then the fiber above q, $F^{-1}(q)$, is an embedded submanifold of M of dimension

$$dim(F^{-1}(q)) = dim(M) - dim(N)$$

and the tangent spaces $T_p\left(F^{-1}(q)\right)$, as subspaces of T_pM, coincide with the kernel of the differential $(dF)_p : T_pM \to T_{F(p)}N$:

$$T_p\left(F^{-1}(q)\right) = Ker(dF)_p \quad ((for\ all\ p \in F^{-1}(q)).$$

Proof. Nice exercise (hint: just look back at our reminder on Analysis). 😢

3.4.6 Back to Euclidean spaces and their embedded submanifolds

Back to Analysis, i.e. for a manifold of type $\mathbb{R}^m$ (or an open inside it), the general/abstract theory of this chapter tells us that the tangent spaces $T_p\mathbb{R}^m$ come with a canonical basis

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p, \tag{3.4.1}$$

which, again, can be looked at using the three view-points:

- T0-description: as speeds, they correspond to the curves parallel to the coordinate axes, $t \mapsto p + te_i$.
- T1-description: with respect to the identity chart, they are represented by the canonical basis $e_1, \ldots, e_m$ of $\mathbb{R}^m$.
- T2-description: as derivations, they are the usual partial derivatives at p.

One obtains an isomorphism that will be treated from now on as an identification:

$$T_p\mathbb{R}^k = \mathbb{R}^k,$$

but we will still keep the notation (3.4.1) for the canonical basis. This is in order to indicate that we are thinking about/using tangent vectors.

Exercise 3.27. Check that, via this identification, the speeds $\frac{d\gamma}{dt}(t)$ discussed in this chapter are identified with the standard ones from Analysis.

As in the previous exercise for curves, we have a similar problem for embedded submanifolds:

Exercise 3.28. For an embedded submanifold $M \subset \mathbb{R}^n$, $p \in M$ we have the corresponding tangent spaces

$$T_pM \subset T_p\mathbb{R}^n = \mathbb{R}^n$$

that can now be described using the three view-points (T0), (T1), (T2). Show that this coincides with the tangent spaces from the Analysis reminder, trying to find as many different arguments as you can; e.g. one using the previous exercise, one using the regular values theorem, etc.

By now, it should also be clear that for smooth maps between embedded submanifolds of Euclidean spaces, the differential discussed in this chapter becomes (identified with) the one from Analysis (e.g. as recalled in Section 1.2 from Chapter 1, Exercise 1.48).

Exercise 3.29. With the identification above, whenever we have a smooth map $f : M \to \mathbb{R}$, we talk about its differentials as maps $(df)_p : T_pM \to \mathbb{R}$.

For $f \in \mathscr{C}^\infty(M)$ and $v \in T_pM$, check that the action of v (as a derivation) on f is precisely

$$\partial_v(f) = (df)_p(v).$$

3.5 Vector fields

We now discuss vector fields. The brief philosophy is:

While for embedded submanifolds $M \subset \mathbb{R}^k$ each single tangent space $T_pM \subset \mathbb{R}^k$ was interesting (e.g. because it reflects the position of M inside $\mathbb{R}^k$ near p), for a general manifold M this is less so. Instead, it is the entire family $\{T_pM\}_{p\in M}$ that is interesting. I.e. not one single tangent vector, but a family $\{v_p\}_{p\in M}$ of vectors (of course, "varying smoothly" with respect to p). I.e. vector fields.

Looking back at the previous sections of this chapter, we see that we insisted in making sense of T_pM as vector spaces independently of the way that M may sit in some larger Euclidean space. Now, just as a vector space, T_pM is rather boring: it is isomorphic to $\mathbb{R}^m$. And the same is true for any finite dimensional vector space V; however, one should keep in mind that, for a bare m-dimensional vector space V, realizing an explicit isomorphism with $\mathbb{R}^m$ amounts to extra choices (namely a basis of V). In particular, for the tangent spaces T_pM, we obtained identifications with $\mathbb{R}^m$ once we fixed a chart χ (and these identifications work for p in the domain of χ only!). Actually this is a "problem" (or better: "interesting phenomena") not only for general Ms, but even for embedded submanifolds $M \subset \mathbb{R}^k$: while isomorphisms

$$\mathbb{R}^m \cong T_pM \subset \mathbb{R}^k$$

can be chosen for each p, often cannot be chosen so that they depend smoothly on $p \in M$. The "hairy ball theorem" implies that this is the case already for $M = S^2 \subset \mathbb{R}^3$:

Exercise 3.30. Assume that for each $p \in S^2$ one can find a linear isomorphism

$$\Phi_p : \mathbb{R}^2 \to T_pS^2 \subset \mathbb{R}^3$$

so that the Φ_ps vary smoothly with respect to p, in the sense that the map

$$\Phi : S^2 \times \mathbb{R}^2 \to \mathbb{R}^3, \quad (p,v) \mapsto \Phi_p(v)$$

is smooth. Show that there exists a nowhere vanishing vector field on S^2, i.e. a smooth map

$$X : S^2 \to \mathbb{R}^3,$$

nowhere vanishing, such that $X(p) \in T_pS^2$ for all $p \in S^2$. Try now to construct such an X! Then state the "hairy ball theorem".

More generally, for an embedded submanifold $M \subset \mathbb{R}^k$, a vector field on M is a function

$$M \ni p \mapsto X(p) \in T_pM \subset \mathbb{R}^k$$

which, when interpreted as a function with values in $\mathbb{R}^k$, is smooth. What if an embedding into some Euclidean space is not fixed (or if we use a different one)?

3.5.1 *Smooth vector fields*

Continuing the last discussion, for general manifolds M: first of all, we can still look at maps

$$X : M \ni p \mapsto X_p \in T_pM$$

(interpreted also as families $X = \{X_p\}_{p\in M}$). These will be called **set-theoretical vector fields** on M. To make sense of their smoothness, we use charts $\chi : U \to \Omega \subset \mathbb{R}^m$ of M. Recall that any such chart induces a basis $\left(\frac{\partial}{\partial\chi_i}\right)_p$ of T_pM hence any set-theoretical vector field X can be written as

$$X_p = \sum_{i=1}^{m} X^i_\chi(\chi(p)) \left(\frac{\partial}{\partial\chi_i}\right)_p \tag{3.5.1}$$

for all $p \in M$, where each X^i_χ is a function

$$X^i_\chi : \Omega \to \mathbb{R}.$$

These will be called **the coordinate functions of X w.r.t. the chart χ**.

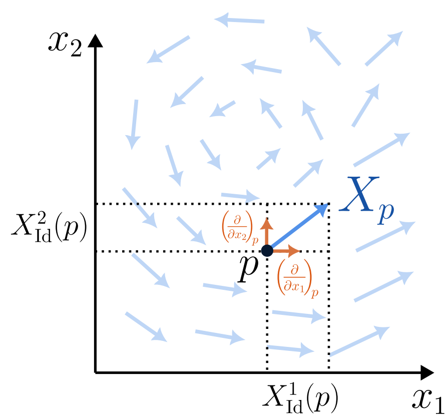


Fig. 3.6 Let M be the Euclidean plane drawn here with the identity chart $\mathrm{Id} = (x_1, x_2)$. The vector field X assigns a vector X_p to every point p. The components of X_p with respect to the basis vectors $\left(\frac{\partial}{\partial x_i}\right)_p$ of the identity chart are its coordinate functions X^i_{Id}.

Definition 3.31. A vector field on a manifold is any set-theoretical vector field X,

$$X : M \ni p \mapsto X_p \in T_pM,$$

with the property that its coordinate functions X^i_χ w.r.t any chart χ are smooth.
We denote by $\mathfrak{X}(M)$ the set of all vector fields on M.

Exercise 3.32. Show that, for the smoothness condition, it is enough to check it only for charts in a/any atlas of M inducing the smooth structure of M.

Note the algebraic structure on $\mathfrak{X}(M)$ that is present right away:

- it is a vector space, with the addition $+$ and multiplication by scalars $\cdot$ defined pointwise:

$$(X+Y)_p := X_p + Y_p, \quad (\lambda \cdot X)_p := \lambda \cdot X_p.$$

- it is a $\mathscr{C}^\infty(M)$-module, i.e. there is an operation of multiplication of vector fields by smooth functions,

$$\mathscr{C}^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (f,X) \mapsto f \cdot X.$$

Again, this is defined pointwise by the obvious formula:

$$(f \cdot X)_p := f(p) \cdot X_p.$$

Example 3.33. (in $\mathbb{R}^m$) Of course, considering the standard basis (3.4.1) at each point one obtains the corresponding standard vector fields on $\mathbb{R}^m$:

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \in \mathfrak{X}(\mathbb{R}^m); \tag{3.5.2}$$

and, using the basic operations described above, one obtains vector fields

$$f_1 \cdot \frac{\partial}{\partial x_1} + \ldots + f_m \cdot \frac{\partial}{\partial x_m} \in \mathfrak{X}(\mathbb{R}^m)$$

for any smooth functions f_i on $\mathbb{R}^m$. And the fact that the standard basis (3.4.1) is actually a basis implies that any vector field on $\mathbb{R}^m$ is of this type. In a more algebraic language: (3.5.2) forms a basis of $\mathfrak{X}(\mathbb{R}^m)$ as a $C^\infty(\mathbb{R}^m)$-module. Of course, the same discussion applies to any open $\Omega \subset \mathbb{R}^m$.

Remark 3.34 (the chart representation). The discussion from the previous example allows one to slightly reinterpret the coordinate functions of a vector field $X \in \mathfrak{X}(M)$ w.r.t. a chart $\chi : U \to \Omega$ discussed above: they will be precisely the coefficients of a vector field X_χ on Ω obtained by pushing forward X (restricted to U) via Ω:

$$\chi_*(X) \in \mathfrak{X}(\Omega), \quad \chi_*(X)_x := (d\chi)_p(X_p) \in T_x\Omega \quad \text{where } p = \chi^{-1}(x).$$

This deserves the name of **the representation of X in the chart χ**. More on "push-forwards" will be discussed in subsection 3.5.6.

Exercise 3.35. In $\mathbb{R}^2$ we consider the vector fields

$$X := \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}, \quad E := x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2).$$

And we also consider the polar coordinates chart $\chi : (x,y) \mapsto (r,\theta)$ where

$$r = r(x,y) \in \mathbb{R}_{>0}, \quad \theta = \theta(x,y) \in (0,2\pi)$$

are determined by the usual formulas:

$$x = r \cdot \cos\theta, \quad y = r \cdot \sin\theta$$

on $U = \mathbb{R}^2 \setminus \{0\}$. Compute the representation of X, Y and E in the chart χ. (since the outcome are vector fields in $\mathbb{R}_{>0} \times (0,2\pi)$ where we use r and θ for the coordinates, please use the notations $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ for the standard basis there).

Example 3.36. (on submanifolds $M \subset \mathbb{R}^n$) When working in an embedded submanifold $M \subset \mathbb{R}^n$ it is, of course, natural to use the coordinates $(x_1, \ldots, x_n)$ that are there anyway, from the ambient space. Actually, there may even no be any other coordinates available without any extra-choices (think e.g. of the spheres).

Similarly, since $T_pM \subset T_p\mathbb{R}^n$, it is natural to write the tangent vector to M as

$$X_p = f_1(p) \cdot \left(\frac{\partial}{\partial x_1}\right)_p + \ldots + f_n(p) \cdot \left(\frac{\partial}{\partial x_n}\right)_p.$$

But one should be aware that the vectors $\frac{\partial}{\partial x_i}$ are, in general, not tangent to M. Actually only some expressions of this type are tangent to M (also, in principle, the functions f_i are defined only for $p \in M$).

A very good example is the one of the spheres $S^m \subset \mathbb{R}^{m+1}$. To have actual manifold coordinates one needs to choose some smooth chart (e.g. stereographic projection), but one also use the coordinates $(x_0, \ldots, x_m)$ of the ambient space. And then, when looking at tangent vectors (and vector fields) one deals with expressions

$$\lambda_0 \cdot \left(\frac{\partial}{\partial x_0}\right)_x + \ldots + \lambda_m \cdot \left(\frac{\partial}{\partial x_m}\right)_x$$

and remembers that such a vector is tangent to the sphere if and only if

$$\lambda_0 \cdot x_0 + \ldots + \lambda_m \cdot x_m = 0.$$

Exercise 3.37. Show that the following defines a smooth vector field on S^1:

$$V_{(x,y)} := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

And make a picture. And try to see the relationship with complex numbers.

Exercise 3.38. Show that the following defines a smooth vector field V on S^3:

$$V_{(x,y,z,t)} := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}.$$

Hence a "hairy ball theorem" does not hold on S^3. Can you go further and find two more, similar and equally interesting vector fields on S^3? (hint: rebaptise V to $\mathbb{I}$, and call the other two $\mathbb{J}$ and $\mathbb{K}$.)

Exercise 3.39. In the last two exercises, in principle, to show that those vector fields are smooth you applied the definition and looked into charts. However, isn't there something more general going on? What if $M \subset N$ is an embedded submanifold and $X \in \mathfrak{X}(N)$ has the property that $X_p \in T_pM$ for all $p \in M$?

Exercise 3.40. On S^m consider the stereographic projection w.r.t. the north pole:

$$\chi : S^m \setminus \{p_N\} \to \mathbb{R}^m, \quad (x_0, \ldots, x_m) \mapsto (u_1, \ldots, u_m)$$

with

$$u_1 = \frac{x_0}{1 - x_m}, \ldots, \; u_m = \frac{x_{m-1}}{1 - x_m}.$$

Find a vector field $X \in \mathfrak{X}(S^m)$ which, represented in this chart, becomes

$$X_\chi = u_1 \cdot \frac{\partial}{\partial u_1} + \ldots + u_m \cdot \frac{\partial}{\partial u_m}.$$

3.5.2 Vector fields as derivations

Next, we look for alternative ways to characterize the smoothness of vector fields; as a bonus, these will reveal more of the structure that vector fields carry. First of all, while tangent vectors could be interpreted as derivations (at given points $p \in M$), for any $X \in \mathfrak{X}(M)$ and $f \in C^{\infty}(M)$ one has $\partial_{X_p}(f) \in \mathbb{R}$ for each $p \in M$, therefore a function

$$L_X(f) : M \to \mathbb{R}, \quad p \mapsto \partial_{X_p}(f) = (df)_p(X_p).$$

This is called **the Lie derivative of f along the vector field X**. Of course, we expect that $L_X(f)$ is again smooth.

Proposition 3.41. *A set-theoretical vector field X on M is smooth if and only if*

$$L_X(f) \in C^{\infty}(M) \text{ for all } f \in C^{\infty}(M).$$

Proof. Assume first that X is smooth in sense of the definition. For $f \in C^{\infty}(M)$, we can check the smoothness of $L_X(f)$ locally: for an arbirary chart χ we have to make sure that $L_X(f) \circ \chi^{-1}$ is smooth. But applying the definitions, we see that this expression is precisely

$$\sum_i X_{\chi}^i \frac{\partial f_{\chi}}{\partial x_i}$$

and then smoothness follows.

For the converse, by the type of arguments we have already seen, it suffices to show that for any point $p \in M$ there exists a chart around p such that the coefficients X_i^{χ} are smooth. For an arbitrary chart $\chi : U \to \Omega \subset \mathbb{R}^m$ around p, we choose $f_i \in \mathscr{C}^{\infty}(M)$ such that, in a smaller neighborhood $U_0 \subset U$ of p, f_i coincides with χ_i. By hypothesis, $L_X(f_i)$ is smooth; but over U_0, this function is precisely X_i^{χ}. Hence taking $\chi_0 := \chi|_{U_0}$, the coefficients $X_i^{\chi_0}$ will be smooth.😀

Hence each vector field X gives rise to an operation

$$L_X : \mathscr{C}^{\infty}(M) \to \mathscr{C}^{\infty}(M).$$

The interpretation of tangent vectors at p as derivations at p can be pushed further to a similar characterization of vector fields. The algebraic objects are **derivations of $\mathscr{C}^{\infty}(M)$**, by which we mean maps

$$L : \mathscr{C}^{\infty}(M) \to \mathscr{C}^{\infty}(M)$$

which are linear and satisfy the derivation low (Leibniz identity):

$$L(fg) = L(f)g + fL(g) \quad \text{for all } f, g \in \mathscr{C}^{\infty}(M).$$

We denote by $\mathrm{Der}(\mathscr{C}^{\infty}(M))$ the (vector) space of such derivations.

Note: again, this is a purely algebraic construction, that applies to any algebra.

Theorem 3.42. *For any $X \in \mathfrak{X}(M)$, the Lie derivative along X,*

$$L_X : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M), \quad f \mapsto L_X(f)$$

is a derivation. Moreover,

$$I : \mathfrak{X}(M) \to Der(\mathscr{C}^\infty(M)), \quad X \mapsto L_X$$

is an isomorphism of vector spaces.

Proof. The fact that L_X is a derivation follows right away from the similar property of the ∂_{X_p}s. The injectivity of I follows from the fact that, if a tangent vector $v_p \in T_pM$ is non-zero, there exists a smooth function f on M such that $\partial_{v_p}(f) \neq 0$ (why?).

For the surjectivity of I, let L be a derivation. Then, for each $p \in M$,

$$X_p : \mathscr{C}^\infty(M) \to \mathbb{R}, \quad f \mapsto L(f)(p)$$

is a derivation at p, hence defines a tangent vector $X_p \in T_pM$. Since $L_X(f) = L(f)$ is smooth for any smooth f, the previous proposition implies that X is smooth. 😀

Exercise 3.43. For the vector field $X \in \mathfrak{X}(S^2)$ that you found in Exercise 3.40, compute $L_X(f)$ for the function

$$f : S^2 \to \mathbb{R}, \quad f(x,y,z) = x + y^2 + z^3.$$

Again, you will probably use here the fact that, although we make reference to f only as a function on S^2, you will be using its extension to the entire ambient space $\mathbb{R}^3$. Is there something more general to learn from this?

Exercise 3.44. Consider the vector field $V \in \mathfrak{X}(S^3)$ from Exercise 3.38, the Hopf map $h : S^3 \to S^2$ (Exercise 2.110) and denote by h_1, h_2 and h_3 its components. Show that L_V kills all these components. Can you reformulate this without referring to the components of h, but to h itself?

3.5.3 The Lie bracket

And here is one clear advantage of the point of view of derivations: the presence of yet another structure on $\mathfrak{X}(M)$. More precisely, given two derivations on $\mathscr{C}^\infty(M)$, necessarily of type L_X and L_Y, their composition

$$L_X \circ L_Y : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$$

is not a derivation anymore. Indeed, what we have is:

$$(L_X \circ L_Y)(fg) = (L_X \circ L_Y)(f)g + f(L_X \circ L_Y)(g) + L_X(f)L_Y(g) + L_Y(f)L_X(g),$$

i.e. a "defect term" $L_X(f)L_Y(g) + L_Y(f)L_X(g)$ which spoils the Leibniz identity. However, the other composition $L_Y \circ L_X$ comes with the same "defect term"; therefore, their difference

$$[L_X, L_Y] := L_X \circ L_Y - L_Y \circ L_X : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$$

is again a derivation.

Definition 3.45. Given $X,Y \in \mathfrak{X}(M)$, we denote by $[X,Y] \in \mathfrak{X}(M)$, called **the Lie bracket of** X **and** Y, the (unique) vector field on M with the property that

$$L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X.$$

The following exercise should convince you of the advantage of the point of view of derivations.

Exercise 3.46. Compute the coordinate functions of $[X,Y]$ with respect to a chart χ, in terms of the coordinates functions of X and Y defined by (3.5.1).

Exercise 3.47. In this exercise we point out the main properties of the Lie bracket operation $[\cdot,\cdot]$ on $\mathfrak{X}(M)$.
First of all, the way it interacts with the other structure present on on $\mathfrak{X}(M)$- show that:

- w.r.t. the vector space structure: it is a bilinear and skew-symmetric operation

$$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M).$$

- w.r.t. the $\mathscr{C}^\infty(M)$-module structure: it satisfies the derivation rule:

$$[X, f \cdot Y] = f \cdot [X,Y] + L_X(f) \cdot Y,$$

  for all $X,Y \in \mathfrak{X}(M)$, $f \in \mathscr{C}^\infty(M)$.

And then a property that $[\cdot,\cdot]$ has on its own- **the Jacobi identity**:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \quad \text{for all } X,Y,Z \in \mathfrak{X}(M).$$

Exercise 3.48. On $\mathbb{R}^3$ we consider the vector fields

$$X^1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad X^2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad X^3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Show that:

$$[X^1,X^2] = X^3, \quad [X^2,X^3] = X^1, \quad [X^3,X^1] = X^2$$

in two different ways:

- using the definition of the Lie bracket via commutators of derivations.
- using the properties of the Lie bracket described in Exercise 3.47 (and the fact that the Lie bracket between any two of the standard vector fields $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ is zero).

Exercise 3.49. Consider the following three vector fields on the sphere S^3:

$$V^1_{(x,y,z)} := \frac{1}{2}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t}\right),$$

$$V^2_{(x,y,z)} := \frac{1}{2}\left(-z\frac{\partial}{\partial x} - t\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + y\frac{\partial}{\partial t}\right),$$

$$V^3_{(x,y,z)} := \frac{1}{2}\left(-t\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + x\frac{\partial}{\partial t}\right).$$

Compute $[V^1, V^2]$, $[V^2, V^3]$ and $[V^3, V^1]$. As a test, please also check the Jacobi identity for V^1, V^2 and V^3.

(But please be aware that we ask you for a computation with vector fields on S^3; so just working formally with symbols may require some explanations- e.g. because simple vector fields like $\frac{\partial}{\partial x}$ are not tangent to S^3. We are not saying the result is not correct at the end, but please explain why what you do is correct. Any way, you should be getting $[V^1, V^2] = V^3$, $[V^2, V^3] = V^1$, $[V^3, V^1] = V^2$.)

3.5.4 The tangent bundle/the configuration space

Back to the the very definition of vector fields and of $\mathfrak{X}(M)$, here is another way to characterize smoothness. The idea is to put all the spaces T_pM (disjointly) in one big space

$$TM := \sqcup_{p\in M} T_pM = \{(p, v_p) : p \in M, v_p \in T_pM\}$$

and make this into a manifold, so that the smoothness of a set-theoretical vector field X is the same as the smoothness of X viewed as a map between manifolds:

$$X : M \to TM, \quad p \mapsto X_p.$$

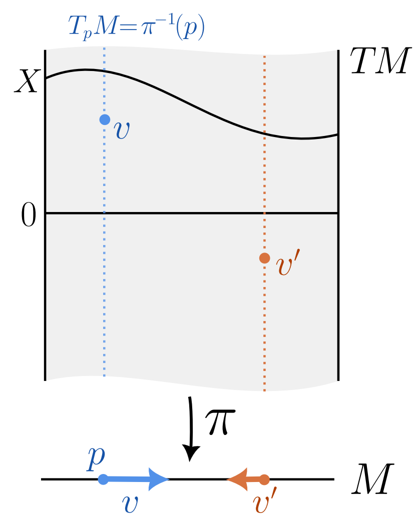


Fig. 3.7 Visualization of the tangent bundle TM of a manifold M (here: an interval). Note that points in TM correspond to vectors in M. The vertical projection map $\pi : TM \to M$ assigns to every vector $v \in T_pM$ the point p at which it is based. Each tangent space T_pM is therefore the preimage $\pi^{-1}(p)$ and also called the *fiber over* p. A vector field can be seen as a map $X : M \to TM$ such that $\pi \circ X = \mathrm{Id}_M$ (we say it is a *section* of TM). A canonical example of this is the *zero section* $0 : M \to TM$ that sends each $p \in M$ to the zero vector based at p.

To put a smooth structure on TM, i.e. to exhibit charts, we need to understand how to parametrize the points of TM by coordinates. Of course, we should start from a chart

$$\chi : U \to \Omega \subset \mathbb{R}^m$$

for M. The point is that such a chart also allows us to parametrize tangent vectors $v_p \in T_pM$, for $p \in U$, by coordinates $\hat{\chi}_i(v_p)$ w.r.t. to the induced basis $\left(\frac{\partial}{\partial \chi_i}\right)_p$ of T_pM: just decompose v_p as

$$v_p = \sum_i \lambda_i \left(\frac{\partial}{\partial \chi_i}\right)_p \quad \text{and set } \hat{\chi}_i(v_p) := \lambda_i. \tag{3.5.3}$$

Therefore, on the subset $TU \subset TM$, we have a natural "chart":

$$\tilde{\chi} : TU \to \Omega \times \mathbb{R}^m \subset \mathbb{R}^{2m},$$
$$\tilde{\chi}(p, v_p) = (\chi_1(p), \ldots, \chi_m(p), \hat{\chi}_1(v_p), \ldots, \hat{\chi}_m(v_p)).$$

Proposition 3.50. *TM can be made into a manifold in a unique way so that, for any chart (U, χ) of M, $(TU, \tilde{\chi})$ is a chart of TM.*
Moreover, this is also the unique smooth structure with the property that a set-theoretical vector field X on M is smooth if and only if it is smooth as a map $X : M \to TM$.

Proof. We first compute the change of coordinates between charts of type $(TU, \tilde{\chi})$. We start with two charts (U, χ) and (U', χ') for M, with change of coordinates $c = \chi' \circ \chi^{-1}$. We want to compute $\tilde{c} := \tilde{\chi}' \circ \tilde{\chi}^{-1}$. Hence we try to write $\tilde{\chi}'(p)$ as a function depending on $\tilde{\chi}(p)$. While the first m coordinates are $c(\chi(p))$, we need to understand the last m. For that we use:

$$\left(\frac{\partial}{\partial \chi_i}\right)_p = \sum_j \frac{\partial c_j}{\partial x_i}(\chi(p)) \left(\frac{\partial}{\partial \chi'_i}\right)_p .$$

(see (3.2.4)). Then, for a tangent vector $p \in M$, replacing this in (3.5.3), we find that

$$\hat{\chi'}_j(v_p) = \sum_i \hat{\chi}_i(v_p) \frac{\partial c_j}{\partial x_i}(\chi(p)).$$

Denoting by $(x, \hat{x})$ the coordinates in $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$, we find that

$$\tilde{c}(x, \hat{x}) = (c(x), \sum_i \hat{x}_i \frac{\partial c}{\partial x_i}(x).$$

Hence the changes of coordinates is smooth.

However, strictly speaking, to show that TM is a manifold, there are still a few things to be done. First of all, we should have made it into a topological space. But, as pointed out in Remark 2.23 from Chapter 2, the topology can actually be recovered from an atlas. Let us give the details, independently of our earlier discussion. Given

$$\text{a chart } \chi : U \to \Omega \text{ for } M, \quad \tilde{\Omega} - \text{open in } \Omega \times \mathbb{R}^m \tag{3.5.4}$$

we would like

$$\tilde{\chi}^{-1}(\tilde{\Omega}) \subset TM$$

to be open in M. We denote by $\mathscr{B}$ the collection of subsets of TM of this type. We will declare a subset $D \subset TM$ to be open if for each $(p, v_p) \in D$ there exists $B \in \mathscr{B}$ such that

$$(p, v_p) \in B \subset D.$$

Of course, what is happening here is that $\mathscr{B}$ is a topology basis, and we consider the associated topology (hence one can also say that a subset of TM is open iff it is an union of members of $\mathscr{B}$).

It should be clear that the resulting topology is Hausdorff. For 2nd countability, note that, in (3.5.4), we can restrict ourselves to χ belonging to an atlas inducing the smooth structure on M and, for each χ, to $\tilde{\Omega}$ belonging to a basis of the Euclidean topology on $\Omega \times \mathbb{R}^m$. As long as the domains of the charts from $\mathscr{A}$ for a basis for the topology of M, the resulting $\mathscr{B}$ would still be a topology basis for the topology we defined on TM. Since M and the $\Omega \times \mathbb{R}^m$s are 2nd countable, we are able to produce a countable basis for the topology on TM.

The fact that a set-theoretical vector field is smooth if and only if it is smooth as a map $X : M \to TM$ should be clear (one just has to check it locally, and then it is really the definition of the smoothness of X). The uniqueness is left as an exercise for the interested student. 😀

Note the object that results from this discussion: TM. It is now a manifold that relates to M by an obvious "projection map"

$$\pi : TM \to M, \quad (p, v_p) \mapsto p.$$

And the fibers $\pi^{-1}(p)$ are vector spaces (precisely the tangent spaces of M). This is precisely what vector bundles are- but we will return to them later on.

3.5.5 Extra-for the interested student: cool stuff with vector fields

Here are some very simple questions about vector fields whose answer is non-trivial question but very exciting. Let us start from the "hairy ball theorem" which can be seen as an interesting property of the two-sphere S^2: it does not admit a nowhere vanishing vector field. The first question in our list is now obvious:

Question: *Which manifolds admit a nowhere vanishing vector field?*

It is worth looking at this question even for the spheres $S^m \subset \mathbb{R}^{m+1}$. Then the question becomes pretty elementary: does there exist a smooth function

$$X : S^m \to \mathbb{R}^{m+1} \setminus \{0\}$$

with the property that

$$x_0 \cdot X_0(x) + x_1 \cdot X_1(X) + \ldots + X_m \cdot X_m(x) = 0 \quad \forall\, x = (x_0, x_1, \ldots, x_m) \in S^{m+1}?$$

For S^1 the answer is clearly yes (think on the picture!) and, as we have already mentioned, for S^2 the answer is no. What about S^3? Well, using quaternions as in Example 2.125 we see that the answer is yes. More precisely, viewing $S^3 \subset \mathbb{H} \cong \mathbb{R}^4$, the multiplication by i, j and k provide three nowhere vanishing (and even linearly independent!) vector fields on S^3:

$$X^i(x) = i \cdot x, \quad X^j(x) = j \cdot x, \quad X^k(x) = k \cdot x.$$

(why are these vector fields on spheres?).

It turns out that, among the spheres, it is precisely the odd dimensional ones that do admit such vector fields.

For general manifolds the answer to the previous question is closely related to the topology of M- namely to the Euler number $\chi(M)$ of M. We will briefly discuss the Euler number later on when we will introduce DeRham cohomology and we will see that $\chi(S^m) = 1 + (-1)^m$. For now, let us mention that this number is, in principle, easy to compute using a triangulation of M, by the Euler formula

$$\chi(M) = \#\text{vertices} - \#\text{edges} + \#\text{faces} - \ldots.$$

(using this you should convince yourself that, indeed, $\chi(S^m) = 1 + (-1)^m$). The answer to the previous question is: a compact manifold M admits a no-where vanishing vector field if and only $\chi(M) = 0$!

What about more than one no-where vanishing vector fields? Of course, the question is interesting only if we look for linearly independent ones- and then the number of such vector fields will be bounded from above by the dimension of the manifold. E.g., on S^3, there are at most three such; and the vector field X^i, X^j and X^k described above show that the upper bound can be achieved. More generally:

Question: *An m-dimensional manifold M is said to be parallelizable if it admits m vector fields which, at each point, are linearly independent.*

For instance, we already know that S^1 and S^3 are parallelizable. One possible explanation for this is the fact that both S^1 as well as S^3 are Lie groups (and they are the only spheres that can be made into Lie groups- see Example 2.125 from Chapter 2). More precisely:

Exercise 3.51. Show that all Lie groups are parallelizable. Here is how to do it. Let G be a k-dimensional Lie group. Consider its tangent space at the identity, $\mathfrak{g} := T_eG$ (a k-dimensional vector space). For $v \in \mathfrak{g}$, we define the vector field $\overrightarrow{v}$ on G by

$$\overrightarrow{v}_g := (dL_g)_e(v)$$

where $L_g : G \to G$ is given by $L_g(h) = gh$ and $(dL_g)_e : \mathfrak{g} \to T_gG$ is its differential.

Show that if $\{v_1, \ldots, v_k\}$ is a basis of the vector space $\mathfrak{g}$, then $\vec{v}_1, \ldots, \vec{v}_k$ are k-linearly independent vector fields on G.

While the only spheres that can be made into Lie groups are S^0, S^1 and S^3, are there any other spheres that are parallelizable? Can one reproduce the argument that we used for S^3 so that it applies to other spheres S^m? We see that we need some sort of "product operation" on $\mathbb{R}^{m+1}$, so that we can define vector fields X^l by

$$X^l(x) = e_l \cdot x, \quad l \in \{1, \ldots, m\}$$

where $\{e_0, e_1, \ldots, e_m\}$ is the canonical basis. We end up again with the question mentioned in Example 2.125 from Chapter 2, of whether $\mathbb{R}^{m+1}$ can be made into a normed division algebra. And, as we mentioned there, the only possibilities were $\mathbb{R}, \mathbb{R}^2$ and R^4 for associative products, and also $\mathbb{R}^8$(octonions) in full generality. However, while the failure of associativity was a problem in making the corresponding sphere into a Lie group, it is not difficult to see that it does not pose a problem in producing the desired vector fields. Therefore: yes, the arguments that used $\mathbb{H}$ works also for octonions, hence also S^7 is parallelizable.

And, as you may expect by now, it turns out that S^0, S^1, S^3 and S^7 are the only spheres that are parallelizable!

So far we have looked at the extreme cases: one single nowhere vanishing vector field, or the maximal number of linearly independent vector fields. Of course, the most general question is:

Question: *Given a manifold M, what is the maximal number of linearly independent vector fields that one can find on M?*

Already for the case of the spheres S^m, this deceivingly simple question turns out to be highly non-trivial. The solution (found half way in the 20th century) requires again the machinery of Algebraic Topology. But here is the answer: write $m+1 = 2^{4a+r}m'$ with m' odd, $a \geq 0$ integer, $r \in \{0,1,2,3\}$. Then the maximal number of linearly independent vector fields on S^m is

$$r(m) = 8a + 2^r - 1.$$

Note that $r(m) = 0$ if m is even- and that corresponds to the fact that there are no nowhere vanishing vector fields on even dimensional spheres. The parallelizability of S^m is equivalent to $r(m) = m$, i.e. $8a + 2^r = 2^{4a+r}m'$, which is easily seen to have the only solutions

$$m = 1, \quad a = 0, \quad r \in \{0,1,2,3\},$$

giving $m = 0, 2, 4, 8$. Hence again the spheres S^0, S^1, S^3 and S^7.

3.5.6 Push-forwards of vector fields

We now discuss how vector fields can be pushed forward from one manifold to another. The best scenario is when we start with a diffeomorphism $F : M \to N$ between two manifolds; then it induces a push-forward operation:

$$F_* : \mathfrak{X}(M) \to \mathfrak{X}(N), \quad X \mapsto F_*(X)$$

where, for $X \in \mathfrak{X}(M)$, $F_*(X) \in \mathfrak{X}(N)$ is defined by describing how it acts on functions $f \in \mathscr{C}^\infty(N)$:

$$L_{F_*(X)}(f) := L_X(f \circ F) \circ F^{-1} \quad \text{for all } f \in \mathscr{C}^\infty(N). \tag{3.5.5}$$

Or, with a bit more insight: F induces a pull-back operation on functions

$$F^* : \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(M), \quad f \mapsto f \circ F;$$

this works for any smooth F but, for diffeomorphisms, this is an isomorphism (which is the inverse?). And then one can just use it to transport derivations on $\mathscr{C}^\infty(M)$ to derivations on $\mathscr{C}^\infty(N)$

$$\begin{array}{ccc} \mathscr{C}^\infty(M) & \xrightarrow{L_X} & \mathscr{C}^\infty(M) \\ \uparrow F^* & & \uparrow F^* \\ \mathscr{C}^\infty(N) & \xrightarrow[L_{F_*X}]{} & \mathscr{C}^\infty(N) \end{array}$$

And that is precisely what formula (3.5.5) does; and you check yourself that the resulting dotted arrow is, indeed, a derivation, i.e. (3.5.5) does define a vector field $F_*(X) \in \mathfrak{X}(N)$. A more explicit description of $F_*(X)$ (that gives $F_*(X)_q \in T_qN$ for each $q \in N$) is obtained using the formula $L_Y(g)(p) = (dg)_p(Y_p)$ for vector fields Y and smooth functions g.

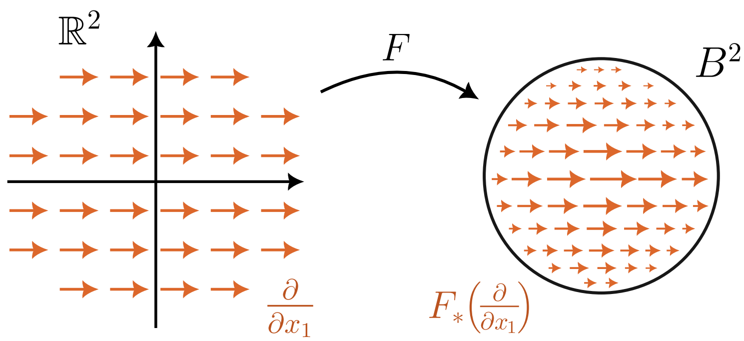


Fig. 3.8 Consider a map $F : \mathbb{R}^2 \to B^2$ that diffeomorphically compresses $\mathbb{R}^2$ into the open disc B^2. The push-forward of the canonical coordinate vector field $\frac{\partial}{\partial x_1}$ along F is exactly the vector field that intuitively gets compressed along with the surrounding space. Can you choose an explicit map F and calculate the components of $F_*(\frac{\partial}{\partial x_1})$ with regard to the canonical identity chart of $B^2 \subseteq \mathbb{R}^2$?

Exercise 3.52. Deduce that, for all $q \in N$,

$$F_*(X)_q = (dF)_p(X_p) \quad \text{where } p = F^{-1}(q). \tag{3.5.6}$$

Also show that, if $G : N \to P$ is another diffeomorphism, then $(G \circ F)_* = G_* \circ F_*$.

Proposition 3.53. *The push forward operation preserves the Lie bracket of vector fields: for any $X_1, X_2 \in \mathfrak{X}(M)$, one has*

$$F_*([X_1, X_2]) = [F_*(X_1), F_*(X_2)].$$

Proof. We start with the definition of $[F_*(X_1), F_*(X_2)]$:

$$L_{[F_*(X_1),F_*(X_2)]}(f) = L_{F_*(X_1)}\left(L_{F_*(X_2)}(f)\right) - L_{F_*(X_2)}\left(L_{F_*(X_1)}(f)\right)$$

(for all $f \in \mathscr{C}^\infty(N)$, in which we plug in the definition of $F_*(X_i)$ to obtain

$$L_{F_*(X_1)}\left(L_{X_2}(f \circ F) \circ F^{-1}\right) - \text{ the similar one} = L_{X_1}\left(L_{X_2}(f \circ F)\right) \circ F^{-1} - \text{ the similar one };$$

using again the formula defining $[X_1, X_2]$, and then the one for F_*, the last expression is

$$L_{[X_1,X_2]}(f \circ F) \circ F^{-1} = L_{F_*([X_1,X_2])}(f).$$

😄

Exercise 3.54. By similar arguments define also a pull-back operation $F^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$. And show that this makes sense not only for diffeomorphisms $F : M \to N$, but for all local diffeomorphisms. Also: is it true that $(G \circ F)^* = G^* \circ F^*$?

The construction of push-forwards cannot be extended for arbitrary smooth maps $F : M \to N$. Instead, one can talk about a vector field $X \in \mathfrak{X}(M)$ being "related via F" to a vector field $Y \in \mathfrak{X}(N)$:

Definition 3.55. Given a smooth map $F : M \to N$, we say that $X \in \mathfrak{X}(M)$ is F**-projectable** to $Y \in \mathfrak{X}(N)$ if

$$(dF)_p(X_p) = Y_{F(p)} \quad \text{for all } p \in M.$$

Note that, when F is a diffeomorphism, this corresponds precisely to $Y = F_*(X)$. However, in general, there may be different vector fields on M that are F-projectable to the same vector field on N (think e.g. of the case when N is a point) or there may be vector fields on M that are not projectable to any vector field on N. However, Proposition 3.53 still has a version that applies to this general setting:

Exercise 3.56. For any smooth map $F : M \to N$, if $X_i \in \mathfrak{X}(M)$ are F-projectable to $Y_i \in \mathfrak{X}(N)$ for $i \in \{1,2\}$, then $[X_1,X_2]$ is F-projectable to $[Y_1,Y_2]$. (Hint: proceed locally and remember the formula that you obtained when you solved Exercise 3.46).

Exercise 3.57. Exercise 3.48 and Exercise 3.49 give you three vector fields X^i on S^2, and V^i on S^3, respectively; you have surely noticed that, mysteriously enough, they satisfy the same types of formulas for their Lie brackets. How are they related?

Well, here is one more piece of data: to move from S^3 to S^2 you can use the Hopf map h from Exercise 2.110. Do some computations and explain what is going on.
(warning: not to get confused by the symbols, better use different letter for the coordinates in the two spheres- e.g. (x,y,z,t) in S^3 and (u,v,w) in S^2; in particular, the vector fields from Exercise 3.48 will be $V^1 = w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}$ etc.)

3.5.7 *Application: the Lie algebra of a Lie group*

We now discuss the infinitesimal counterpart of Lie groups.

Definition 3.58. A Lie algebra is a vector space $\mathfrak{a}$ endowed with an operation

$$[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$$

which is bi-linear, antisymmetric and which satisfies the Jacobi identity:

$$[[u,v],w] + [[v,w],u] + [[w,u],v = 0 \quad \text{for all } u,v,w \in \mathfrak{a}.$$

Example 3.59 (commutators of matrices). The space $\mathscr{M}_{n\times n}(\mathbb{R})$ of $n \times n$ matrices, together with the commutator of matrices,

$$[A,B] := AB - BA$$

is a Lie algebra. More generally, for any vector space V, the space $\mathrm{Lin}(V,V)$ of linear maps from V can be made into a Lie algebra by the same formula (just that AB becomes the composition of A and B).

Example 3.60 (vector fields). As we have already pointed out, for any manifold M, the space of vector fields $\mathfrak{X}(M)$ endowed with the Lie bracket of vector fields becomes a (infinite dimensional) Lie algebra.

Exercise 3.61. What is the relationship between the previous two examples?

We now explain how a Lie group G gives rise to a finite dimensional Lie algebra $\mathfrak{g}$. As a linear space, it is defined as the tangent space of G at the identity element $e \in G$:

$$\mathfrak{g} := T_e G.$$

For the Lie bracket, we will relate $\mathfrak{g}$ to vector fields on G, and then use the Lie bracket of vector fields. Hence, as a first step, we note that any $v \in \mathfrak{g}$ gives rise to a vector field

$$\vec{v} \in \mathfrak{X}(G).$$

defined as follows. For $g \in G$, to define $\vec{v}_g \in T_g G$, we use the left translation by g:

$$L_g : G \to G, \quad L_g(h) = gh.$$

This is smooth (and even a diffeomorphism!) and make use of its differential at e, $(dL_g)_e : \mathfrak{g} \to T_g G$. Define then

$$\vec{v}_g := (dL_g)_e(v) \in T_g G.$$

Exercise 3.62. Show that, indeed, $\vec{v}$ is smooth.

Proposition 3.63. *With the same notations as above (a Lie group G and $\mathfrak{g} := T_e G$), for any $u, v \in \mathfrak{g}$, the Lie bracket $[\vec{u}, \vec{v}] \in \mathfrak{X}(G)$ is again of type $\vec{w}$ for some $w \in \mathfrak{g}$. Denoting the resulting w by $[u, v] \in \mathfrak{g}$, we obtain an operation*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that makes $\mathfrak{g}$ into a Lie algebra.

Proof. Note that the map

$$\mathrm{j} : \mathfrak{g} \to \mathfrak{X}(G), \quad v \mapsto \vec{v} \tag{3.5.7}$$

is injective. That is simply because $\vec{v}_e = v$. This implies the uniqueness of w; and it also indicates how to define it:

$$w := [\vec{u}, \vec{v}]_e.$$

We still have to show that $[\vec{u}, \vec{v}] = \vec{w}$. For that we note that

$$(L_a)_*(\vec{v}) = \vec{v}$$

for all $a \in G$ and all $v \in \mathfrak{g}$, where we use the push-forward as defined in subsection 3.5.6. Indeed, using the formula (3.5.6) for the push-forward and then the definition of $\vec{v}$:

$$(L_a)_*(\vec{v})_g = (dL_a)_{a^{-1}g}\left((dL_{a^{-1}g})_e(v)\right)$$

and then, using the chain rule and $L_a \circ L_{a^{-1}g} = L_g$, we end up with $\vec{v}_g$.

Since $(L_a)_*(\vec{v}) = \vec{v}$ and similarly for u, Proposition 3.53 implies that $[\vec{u}, \vec{v}] = (L_a)_*([\vec{u}, \vec{v}])$. Evaluating this at $a \in G$ and using again formula formula (3.5.6) (where $F = L_a, q = a$ so that $p = L_a^{-1}(a) = e$):

$$[\vec{u}, \vec{u}]_a = (dL_a)_e([\vec{u}, \vec{u}]_e) = (dL_a)_e(w_e) = \vec{w}_a\ .$$

To fact that the resulting operation $[\cdot,\cdot]$ on $\mathfrak{g}$ is a bilinear and skew-symmetric is immediate. For the Jacobi identity, since the map j from (3.5.7) is injective and satisfies $j([v,w]) = [j(v), j(w)]$, it suffices to remember that the Lie bracket of vector fields does satisfy the Jacobi identity. 😀

Definition 3.64. The Lie algebra of the Lie group G is defined as $\mathfrak{g} = T_e G$ endowed with the Lie algebra structure from the previous proposition.

We now compute some examples.

Example 3.65. We start with the general linear group $GL_n(\mathbb{R})$, and we claim that its Lie algebra is just the algebra $\mathscr{M}_{n\times n}(\mathbb{R})$ of $n \times n$ matrices endowed with the commutator bracket.

Proof. As we already mentioned, $\mathscr{M}_{n\times n}(\mathbb{R})$ is seen as a Euclidean space; we denote coordinates functions by

$$x^i_j : \mathscr{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$$

(sending a matrix to the element on the position (i,j)). We also remarked already that, while $GL_n = GL_n(\mathbb{R})$ sits openly inside $\mathscr{M}_{n\times n}(\mathbb{R})$, its tangent space at the identity matrix I (and similarly at any point) is canonically identified with $\mathscr{M}_{n\times n}(\mathbb{R})$: any $X \in \mathscr{M}_{n\times n}(\mathbb{R})$ is identified with the speed at $t = 0$ of $t \mapsto (I + tX)$. After left translating, we find that the corresponding vector field $\vec{X} \in \mathfrak{X}(GL_n)$ is given, at an arbitrary point $A \in GL_n$, by

$$\vec{X}_A = \frac{d}{dt}\Big|_{t=0} A\cdot(I+tX) \in T_A GL_n. \tag{3.5.8}$$

Let $X, Y \in \mathscr{M}_{n\times n}(\mathbb{R})$ and let $[X,Y] \in \mathscr{M}_{n\times n}(\mathbb{R})$ be the Lie algebra bracket- hence defined by $\overrightarrow{[X,Y]} = [\vec{X}, \vec{Y}]$, where the last bracket is the Lie bracket of vector fields on GL_n. To compute it, we use the defining equation for the Lie bracket of two vector fields:

$$L_{[\vec{X},\vec{Y}]}(F) = L_{\vec{X}} L_{\vec{Y}}(F) - L_{\vec{Y}} L_{\vec{X}}(F) \tag{3.5.9}$$

for all smooth functions

$$F : GL_n \to \mathbb{R}.$$

From the previous description of $\vec{X}$ we deduce that

$$L_{\vec{X}}(F)(A) = \frac{d}{dt}\Big|_{t=0} F(A\cdot(I+tX)) = \sum_{i,j,k} \frac{\partial F}{\partial x^i_j}(A) A^i_k X^k_j.$$

In particular, applied to a coordinate function $F = x^i_j$, we find

$$L_{\vec{X}}(x^i_j)(A) = \sum_k A^i_k X^k_j$$

or, equivalently, $L_{\vec{X}}(x^i_j) = \sum_k x^i_k X^k_j$. Therefore

$$L_{\vec{Y}}(L_{\vec{X}}(x^i_j)) = \sum_k L_{\vec{Y}}(x^i_k) X^k_j = \sum_{k,l} x^i_l Y^l_k X^k_j = \sum_l x^i_l (Y\cdot X)^l_j,$$

which is precisely $L_{\overrightarrow{Y\cdot X}}(x^i_j)$. Since this holds for all coordinate functions, we have

$$L_{\vec{Y}} \circ L_{\vec{X}} = L_{\overrightarrow{Y\cdot X}};$$

we deduce that $[\vec{X}, \vec{Y}] = \overrightarrow{Y\cdot X} - \overrightarrow{X\cdot Y}$ and then that $[X,Y]$ is $X\cdot Y - Y\cdot X$, i.e. the usual commutator of matrices.

Example 3.66. For the classical subgroups of GL_n from Example 2.128, since $T_I \mathrm{GL}_n = \mathscr{M}_{n\times n}$, their Lie algebras will be subspaces of $\mathscr{M}_{n\times n}(\mathbb{R})$. They can be described explicitly in each case. Let us look here at $O(n)$. When we discuss its smooth structure (in Example 2.129) we made use of the regular value theorem writing $O(n) = f^{-1}(\{I\})$ and computing df at arbitrary points. In particular, since $(df)_I(X) = X + X^T$, one finds that

$$T_I O(n) = \{X \in \mathscr{M}_{n\times n}(\mathbb{R}) : X + X^T = 0\},$$

the space of skew-symmetric matrices, usually denoted $o(n)$. Similar computations can be done for the other groups, leading to the following list:

- for $O(n)$ and $SO(n)$ we obtain
$$o(n) := \{X \in \mathscr{M}_{n\times n}(\mathbb{R}) : X + X^T = 0\}.$$
- for $SL_n(\mathbb{R})$ we obtain
$$sl_n(\mathbb{R}) := \{A \in \mathscr{M}_{n\times n}(\mathbb{R}) : Tr(X) = 0\}.$$
- $U(n)$ we obtain
$$u(n) := \{X \in \mathscr{M}_{n\times n}(\mathbb{C}) : X + X^* = 0\}.$$
- for $SU(n)$ we obtain
$$su(n) := \{X \in \mathscr{M}_{n\times n}(\mathbb{C}) : X + X^* = 0, Tr(X) = 0\}.$$

For more general closed subgroups $G \subset \mathrm{GL}_n$ (see Theorem 2.131) their Lie algebras appeared implicitly in the proof of Theorem 2.131): it follows that they are precisely the spaces (2.5.12) that were used there

$$T_I G = \{X \in \mathscr{M}_{n\times n} : \exp(tX) \in G \text{ for all } t\}. \tag{3.5.10}$$

What about the Lie brackets? We claim that, in all these examples, i.e. when G is one of the classical groups from Example 2.128 or, if you are comfortable with Theorem 2.131, then for all closed subgroups of GL_n, the resulting Lie bracket on $\mathfrak{g} = T_I G$ is still the commutator bracket. Notice that the fact that $\mathfrak{g}$ was closed under the commutator bracket operation was already pointed out in the previous chapter, in Exercise 2.138.

Proof. Note that for $X \in \mathfrak{g}$ we have two induced vector fields: one on G, denoted as above by $\overrightarrow{X} \in \mathfrak{X}(G)$ and then, since $\mathfrak{g} \subset \mathscr{M}_{n\times n}(\mathbb{R})$ we will have a similar one on GL_n; to distinguish the two we will denote the second one by $\overset{\longmapsto}{X} \in \mathfrak{X}(GL_n)$. Note that, denoting by $i : G \hookrightarrow GL_n$ the inclusion, we are in the situation of Definition 3.55: $\overrightarrow{X}$ is i-projectable to $\overset{\longmapsto}{X}$. Hence, by Exercise 3.56, $[\overrightarrow{X}, \overrightarrow{Y}]$ is i-projectable to $[\overset{\longmapsto}{X}, \overset{\longmapsto}{Y}]$ (for all $X, Y \in \mathfrak{g}$). By the previous example, the last expression is $\overset{\longmapsto}{[X,Y]_{\mathrm{comm}}}$ where, for clarity, we denote by $[X,Y]_{\mathrm{comm}}$ the commutator bracket of the matrices X and Y. Hence, for the Lie bracket of X and Y as elements of the Lie algebra $\mathfrak{g}$ of G we find

$$[X,Y] = [\overrightarrow{X}, \overrightarrow{Y}]_I = \left(\overset{\longmapsto}{[X,Y]_{\mathrm{comm}}}\right)_I = [X,Y]_{\mathrm{comm}}.$$ 😀

Remark 3.67 (For the interested students: Lie's fundamental theorems). The Lie algebra $\mathfrak{g}$ of a Lie group G contains (almost) all the information about G! This may sound surprising at first, since $\mathfrak{g}$ is only "the linear approximation" of G and only around the identity $e \in G$. The fact that it is enough to consider only the identity e can be explained by the fact that, using the group structure (and the resulting left translations) one can move around from e to any other element in g. The fact that "the linear approximation" contains (almost) all the information is more subtle and the real content of this is to be found on the Lie bracket structure of $\mathfrak{g}$.

And here is what happens precisely:

- As we have seen, any Lie group has an associated (finite dimensional) Lie algebra.
- Any finite dimensional Lie algebra $\mathfrak{g}$ comes from a Lie group.
- For any finite dimensional Lie algebra $\mathfrak{g}$ there exists and is unique a Lie group $G(\mathfrak{g})$ whose Lie algebra is $\mathfrak{g}$ and which is 1-connected (i.e. both connected as well as simply-connected). The Lie group $G(\mathfrak{g})$ can be constructed explicitly out of $\mathfrak{g}$. And any other connected Lie group which has $\mathfrak{g}$ as Lie algebra is a quotient of $G(\mathfrak{g})$ by a discrete subgroup.
- Explanation for the 1-connectedness condition: starting with a Lie group G, the connected component of the identity element, is itself a Lie group with the same Lie algebra as G. And so is the universal cover of G. I.e. one can always replace G by a Lie group that is 1-connected without changing its Lie algebra.

- the construction $G \mapsto \mathfrak{g}$ gives an equivalence between (the category of) 1-connected Lie groups and (the category of) finite dimensional Lie algebras. At the level of morphisms even more is true: while a morphism of Lie groups $F: G \to H$ induces a morphism of the corresponding Lie algebras, $f: \mathfrak{g} \to \mathfrak{h}$, if G is 1-connected (but H is arbitrary), then F can be recovered from f (i.e.: any morphism f of Lie algebras comes from a unique morphism F of Lie groups!). That is remarkable since f is just the linearization of F at the identity $f = (dF)_e$.

3.6 The flow of a vector field

The brief philosophy is:

If you live on a manifold M (... not necessarily the sphere) and your are sitting at a point p, then a tangent vector $X_p \in T_pM$ gives you a direction in which you can start walking. A vector field however gives you an entire path to walk on. In the ideal situation (no friction etc), and if you think of a vector field as describing the wind blowing, its integral curve at p is the trajectory that you will follow, blown by the wind.

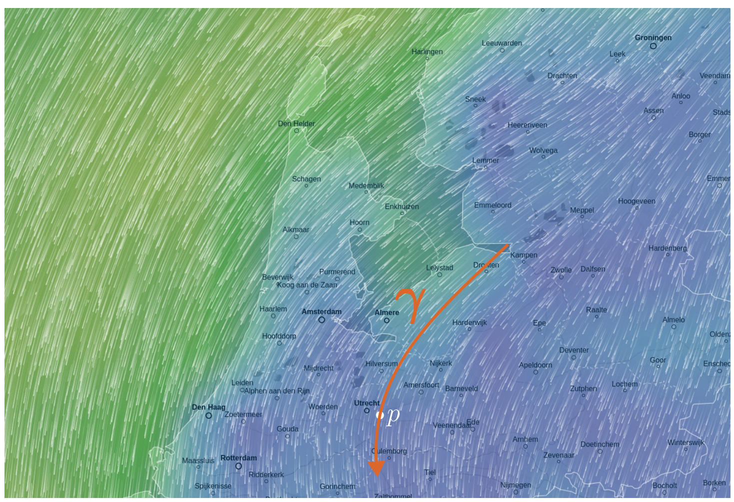


Fig. 3.9 Wind speeds as a vector field together with an integral curve γ passing through the Uithof p.

3.6.1 Integral curves

We first make precise what it means for "a trajectory (path) to follow a vector field".

Definition 3.68. Given a vector field $X \in \mathfrak{X}(M)$, **an integral curve of** X is any curve $\gamma: I \to M$ defined on some open interval $I \subset \mathbb{R}$ ($I = \mathbb{R}$ not excluded) such that

$$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad \text{for all } t \in I.$$

We say that γ **starts at** p if $0 \in I$ and $\gamma(0) = p$.

Example 3.69. Consider $M = \mathbb{R}^2$ and the vector field X given by

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Then a curve in $\mathbb{R}^2$, written as $\gamma(t) = (x(t), y(t))$, is an integral curve if and only if

$$\dot{x}(t) = 1, \quad \dot{y}(t) = 1.$$

Hence we find that the integral curves of X are those of type

$$\gamma(t) = (t + a, t + b)$$

with $a, b \in \mathbb{R}$ constants. Prescribing the starting point of γ is the same thing as fixing the constants a and b (and then we have a unique integral curve).

A more interesting vector field is:

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Then the equations to solve are $\dot{x} = y$ and $\dot{y} = -x$. Hence $\ddot{x} = -x$ from which one can deduce (or guess, and then use the uniqueness recalled below) that

$$x(t) = a\cos t + b\sin t, \quad y(t) = b\cos t - a\sin t,$$

for some (arbitrary) constants $a, b \in \mathbb{R}$ (and, again, prescribing the starting point of γ is the same thing as fixing the constants a and b).

Note that in both examples the integral curves that we were obtaining are defined on the entire $I = \mathbb{R}$. However, this need not always be the case. For instance, for

$$X = -x^2\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

starting with any constants $a, b \in \mathbb{R}$ one has the integral curve given by

$$\gamma(t) = \left(\frac{a}{at+1},\ be^{-t}\right);$$

but, for $a > 0$, the largest interval containing 0 on which this is defined is $(-\frac{1}{a}, \infty)$. However, still as above, for any point $p = (a, b) \in \mathbb{R}^2$, one finds an integral curve that starts at p.

Of course, in the previous examples, the existence of integral curves starting at a given (arbitrary) point is no accident:

Proposition 3.70 (local existence and uniqueness). *Given a vector field $X \in \mathfrak{X}(M)$:*

1. *for any $p \in M$ there exists an integral curve $\gamma : I \to M$ of X that starts at p.*
2. *any two integral curves of X that coincide at a certain time t_0 must coincide in a neighborhood of t_0.*

Since this is a local statement, we just have to look in a chart. Then, as we shall see, what we end up with is a system of ODEs and the previous proposition becomes the standard local existence/uniqueness result for ODEs. Here are the details. Fix a chart $\chi : U \to \Omega \subset M$ of M around p. Then, as in (3.5.1), X can be written on U as

$$X_p = \sum_{i=1}^{m} X_\chi^i(\chi(p)) \left(\frac{\partial}{\partial \chi_i}\right)_p$$

with the coordinate functions X_χ^i of X w.r.t. χ being smooth functions on Ω. Also, a curve γ in U becomes, via χ, a curve γ_χ in Ω. Since $\gamma(t) = \chi^{-1}(\gamma_\chi(t))$, its derivatives becomes

$$\frac{d\gamma}{dt}(t) = \frac{d\gamma_\chi^i}{dt}(t) \left(\frac{\partial}{\partial \chi_i}\right)_{\gamma(t)}.$$

Hence the integral curve condition becomes the system of ODEs:

$$\frac{d\gamma_\chi^i}{dt}(t) = X_\chi^i(\gamma_\chi^1(t), \ldots, \gamma_\chi^m(t)) \quad i \in \{1, \ldots, m\}$$

or, more compactly,

$$\frac{d\gamma_\chi}{dt}(t) = F_\chi(\gamma_\chi(t)),$$

where $F_\chi = (X_\chi^1, \ldots, X_\chi^m) : \Omega \to \mathbb{R}^m$. And here is the standard result on such ODEs:

Theorem 3.71. *Let $F : \Omega \to \mathbb{R}^m$ be a smooth function. Then, for any $x \in \Omega$, the following ordinary differential equation with initial condition:*

$$\frac{d\gamma}{dt}(t) = F(\gamma(t)), \quad \gamma(0) = x \tag{3.6.1}$$

has a solution γ defined on an open interval containing $0 \in \mathbb{R}$, and any two such solutions must coincide in a neighborhood of 0.
Furthermore, for any $x_0 \in \Omega$ there exists an open neighborhood Ω_0 of x_0 in Ω, $\varepsilon > 0$ and a smooth map

$$\phi : (-\varepsilon, \varepsilon) \times \Omega_{x_0} \to \Omega,$$

such that, for any $x \in \Omega_0$, $\phi(\cdot, x) : (-\varepsilon, \varepsilon) \to \Omega$ is a solution of (3.6.1).

We see that this the first part of the theorem immediately implies (well, it is actually the same as) the previous proposition. For a more global version of the proposition, we have to look at maximal integral curves.

Definition 3.72. Given a vector field $X \in \mathfrak{X}(M)$, a **maximal integral curve** of X is any integral curve $\gamma : I \to M$ which admits no extension to a strictly larger interval $\tilde{I}$ and which is still an integral curve of X.

With this, the local result implies quite easily the following one, with a more global flavour.

Corollary 3.73. *Given a vector field $X \in \mathfrak{X}(M)$, for any $p \in M$ there exists a unique maximal integral curve of X $\gamma_p : I_p \to M$ that starts at p.*

Proof. The main remark is that if γ_1 and γ_2 are two integral curves defined on intervals I_1 and I_2, respectively, then the set I of points in $I_1 \cap I_2$ on which the two coincide is closed in $I_1 \cap I_2$ (because it is defined by an equation) and open by the second part of the previous proposition. Hence, since $I_1 \cap I_2$ is connected, if $I \neq \emptyset$ (i.e. the two coincide at some t), then $I = I_1 \cap I_2$ (i.e. the two must coincide on their common domain of definition). Therefore, to obtain I_p and γ_p we can just put together all the integral curves that start at p. 😀

The best possible scenario is:

Definition 3.74. We say that $X \in \mathfrak{X}(M)$ is **complete** if all the maximal integral curves are defined on the entire $\mathbb{R}$.

As we shall see below, all the vector fields that are compactly supported (i.e. which vanish outside a compact subset of M) are complete. In particular, when M is compact, all vector fields on M are complete.

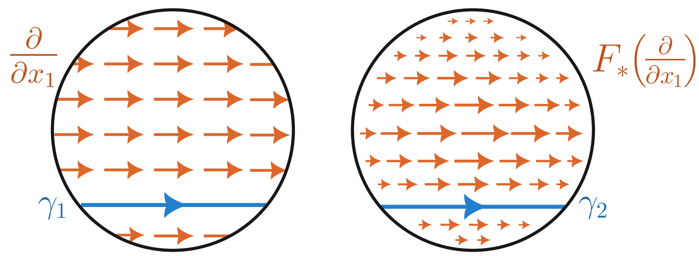


Fig. 3.10 Take M to be the open unit disc in $\mathbb{R}^2$ and consider the vector fields $\frac{\partial}{\partial x_1}$ and $F_*(\frac{\partial}{\partial x_1})$, where F is as in Figure 3.8. Note that the marked maximal integral curves γ_1 and γ_2 with respect to each of these fields have the same image, but must be parametrized differently: γ_1 moves at constant (unit) speed and is defined on a finite open interval, while γ_2 is defined on all of $\mathbb{R}$ and moves slower and slower as one tends towards the boundary (within the ambient $\mathbb{R}^2$) of M. Since we can't extend γ_1, $\frac{\partial}{\partial x_1}$ is not a complete vector field while $F_*(\frac{\partial}{\partial x_1})$ is. As pushing a vector field along a diffeomorphism preserves completeness, $\frac{\partial}{\partial x_1}$ *is* complete when regarded as a vector field on all of $\mathbb{R}^2$. Note that an open boundary within an ambient space from within a manifold looks like it stretches towards infinity, and intuitively, a vector field is complete if it doesn't grow so fast when moving towards that infinity that you reach it in a finite time when you follow along.

Exercise 3.75. Consider the vector field on the sphere $X \in \mathfrak{X}(S^2)$ given by

$$X_{(x,y,z)} = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Draw it in a picture. Then compute all its integral curves and show them in the picture as well. Is there something more general happening (with the integral curves) at points where a vector field vanishes?

Exercise 3.76. If $\gamma : \mathbb{R} \to M$ is an integral curve of a vector field $X \in \mathfrak{X}(M)$ such that there exists $t_0, t_1 \in \mathbb{R}$ distinct such that $\gamma(t_0) = \gamma(t_1)$, show that γ is periodic, i.e. there exists $T \in \mathbb{R}_{>0}$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. (Hint: uniqueness of integral curves).

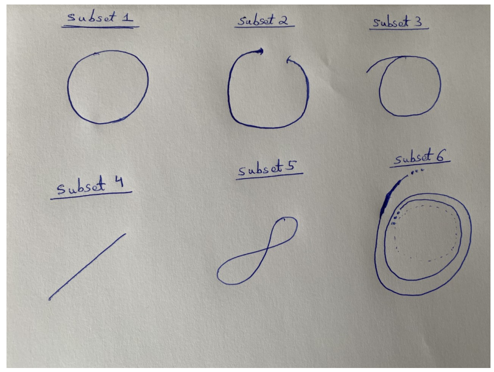
Exercise 3.77. In this exercise $\gamma : \mathbb{R} \to M$ is an integral curve of a vector field $X \in \mathfrak{X}(M)$ and we assume that γ is periodic, in the sense that there exists $T \in \mathbb{R}_{>0}$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. Such a T will be called a period of γ.

1. Show that any closed subgroup Γ of $(\mathbb{R}, +)$, different from $\mathbb{R}$, must be of type $\mathbb{Z} \cdot r_0$ for some $r_0 \in \mathbb{R}$.
2. Show that one can talk about "the period of γ", i.e. there exists a smallest period T_0 of γ.
3. With T_0 as above, show that

$$\overline{\gamma} : S^1 \to M, \quad e^{i\cdot\theta} \mapsto \gamma\left(\frac{T\cdot\theta}{2\pi}\right)$$

is an embedding of S^1 in M.

Exercise 3.78 (from the 2019 exam). Which of the following subsets of $\mathbb{R}^2$ can be the image of an integral curve of a vector field $X \in \mathfrak{X}(\mathbb{R}^2)$. (Explanation: subset 1 is a circle, subset 2 is an open part of a circle, subset 4 is an infinite line, subset 6 describes a spiral approaching a circle. Please explain you answer; but when you describe a vector field, you do not have to write down a formula- just draw it on the picture)



Exercise 3.79 (from the 2019 exam). On the 2-sphere we consider the following curve:

$$\gamma : \mathbb{R} \to S^2, \quad \gamma(t) = \left(\frac{t}{\sqrt{1+t^2}}, \frac{\cos t}{\sqrt{1+t^2}}, \frac{\sin t}{\sqrt{1+t^2}} \right).$$

(a) draw a picture of γ.
(b) find a vector field X on S^2 (explicit formulas!) for which γ is an integral curve.
(c) did you check how smooth your vector field is? please do!

3.6.2 Flows

We now put together all the maximal integral curves in one object: the flow of X. Therefore, we define:

- **the domain of the flow** of X as

$$\mathscr{D}(X) := \{(p,t) \in M \times \mathbb{R} : t \in I_p\} \subset M \times \mathbb{R}.$$

- **the flow of** X is defined as the resulting map

$$\phi_X : \mathscr{D}(X) \to M, \quad \phi_X(p,t) := \gamma_p(t).$$

For complete vector fields one has $\mathscr{D}(X) = M \times \mathbb{R}$. What we can say in general is that

$$M \times \{0\} \subset \mathscr{D}(X) \subset M \times \mathbb{R}.$$

and $\mathscr{D}(X)$ is open in $M \times \mathbb{R}$. Actually, applying the second part of the Theorem 3.71, we deduce:

Corollary 3.80. *For any $X \in \mathfrak{X}(M)$, the domain $\mathscr{D}(X)$ is open in $M \times \mathbb{R}$ and the flow ϕ_X is a smooth map.*

One of the main uses of the flow of a vector field X comes from the maps one obtains whenever one fixes a time t; we will use the notation:

$$\phi_X^t(\cdot) := \phi_X(\cdot, t)$$

The best scenario is when X is complete, when each ϕ^t will be defined on the entire M:

Theorem 3.81. *If $X \in \mathfrak{X}(M)$ is complete then $\phi_X^t : M \times \mathbb{R} \to M$ satisfy:*

$$\phi_X^t \circ \phi_X^s = \phi_X^{t+s}, \quad \phi_X^0 = Id$$

(for all $t, s \in \mathbb{R}$). In particular, each ϕ_X^t is a diffeomorphism and

$$\left(\phi_X^t\right)^{-1} = \phi_X^{-t}.$$

Proof. Fixing $x \in M$ and $s \in \mathbb{R}$, the two curves

$$t \mapsto \phi_X^{t+s}(p), \quad t \mapsto \phi_X^t(\phi_X^s(p))$$

are both integral curves of X and they coincide at $t = 0$- hence they coincide everywhere. That ϕ_X^0 is the identity is clear; we see that the last part follows by taking $s = -t$ in the first part. 😀

For general (possibly non-complete) vector fields one has to be a bit more careful.

Definition 3.82. For $X \in \mathfrak{X}(M)$ and $t \in \mathbb{R}$, define **the flow of X at time t** as the map

$$\phi_X^t : \mathscr{D}_t(X) \to M, \quad p \mapsto \phi_X^t(p) := \phi_X(p,t) = \gamma_p(t),$$

defined on:

$$\mathscr{D}_t(X) := \{p \in M : t \in I_p\}.$$

Theorem 3.83. *For any $X \in \mathfrak{X}(M)$:*

1. *for any $t \in \mathbb{R}$, the domain $\mathscr{D}_t(X)$ of ϕ_X^t is open in M, ϕ_X^t takes values in $\mathscr{D}_{-t}(X)$, and*

$$\phi_X^t : \mathscr{D}_t(X) \to \mathscr{D}_{-t}(X)$$

   is a diffeomorphism.
2. *For $t, s \in \mathbb{R}$ one has*

$$\phi^t \circ \phi^s = \phi^{t+s}$$

   in the sense that, for each $p \in M$ on which the left hand side is defined, also $\phi^{t+s}(p)$ is defined and the two expressions coincide.

Proof. The fact that $\mathscr{D}_t(X)$ is open and ϕ_X^t is smooth follows right away from the previous the standard local result (Theorem 3.71). For the last part we consider again the two curves from the proof of Theorem 3.81; this time we have to be more careful with their domain of definition. Hence we fix $p \in M$, $s \in I(p)$ and we look at the curves:

$$a : -s + I(p) \to M, \quad a(t) = \phi_X^{t+s}(p),$$

$$b : I(\phi_X^s(p)) \to M, \quad b(t) = \phi_X^t(\phi_X^s(p)).$$

Note that both a and b are integral curves of X, and their are both maximal (the second one is maximal from its very definition; for the first one, if $\tilde{a}$ was an extension to $J \supset -s + I(p)$, then $s + J \ni t \mapsto \tilde{a}(t-s)$ would be an integral curve defined on $s+J \supset I(p)$, and starting at p). We then obtain the last part of the theorem together with the equality

$$I(\phi_X^s(p)) = I(p) - s$$

for all s. Since $0 \in I(p)$ we find that $-s \in I(\phi_X^s(p))$, hence

$$\phi_X^s(\mathscr{D}_s) \subset \mathscr{D}_{-s}$$

and, for $p \in \mathscr{D}_s$, $\phi_X^{-s}(\phi_X^s(p)) = p$. We still have to show that, in the previously centered inclusion, equality holds. Using the inclusion for $-s$ instead of s and applying ϕ_X^s we obtain

$$\phi_X^s\left(\phi_X^{-s}(\mathscr{D}_{-s})\right) \subset (\phi_X^s(\mathscr{D}_s);$$

but the first term equals to $\phi_X^0(\mathscr{D}_{-s}) = \mathscr{D}_{-s}$, hence we obtain the reverse inclusion and this finishes the proof.

Exercise 3.84. Show that after multiplying a vector field $X \in \mathfrak{X}(M)$ by a real number λ, one has

$$\mathscr{D}(\lambda \cdot X) = \{(x,t) \in M \times \mathbb{R} : (x, \lambda t) \in \mathscr{D}(M)\}, \quad \phi_{\lambda \cdot X}(x,t) = \phi_X(x, \lambda \cdot t).$$

Exercise 3.85. For the vector field $X \in \mathfrak{X}(S^2)$ that you found in Exercise 3.40, compute its flow. Again, you will probably use the ambient space $\mathbb{R}^3$ to carry out some of the computations (e.g. you may end up computing flows in $\mathbb{R}^3$ and then noticing that they do not leave the sphere). So, again: is there something more general to learn from this?

Exercise 3.86. Consider the vector field V from Exercise 3.38 and the Hopf map $h : S^3 \to S^2$ from Exercise 2.110. Do the following:

(a) Compute the flow of V.
(b) If you did not do it already, write the flow making use of the fact that $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$.
(c) Check that h is constant on the integral curves of V, i.e. $h(\phi_V^t(p)) = h(p)$ for all $p \in S^2$ and all $t \in \mathbb{R}$.
(d) Derive now the main conclusion of Exercise 3.44: that the differential of h kills V.
(e) What can you say about the fibers of the Hopf map h (for each $q \in S^2$ one talks about the fiber of h above q which is simply the pre-image $h^{-1}(p) = \{p \in S^3 : h(p) = q\}$)?

Exercise 3.87. If you want more computations of flows as for V in the previous exercise: look also at the vector fields V^1, V^2 and V^3 from Exercise 3.49.

Exercise 3.88 (from the 2018 exam). This is about vector fields on $\mathbb{R}^3$. Do the following:

(a) find a vector field X for which $\gamma(t) = (e^{2t}, e^t \sin t, e^t \cos t)$ is an integral curve.
(b) find one more such vector field, but different from X.
(c) for one of the vector fields you found, compute its flow.

Exercise 3.89. Do again 3.76, but using now the properties of the flow ϕ_X^t.

3.6.3 Completeness

The notion of completeness was already mentioned before as "the best possible scenario": $X \in \mathfrak{X}(M)$ is complete if all the maximal integral curves are defined on the entire $\mathbb{R}$ or, equivalently, the domain of its flow is the entire $M \times \mathbb{R}$:

$$\mathscr{D}(X) = M \times \mathbb{R}.$$

Theorem 3.90. *If M is compact then any vector field $X \in \mathfrak{X}(M)$ is complete. More generally, for any M, any $X \in \mathfrak{X}(M)$ that is compactly supported (i.e. is zero outside some compact subset of M) is complete.*

Proof. When M is compact, since $\mathscr{D}(X)$ is an open in $M \times \mathbb{R}$ containing $M \times \{0\}$, the tube lemma implies that there exists $\varepsilon > 0$ such that:

$$M \times (-\varepsilon, \varepsilon) \subset \mathscr{D}(X). \tag{3.6.2}$$

(proof: for any $p \in M$, using that ϕ_X is continuous at $(p,0)$ and that $\mathscr{D}(X)$ is open, find $\varepsilon_p > 0$ and U_p open containing p with $U_p \times (-\varepsilon_p, \varepsilon_p) \subset \mathscr{D}(X)$. Then $\{U_p : p \in M\}$ is an open cover of M hence, by compactness, M is covered by a finite number of them, say the ones corresponding to $p_1, \ldots, p_k \in M$; set ε as the smallest of the epsilons corresponding to the points p_i.)

We claim that the existence of ε such that the inclusion (3.6.2) holds implies (without further using the compactness of M!) that X is complete. Indeed, we would have that $\phi_X^t(p)$ is defined for all $t \in (-\varepsilon, \varepsilon)$ and all $p \in M$. But then so would be $\phi_X^t(\phi_X^t(p))$, hence $\phi_X^{2t}(p)$, hence (3.6.2) holds also with 2ε instead of ε. Repeating the process, we find that $\mathbb{R} \times M \subset \mathscr{D}(X)$, hence X must be complete.

For the last part (M general and X is compactly supported) it is enough to remark that an inclusion of type (3.6.2) can still be achieved: if X is zero outside the compact K then the tube lemma implies that $(-\varepsilon, \varepsilon) \times K \subset \mathscr{D}(X)$ for some $\varepsilon > 0$, while all the points (t, p) with $p \notin K$ are clearly in $\mathscr{D}(X)$ since X vanishes at p (hence the integral curve of X starting at p is simply $\gamma(t) = p$, defined for all t). 😀

For latter reference, let us also cast the last part of the argument into:

Corollary 3.91. *If $X \in \mathfrak{X}(M)$ has the property that $M \times (-\varepsilon, \varepsilon) \subset \mathscr{D}(X)$ for some $\varepsilon > 0$, then X is complete.*

Exercise 3.92. Let $F : M \to N$ be a smooth function. If $X \in \mathfrak{X}(M)$ is F-projectable to $Y \in \mathfrak{X}(N)$ (see Definition 3.55) show that, for any $t \in \mathbb{R}$ one has

$$F(\mathscr{D}_t(X)) \subset \mathscr{D}_t(Y)$$

and, on $\mathscr{D}_t(X)$,

$$F \circ \phi_X^t = \phi_Y^t \circ F.$$

3.6.4 Lie derivatives along vector fields

The general philosophy of taking Lie derivatives L_X along vector fields is rather simple:

for any "type of objects" ξ on manifolds M (functions, vector fields, etc etc), which are natural (in the sense that a diffeomorphism $\phi : M \to N$ allows one pull-back the objects from N to ones on M), the Lie derivative $\mathscr{L}_X(\xi)$ of ξ along X measures the variation of ξ along the flow of X:

$$\mathscr{L}_X(\xi) := \left.\frac{d}{dt}\right|_{t=0} (\phi_X^t)^*(\xi) = \lim_{t \to 0} \frac{(\phi_X^t)^*(\xi) - \xi}{t}. \tag{3.6.3}$$

This is a very general principle that is applied over and over in Differential Geometry, depending on the "type of objects" one is interested in. Here we illustrate this principle for functions and vector fields, and compute the outcome. For functions, there is an obvious pull-back operation associated to any smooth function $F : M \to N$ (diffeomorphism or not):

$$F^* : \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(M), \quad F^*(f) = f \circ F.$$

Therefore, when $F = \phi_X^t : M \to M$ is the flow of a complete vector field, the guiding equation (3.6.3) for $\xi = f \in \mathscr{C}^\infty(M)$ makes sense pointwise as

$$\mathscr{L}_X(f)(p) := \left.\frac{d}{dt}\right|_{t=0} f(\phi_X^t(p)). \tag{3.6.4}$$

Even more, written in this way, the completeness of X is not even necessary for this formula to make sense (indeed, we only need $\phi_X^t(p)$ to be defined for t near 0, which is always the case).

Exercise 3.93. For the vector field $X \in \mathfrak{X}(S^2)$ from Exercise 3.40 you have computed its flow in Exercise 3.85. Take $f(x,y,z) = x + y^2 + z^3$ interpreted as a smooth function on S^2 and compute now $\mathscr{L}_X(f)$. Did you get the same result as in Exercise 3.43?

Proposition 3.94. *For $X \in \mathfrak{X}(M)$ and $f \in \mathscr{C}^\infty(M)$, $\mathscr{L}_X(f)$ defined by (3.6.4) is precisely $L_X(f) \in \mathscr{C}^\infty(M)$.*

Proof. The expression in the right hand side of (3.6.4) is precisely

$$(df)_p\left(\frac{d}{dt}\bigg|_{t=0} \phi_X^t(p)\right) = (df)_p(X_p) = L_X(f)(p).$$

😀

The discussion is similar for vector fields. In this case, a diffeomorphism $F : M \to N$ allows one to pull-back vector fields on N to vector fields on M,

$$F^* : \mathfrak{X}(N) \to \mathfrak{X}(M)$$

where, for $Y \in \mathfrak{X}(N)$, its pull-back $F^*(Y) \in \mathfrak{X}(M)$ is defined by

$$F^*(Y)_p := (dF^{-1})_{F(p)}(Y_{F(p)}).$$

With this, when $F = \phi_X^t : M \to M$ is the flow of a complete vector field, the guiding equation (3.6.3) for $\xi = Y \in \mathfrak{X}(M)$ makes sense pointwise as

$$\mathscr{L}_X(Y)(p) := \frac{d}{dt}\bigg|_{t=0} (d\phi_X^{-t})_{\phi_X^t(p)}(Y_{\phi_X^t(p)}). \tag{3.6.5}$$

And, as for functions, this formula makes sense without any completeness assumption on X.

Proposition 3.95. *For $X \in \mathfrak{X}(M)$ and any other $Y \in \mathfrak{X}(M)$, $\mathscr{L}_X(Y)$ defined by (3.6.5) is precisely the Lie bracket $[X,Y] \in \mathfrak{X}(M)$.*

Proof. Let $Z = \mathscr{L}_X(Y)$. For $f \in \mathscr{C}^\infty(M)$ we compute $L_Z(f)(p) = (df)_p(Z_p)$ and we find

$$\frac{d}{dt}\bigg|_{t=0} (df)_p\left((d\phi_X^{-t})_{\phi_X^t(p)}(Y_{\phi_X^t(p)})\right) = \frac{d}{dt}\bigg|_{t=0} d\left(f \circ \phi_X^{-t}\right)_{\phi_X^t(p)}\left(Y_{\phi_X^t(p)}\right).$$

Note that $f \circ \Phi_X^{-t}$ is f at $t = 0$ and has the derivative w.r.t. t at $t = 0$ equal to $-L_X(f)$, hence

$$f \circ \Phi_X^{-t} = f - tL_X(f) + t^2 \cdot ?, \tag{3.6.6}$$

where ? is smooth. Continuing the computation started above, we find

$$\frac{d}{dt}\bigg|_{t=0} (df - td(L_X(f)))_{\phi_X^t(p)}\left(Y_{\phi_X^t(p)}\right) = \frac{d}{dt}\bigg|_{t=0} (df)_{\phi_X^t(p)}\left(Y_{\phi_X^t(p)}\right) - (dL_X(f))(Y_p) =$$

$$\frac{d}{dt}\bigg|_{t=0} L_Y(f)(\phi_X^t(p)) - L_Y(L_X(f))(p) = L_X(L_Y(f))(p) - L_Y(L_X(f))(p),$$

i.e. $L_Z(f)(p) = L_{[X,Y]}(f)(p)$ for all p and f. This implies that $Z = [X,Y]$. 😀

We obtain the following characterization of the commutation relation $[X,Y]=0$; for simplicity, we restrict here to the case of complete vector fields.

Corollary 3.96 (pairwise commuting vector fields). *For $X,Y \in \mathfrak{X}(M)$ complete, one has:*

$$[X,Y]=0 \iff \Phi_X^t \circ \Phi_Y^s = \Phi_Y^s \circ \Phi_X^t \quad \forall\, t,s \in \mathbb{R}.$$

Exercise 3.97. Returning to Exercise 3.48 compute $[X^1,X^2]$ in yet another way, now using flows (that you have to compute!).

Exercise 3.98. Using the previous Proposition and Exercise 3.92 prove again that if $F: M \to N$ is smooth and $X_1, X_2 \in \mathfrak{X}(M)$ are F-projectable to Y_1 and $Y_2 \in \mathfrak{X}(N)$, respectively, then $[X_1,X_2]$ is F-projectable to $[Y_1,Y_2]$.

3.6.5 Application: the exponential map for Lie groups

Proposition 3.99. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $u \in \mathfrak{g}$, the vector field $\vec{u}$ is complete and its flow $\phi^t_{\vec{u}}$ satisfies*

$$\phi^t_{\vec{u}}(ag) = a\phi^t_{\vec{u}}(g).$$

In particular, the flow can be reconstructed from what it does at time $t=1$, at the identity element of G, i.e. from

$$exp: \mathfrak{g} \to G, \quad exp(u) := \phi^1_{\vec{u}}(e),$$

by the formula

$$\phi^t_{\vec{u}}(a) = a\, exp(t\,u).$$

Moreover, the exponential is a local diffeomorphism around the origin: it sends some open neighborhood of the origin in $\mathfrak{g}$ diffeomorphically into an open neighborhood of the identity matrix in G.

Proof. For the first part the key remark is that if $\gamma: I \to G$ is an integral curve of $\vec{u}$ defined on some interval I, i.e. if

$$\frac{d\gamma}{dt}(t) = \vec{u}\,(\gamma(t)) \quad \forall\, t \in I$$

then, for any $a \in G$, the left translate $a \cdot \gamma : t \mapsto a \cdot \gamma(t)$ is again an integral curve (of the same $\vec{u}$); this follows by writing $a \cdot \gamma = L_a \circ \gamma$ and using the chain rule. Hence, if γ_g is the maximal integral curve starting at $g \in G$ then $a \cdot \gamma_g$ will be the one starting at $a \cdot g$- and that proves the first identity (for as long as both terms are defined). And we also obtain that $I_g = I_{ag}$ for all $a,g \in G$, hence $I_g = I_e$ for all $g \in G$. Using Corollary 3.91, we obtain that $\vec{u}$ is complete.

We are left with the last part. For that it suffices to show that $(d\exp)_0$ is an isomorphism. But

$$(d\exp)_0 : \mathfrak{g} \to T_eG = \mathfrak{g}$$

sends $u \in \mathfrak{g}$ to

$$\left.\frac{d}{dt}\right|_{t=0} \exp(tu) = \left.\frac{d}{dt}\right|_{t=0} \phi^t_{\vec{u}}(e) = \vec{u}_e = u,$$

i.e. the differential is actually the identity map. 😀

Example 3.100. In Example 3.65 we have seen that the Lie algebra of the general linear group $GL_n(\mathbb{R})$ is just the algebra $\mathscr{M}_{n\times n}(\mathbb{R})$ of $n \times n$ matrices endowed with the commutator bracket. We claim that the resulting exponential map is just the usual exponential map for matrices:

$$\mathrm{Exp}: \mathscr{M}_{n\times n}(\mathbb{R}) \to GL_n(\mathbb{R}),\ X \mapsto e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

(where we work with the conventions that $X^0 = I$ the identity matrix and $0! = 1$). Let us compute $\exp(X)$ for an arbitrary $X \in \mathscr{M}_{n\times n}(\mathbb{R}) = T_I GL_n$; according to the definition, it is $\gamma(1)$, where γ is the unique path in GL_n satisfying

$$\gamma(0) = I, \ \frac{d\gamma}{dt}(t) = \vec{X}\,(\gamma(t)).$$

The last equality takes place in $T_{\gamma(t)}GL_n$, which is canonically identified with $\mathscr{M}_{n\times n}(\mathbb{R})$ (as in Example 3.65 , $Y \in \mathscr{M}_{n\times n}(\mathbb{R})$ is identified with the speed at $\varepsilon = 0$ of $\varepsilon \mapsto \gamma(t) + \varepsilon Y$). By this identification, $\frac{d\gamma}{dt}(t)$ goes to the usual derivative of γ (as a path in the Euclidean space $\mathscr{M}_{n\times n}(\mathbb{R})$) and $\vec{X}\,(\gamma(t))$ goes to $\gamma(t)\cdot X$. Hence γ is the solution of

$$\gamma(0) = I, \ \gamma'(t) = X \cdot \gamma(t),$$

which is precisely what $t \mapsto e^{tX}$ does. We deduce that $\exp(X) = e^X$.

Given the fact that the main ingredient in proving Theorem 2.131 was the exponential for matrices, which we have just generalized, it is now just a simple observation that the same proof applied in the greater generality, giving rise to:

Theorem 3.101. *Any closed subgroup of a Lie group is automatically an embedded submanifold and, therefore, becomes a Lie group.*

3.6.6 Extra-exercises

Exercise 3.102. (exercise in the 2019-2020 retake) On $\mathbb{R}^2$ compute the Lie bracket $[X,Y]$ of the following vector fields:

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Exercise 3.103 (part of the 2019/2020 retake exam). The 2019/2020 retake exam consisted on 20 questions on the 2-torus

$$\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$$

(equipped with the product manifold structure). Here one considers the vector field $\frac{\partial}{\partial\theta_1}$ given by

$$\left(\frac{\partial}{\partial\theta_1}\right)_{(z_1,z_2)} := \frac{d}{dt}\Big|_{t=0} (e^{i\cdot t}\cdot z_1, z_2) \in T_{(z_1,z_2)}\mathbb{T}^2,$$

and similarly we define $\frac{\partial}{\partial\theta_2}$. For any smooth function $f : \mathbb{T}^2 \to \mathbb{R}$, we use the notations $\frac{\partial f}{\partial\theta_i}$ for the resulting Lie derivatives (again functions on the torus):

$$\frac{\partial f}{\partial\theta_1}(z_1,z_2) = \frac{d}{dt}\Big|_{t=0} f(e^{i\cdot t}\cdot z_1, z_2), \quad \text{etc.}$$

We will also consider the more general vector fields on the torus:

$$X^f := \frac{\partial}{\partial\theta_1} + f\cdot\frac{\partial}{\partial\theta_2} \in \mathfrak{X}(\mathbb{T}^2). \qquad (*)$$

defined for any smooth function f on $\mathbb{T}^2$.

Recall also that for any $R > r > 0$, one has a "concrete model" of $\mathbb{T}^2$:

$$T^2_{R,r} := \{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\},$$

which is the image of the map

$$F:\mathbb{T}^2\to\mathbb{R}^3,\quad (e^{i\theta_1},e^{i\theta_2})\mapsto((R+r\cdot\cos\theta_1)\cdot\cos\theta_2,(R+r\cdot\cos\theta_1)\cdot\sin\theta_2,r\cdot\sin\theta_1).$$

You may assume (e.g. from Topology) that you know already that F is a homeomorphism between $\mathbb{T}^2$ and $T^2_{R,r}$. Here are eight questions from the exam:

1. For $i\in\{1,2\}$ compute $(dF)_p(\frac{\partial}{\partial\theta_i})$ at arbitrary points $p=(e^{i\cdot\theta_1},e^{i\cdot\theta_2})\in\mathbb{T}^2$ and show that F is an immersion at each point.
2. Deduce that F is a diffeomorphism between $\mathbb{T}^2$ and $T^2_{R,r}$.
3. We want to compute the vector fields $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$ by moving them to $T^2_{R,r}$ (via F) and decomposing them w.r.t. the basis $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ of the tangent spaces of $\mathbb{R}^3$. In other words, we want to compute $F_*\left(\frac{\partial}{\partial\theta_1}\right)$ and $F_*\left(\frac{\partial}{\partial\theta_2}\right)$. Show that
$$F_*\left(\frac{\partial}{\partial\theta_2}\right)_q=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y},\quad\text{for all } q=(x,y,z)\in T^2_{R,r},$$
and find a similar formula for $F_*\left(\frac{\partial}{\partial\theta_1}\right)$.
4. Compute the flows of $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$ and then deduce that $[\frac{\partial}{\partial\theta_1},\frac{\partial}{\partial\theta_2}]=0$.
5. Looking at the vector fields of type $(*)$, show that for any two smooth functions f,g on $\mathbb{T}^2$, there exists another smooth function h such that $\frac{\partial}{\partial\theta_1}+[X^f,X^g]=X^h$.
6. For each real number $\lambda\in\mathbb{R}$ consider the vector field X^λ (i.e. $(*)$ obtained when f is the constant function λ). Compute the maximal integral curve γ of X^λ starting at $(1,1)\in\mathbb{T}^2$.
7. With γ as above (and λ arbitrary constant) show that:
    - the image of γ is always an immersed submanifold of $\mathbb{T}^2$
    - but, if λ is irrational, then that image is not an embedded submanifold of $\mathbb{T}^2$.

13. Show that there do not exist smooth functions f,g on $\mathbb{T}^2$ such that $\frac{\partial g}{\partial\theta_1}-\frac{\partial f}{\partial\theta_2}=1$.

Exercise 3.104. Is the sum of two complete vector fields on $\mathbb{R}^m$ again complete? Or the multiplication of a complete vector field by an arbitrary smooth function?

Chapter 4
Differential forms

4.1 Differential forms of degree one (1-forms)

4.1.1 Cotangent vectors

For a (real) vector space V, its dual V^* is the vector space consisting of all linear maps $\xi : V \to \mathbb{R}$. While the elements of V are vectors of V, the elements of V^* are thought of as "covectors of V". And vectors and covectors pair via the evaluation map

$$\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}, \quad (\xi, v) \mapsto \langle \xi, v \rangle := \xi(v).$$

Moreover, any basis $e_1, \ldots, e_m$ of V gives rise to the **dual basis**- the basis $e^1, \ldots, e^m$ of V^* uniquely characterised by the conditions

$$\langle e^i, e_j \rangle = \delta^i_j.$$

Ecplicitly, $e^i : V \to \mathbb{R}$ takes $v \in V$ to the i-the coefficient of v w.r.t. the given basis. In particular, $\dim V^* = \dim V$.

Differential 1-forms on a manifold M arise similarly, in duality with vector fields. First of all, for each $p \in M$,

we define **the cotangent space** of M at p as the dual of the tangent space

$$T^*_p M := (T_p M)^*.$$

The elements of $T^*_p M$ are called **cotangent vectors at** p.

Example 4.1. For any smooth function $f : M \to \mathbb{R}$ its differential at any point $p \in M$,

$$(df)_p : T_p M \to \mathbb{R}, \quad v = \frac{d\gamma}{dt}(0) \mapsto (df)_p(v) = \partial_v(f) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)),$$

can now be interpreted as a cotangent vector at p,

$$(df)_p \in T^*_p M.$$

Of course, for this to make sense we only need f to be defined on an open containing p.

While a chart χ around p gives rise to a basis of $T_p M$,

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p \tag{4.1.1}$$

it will also give rise to a basis of T_p^*M, denoted

$$(d\chi_1)_p, \ldots, (d\chi_m)_p. \tag{4.1.2}$$

As the notation suggests, each member $(d\chi_i)_p \in T_p^*M$ is obtained by applying the previous example to the component $\chi_i : U \to \mathbb{R}$ of χ. Unraveling the definitions one obtains:

$$\left\langle (d\chi_i)_p, \left(\frac{\partial}{\partial \chi_j}\right)_p \right\rangle = \delta_j^i$$

for all i and j. This implies that (4.1.2) is indeed a basis of T_p^*M- actually the dual basis to (4.1.1).

4.1.2 Differential 1-forms

Similar to set-theoretical vector fields on M (subsection 3.5.1), but dually, we now look at maps

$$\omega : M \ni p \mapsto \omega_p \in T_p^*M$$

and, to make sense of their smoothness, we look in arbitrary charts $\chi : U \to \Omega \subset \mathbb{R}^m$, we decompose ω as

$$\omega_p = \sum_i \omega_\chi^i(\chi(p))(d\chi_i)_p \quad (\text{for } p \in U),$$

and we consider of the resulting **coordinate functions of** ω w.r.t. the chart χ

$$\omega_\chi^i : \Omega \to \mathbb{R}.$$

Definition 4.2. A **differential** 1**-form** on M is any map

$$\omega : M \ni p \mapsto \omega_p \in T_p^*M$$

whose coordinate functions w.r.t. any chart are smooth.

We denote by $\Omega^1(M)$ the set of such 1-forms.

As for vector fields (the discussion following Definition 3.31), $\Omega^1(M)$ carry the following algebraic structure:

- it is a vector space, with the addition $+$ and multiplication by scalars $\cdot$ defined pointwise:

$$(\omega + \omega')_p := \omega_p + \omega'_p, \quad (\lambda \cdot \omega)_p := \lambda \cdot \omega_p.$$

- it is a $\mathscr{C}^\infty(M)$-module, i.e. there is an operation of multiplication of vector fields by smooth functions,

$$\mathscr{C}^\infty(M) \times \Omega^1(M) \to \Omega^1(M), \quad (f, \omega) \mapsto f \cdot \omega.$$

  where $(f \cdot \omega)_p := f(p) \cdot \omega_p$.

Example 4.3 (differentials of functions). Example 4.1 can now be upgraded: we see that for any function $f \in \mathscr{C}^\infty(M)$ its differential now becomes a 1-form

$$df \in \Omega^1(M)$$

(why is it smooth?). In this way, the differentiation of $\mathbb{R}$-valued functions becomes a linear operator

$$d : \mathscr{C}^\infty(M) \to \Omega^1(M).$$

Exercise 4.4. Show that, for any $f, g \in C^\infty(M)$, one has:

$$d(fg) = f \cdot dg + g \cdot df.$$

Exercise 4.5. Show that, for a 1-form $\omega \in \Omega^1(M)$ to be exact, i.e. to be of type df for some smooth function f, a necessary (but not sufficient) condition is that

$$\omega([X,Y]) = L_X(\omega(Y)) - L_Y(\omega(X)) \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

Example 4.6 (expressions of type $\sum f_i dg_i$). Using the vector space and C^∞-module operations on $\Omega^1(M)$, as well as differentials of functions, we see that any expressions of type

$$\sum_{i=1}^{k} f_i \cdot dg_i \qquad \text{with } f_i, g_i \in C^\infty(M)$$

now make sense as 1-forms. One can actually show that any 1-form is representable by such an expression; however, one should be careful with the fact that representations are not unique (this is clearly shown already by the derivation property from Exercise 4.4).

Moving to $\mathbb{R}^m$ (which we do anyway when we look in a chart), one obtains the analogue of Example 3.33: any 1-form ω on $\mathbb{R}^m$ can be uniquely written as

$$\omega = \sum f_i \cdot dx_i \tag{4.1.3}$$

with $f_i \in \mathscr{C}^\infty(\mathbb{R}^m)$, and where x_i are the coordinate functions interpreted as elements of $\mathscr{C}^\infty(\mathbb{R}^m)$.

Exercise 4.7. Show that:

(a) For any function $f \in \mathscr{C}^\infty(\mathbb{R}^m)$, one has

$$df = \sum_i \frac{\partial f}{\partial x_i} \cdot dx_i.$$

(b) Show that the 1-form on $\mathbb{R}^2$ given by

$$\omega = y dx - x dy$$

is not exact, i.e. cannot be written as df for some smooth function f.

Exercise 4.8. Show that for any $f \in \mathscr{C}^\infty(M)$ and any chart (U, χ) one has, on U,

$$df = \sum_i \frac{\partial f}{\partial \chi_i} \cdot d\chi_i.$$

Deduce that for any two coordinate charts (U, χ) and (U', χ') one has, on $U \cap U'$,

$$d\chi_i' = \sum_j \frac{\partial \chi_i'}{\partial \chi_j} \cdot d\chi_j. \tag{4.1.4}$$

Then, using the duality between $\frac{\partial}{\partial \chi_i}$ and $d\chi_i$, show again that, over $U \cap U'$, one has:

$$\frac{\partial}{\partial \chi_i} = \sum \frac{\partial \chi_j'}{\partial \chi_i} \cdot \frac{\partial}{\partial \chi_j'}.$$

(here we are also using the notation introduced in subsection 3.4.3 - see the formula (*) there).

Exercise 4.9. Show that there is a unique 1-form on S^1, denoted by $d\theta$, such that $p^* d\theta = dt \in \Omega^1(\mathbb{R})$, where

$$p : \mathbb{R} \to S^1, \quad t \mapsto e^{it}.$$

Then try to prove that, despite the notation that we use, $d\theta$ is actually not exact.
(the most elegant argument for the last part is via Stokes theorem, to be discussed later).

4.1.3 Pull-backs of 1-forms

And here is one very nice feature of 1-forms: unlike vector fields, 1-forms can be pulled-back along *any* smooth map. We start here with the simplest instance of this operation: the restriction to submanifolds.

Example 4.10 (restrictions to submanifolds). For a submanifold $M \subset N$, dual to the inclusions $T_pM \subset T_pN$, there is a restriction operation of 1-forms on N to 1-forms on M,

$$\text{restr} : \Omega^1(N) \to \Omega^1(M), \quad \omega \mapsto \omega|_M,$$

where $\omega|_M$ is defined by

$$(\omega|_M)_p (X_p) := \omega_p(X_p)$$

for all $X_p \in T_pM$. Note that the right hand side makes sense precisely because $T_pM \subset T_pN$.

This operation is particularly useful when M is an embedded submanifold of a Euclidean space $\mathbb{R}^k =: N$: one can produce 1-forms on M by starting with 1-forms on $\mathbb{R}^k$ (necessarily of type (4.1.3)); and, although it is not completely obvious, one can prove that any 1-form on M arises in this way. In other words, one can "represent" 1-forms on M by 1-forms (4.1.3) on $\mathbb{R}^k$. E.g., for $M = S^2$, we can now talk about

$$(x^2 + y) \cdot dx + dy + xyz \cdot dz \quad \text{on } S^2,$$

by which we mean the restriction of $(x^2 + y) \cdot dx + dy + xyz \cdot dz$ from $\mathbb{R}^3$ to $S^2 \subset \mathbb{R}^3$.

However, one should be aware that *different forms on $\mathbb{R}^k$ may give the same restriction to M*. E.g., for $S^2 \subset \mathbb{R}^3$, there are expressions that look like being non-zero (and they are non-zero as 1-forms on $\mathbb{R}^3$) but which represent the zero form on S^2. For instance,

$$x \cdot dx + y \cdot dy + z \cdot dz \quad \text{on } S^2$$

is zero. Indeed, viewing it as a 1-form on S^2, it means we evaluate it only on vectors tangent to S^2; and, for $p = (x, y, z) \in S^2$ and an arbitrary tangent vector

$$V_p = a \left(\frac{\partial}{\partial x}\right)_p + b \left(\frac{\partial}{\partial y}\right)_p + c \left(\frac{\partial}{\partial z}\right)_p \in T_pS^2$$

one has

$$(x \cdot dx + y \cdot dy + z \cdot dz)(V_p) = ax + by + cz = 0$$

since V was tangent to S^2. In other words,

$$(x \cdot dx + y \cdot dy + z \cdot dz)|_{S^2} = 0.$$

Exercise 4.11. Prove this last formula by starting from $x^2 + y^2 + z^2 = 1$ on S^2 and using the derivation property from Exercise 4.4.

Exercise 4.12. Show that, conversely, if $M \subset \mathbb{R}^3$ is a compact and connected submanifold with the property that

$$(xdx + ydy + zdz)|_M = 0,$$

then M is one of the spheres centred at the origin.

Exercise 4.13. For the torus $T_{R,r}$ as in (2.5.6), show that the restrictions of the standard 1-forms dx, dy, dz from $\mathbb{R}^3$ to $T^2_{R,r}$ satisfy:

$$x \cdot dx + y \cdot dy + z \cdot dz = \frac{R}{\sqrt{x^2+y^2}}(x \cdot dx + y \cdot dy) \quad (\text{on } T^2_{R,r}).$$

Exercise 4.14 (part of the 2019/2020 exam). Do now the last part of the exam Exercise 2.73:

(d) give an example of a 1-form θ on S^3, different from the zero form, such that $\theta|_{M_0} = 0$.

Exercise 4.15. Show that, for the 1-form $d\theta$ on S^1 discussed in Exercise 4.9

$$d\theta = (ydx - xdy)|_{S^1}.$$

Exercise 4.16. Which one of the following 2-forms on S^2:

$$\theta_1 = xdy - ydx + zdz, \quad \theta_2 = xdy + ydx + zdz \quad (\text{restricted to } S^2)$$

is closed? But exact?

And, as we already mentioned, the restriction operation is a particular case of pull-backs, corresponding to inclusions $i : M \hookrightarrow N$. To explain this, we start with an arbitrary smooth map $F : M \to N$ and $\omega \in \Omega^1(N)$. Then at each point $p \in M$ we can apply the differential $(dF)_p : T_pM \to T_{F(p)}N$ followed by $\omega_{F(p)} : T_{F(p)}N \to \mathbb{R}$ to obtain a covector on M at p:

$$T_pM \ni X_p \mapsto \omega_{F(p)}((dF)_p(X_p)) \in \mathbb{R}$$

(compare with vector fields!). The resulting 1-form on M is denoted by

$$F^*(\omega) \in \Omega^1(M)$$

and is called **the pull-back of ω by F**. Therefore, any smooth map $F : M \to N$ gives rise to a pull-back operation

$$\boxed{F^* : \Omega^1(N) \to \Omega^1(M).}$$

Exercise 4.17. Show that:

(a) indeed, $F^*(\omega)$ is smooth.
(b) if $G : N \to P$ is another smooth map, $(G \circ F)^* = F^* \circ G^*$.
(c) when $M \subset N$ and F is the inclusion $F^*(\omega) = \omega|_M$.

Exercise 4.18. Show that for any $F : M \to N$ smooth, the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{C}^\infty(N) & \xrightarrow{d} & \Omega^1(N) \\ \downarrow F^* & & \downarrow F^* \\ \mathscr{C}^\infty(M) & \xrightarrow{d} & \Omega^1(M) \end{array}$$

Exercise 4.19. Since $\mathbb{P}^n$ does not sit canonically in any Euclidean space, the principle of representing 1-forms on $\mathbb{P}^n$ by 1-forms on Euclidean spaces (via restrictions) does not work right away. However, one can relate $\mathbb{P}^n$ to a nicer space, namely to the sphere, via the canonical map

$$H : S^n \to \mathbb{P}^n$$

which sends a point $x \in S^n$ to the line l_x through the origin and x. In this way, any 1-form $\omega \in \Omega^1(\mathbb{P}^n)$ can be "lifted" to a form on S^n:

$$\widetilde{\omega} := H^*(\omega) \in \Omega^1(S^n).$$

We will also make use of the map

$$\tau : S^n \to S^n, \quad \tau(x) = -x.$$

Show that:

(a) for such forms $\widetilde{\omega}$, one has $\tau^*(\widetilde{\omega}) = \widetilde{\omega}$. In other words, the pull-back $\widetilde{\omega}$ always belongs to the subspace of $\Omega^1(S^n)$ consisting of τ-invariant 1-forms,

$$\Omega^1(S^n)^\tau := \left\{ \eta \in \Omega^1(S^n) : \tau^*\eta = \eta \right\}.$$

(b) Conversely, any τ-invariant 1-form on the sphere, $\eta \in \Omega^1(S^n)^\tau$, is necessarily of type $\widetilde{\omega}$ for some $\omega \in \Omega^1(\mathbb{P}^n)$.
(c) Conclude that one obtains a bijection

$$H^* : \Omega^1(\mathbb{P}^n) \xrightarrow{\sim} \Omega^1(S^n)^\tau.$$

4.1.4 1-forms as $C^\infty(M)$-linear functionals eating vector fields

While covectors of a vector space V take vectors to real numbers, 1-forms take vector fields to smooth functions: given ω as above and $X \in \mathfrak{X}(M)$, evaluating ω_p on X_p for each $p \in M$ we obtain a smooth function

$$\omega(X) \in \mathscr{C}^\infty(M);$$

(why smooth?). When we vary X it is clear that the resulting map, still denoted by ω,

$$\boxed{\omega : \mathfrak{X}(M) \to \mathscr{C}^\infty(M)}$$

is $C^\infty(M)$-linear, i.e. it is linear and

$$\omega(f \cdot X) = f \cdot \omega(X) \quad \text{for all } f \in \mathscr{C}^\infty(M), X \in \mathfrak{X}(M).$$

We denote by

$$\mathrm{Hom}_{\mathscr{C}^\infty}(\mathfrak{X}(M), \mathscr{C}^\infty(M))$$

the space of all $\mathscr{C}^\infty(M)$-linear maps from $\mathfrak{X}(M)$ to $\mathscr{C}^\infty(M)$.

Theorem 4.20. *Interpreting any $\omega \in \Omega^1(M)$ as a map $\mathfrak{X}(M) \to \mathscr{C}^\infty(M)$, we obtain an isomorphism*

$$\Omega^1(M) \cong Hom_{\mathscr{C}^\infty}(\mathfrak{X}(M), \mathscr{C}^\infty(M)).$$

Proof. Note first that, by the same arguments as in Proposition 3.41 from Chapter 3, for a set-theoretical 1-form ω its smoothness is equivalent to the fact that $\omega(X)$ is a smooth function on M for all $X \in \mathfrak{X}(M)$.

Another ingredient that we need in the proof is the fact that, for each $p \in M$, the evaluation of vector fields at p,

$$\mathrm{ev}_p : \mathfrak{X}(M) \to T_pM, \quad X \mapsto X_p, \tag{4.1.5}$$

is surjective. To show this, we start with an arbitrary tangent vector $v \in M$ (p is fixed now). Using a coordinate chart around p, we find a vector field Y defined on an open neighborhood U of p such that $Y_p = v$. As in the proof of Lemma 3.20 from Chapter 3, choose a smooth function η on M, supported inside U and with $\eta(p) = 1$. Then $X := \eta \cdot Y$ is well-defined and smooth on M and $X_p = \eta(p) \cdot v = v$.

We now move to the actual proof. For the injectivity of the map we have to show that, if $\omega_p(X_p) = 0$ for all $X \in \mathfrak{X}(M)$ and all $p \in M$, then $\omega_p = 0$ for all $p \in M$. But this clearly follows from the surjectivity of ev_p.

We now prove the surjectivity of the map in the statement. Let $A : \mathfrak{X}(M) \to \mathscr{C}^\infty(M)$ be a $\mathscr{C}^\infty(M)$-linear map. We fix $p \in M$ arbitrary and we define

$$\omega_p : T_pM \to \mathbb{R}, \quad \omega_p(v) := A(Y)(p)$$

where A is any vector field with $Y_p = v$. If we check that this definition does not depend on the choice of Y then we are done: linearity of each ω_p follows right away, while the resulting set-theoretical 1-form is also smooth since, by construction, $\omega(X) = A$.

To see that $A(Y)(p)$ does not depend on the choice of Y, assume that Y' is another vector field with $Y_p^{'} = v$. Then $X := Y - Y'$ vanishes at p and we have to show that $A(X)(p) = 0$. Using the $\mathscr{C}^\infty(M)$-linearity of A, this will follow from the following:

Lemma 4.21. *If a vector field $X \in \mathfrak{X}(M)$ vanishes at a point $p \in M$, then we can write*

$$X = \sum_{j=0}^{k} f_j \cdot X^j$$

with $k \geq 1$ integer, $X_j \in \mathfrak{X}(M)$ and f_j smooth functions that vanish at p.

Proof. We first look in a coordinate chart (U, χ) around p which sends p to $\chi(p) = 0 \in \mathbb{R}^m$. Then writing

$$X|_U = \sum_{i=1}^{m} F_i \cdot \frac{\partial}{\partial \chi_i},$$

the coefficients F_i are smooth functions on U that vanish at p. By the the proof of Theorem 3.25 from Chapter 3 applied to $F_i \circ \chi^{-1}$, we see that each F_i can be written as a finite sum $\sum_j \chi_j \cdot g_i^j$ with $g_i^j : U \to \mathbb{R}$ smooth. From this we immediately deduce that we can find some smooth functions $f_1, \ldots, f_k$ on U and vector fields $Y_1, \ldots, Y_k$ (on U) such that

$$X|_U = \sum_{j=1}^{k} f_j \cdot Y^j.$$

To go from U to the entire M, the main idea is to write $X = \eta \cdot X + (1 - \eta) \cdot X$, for some carefully chosen smooth function η.

Note that, after eventually making U smaller, we may assume that $f_i = \tilde{f}_i|_U$ for some smooth function $\tilde{f}_i \in \mathscr{C}^\infty(M)$; to see this, just use the second dotted item in the proof of Lemma 3.21 from Chapter 3. Furthermore, as in that proof, we choose a smooth function $\eta \in C^\infty(M)$ which is supported inside U and is 1 in some neighborhood V of p. Since η is supported inside U, each $\eta|_U \cdot Y^j \in \mathfrak{X}(U)$ can be promoted to a vector field $X^j \in \mathfrak{X}(M)$ by declaring it to be zero outside U. With the inspiration from the "main idea" mentioned above, we now remark that

$$X = (1 - \eta) \cdot X + \sum_{j=1}^{k} \tilde{f}_j \cdot \tilde{X}^j$$

on the entire M. Indeed, at $p \in U$ the right hand side is

$$(1 - \eta(p)) \cdot X_p + \eta(p) \cdot \sum_{j=1}^{k} f_j(p) \cdot X_p^j = (1 - \eta(p)) \cdot X_p + \eta(p) \cdot X_p = X_p,$$

while at p outside U we obtain $(1 - 0) \cdot X_p + \sum_j \tilde{f}_j(p) \cdot 0 = X_p$.

Therefore the desired decomposition is achieved if we take $f_0 = 1 - \eta$ and $X^0 = X$. 😀

And, still as in the case of tangent vectors, one can form the cotangent bundle

$$T^*M := \sqcup_{p \in M} T_p^*M = \{(p, \omega_p) : p \in M, \omega_p \in T_p^*M\}$$

and put a smooth structure on T^*M so that the smoothness of a set-theoretical 1-form ω is equivalent to the smoothness of ω as a map from M to T^*M.

Exercise 4.22. Fill in the details, i.e. state and prove a version of Proposition 3.50 but for T^*M instead of TM.

4.2 Differential forms of arbitrary degree

4.2.1 The relevant linear algebra

The passage from 1-forms to arbitrary k-forms is basically a passage from linear maps $T_pM \to \mathbb{R}$ to k-multilinear skew-symmetric maps. Here we discuss the linear algebra involved.

Definition 4.23. Given a vector space V and $k \in \mathbb{N}$ a natural number, a k-covector on V is any map

$$\omega : \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}$$

that is multilinear (i.e. linear in each argument). We say that ω if a linear k-form on V if

$$\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \text{sign}(\sigma)\omega(v_1, \ldots, v_k)$$

for any $v_1, \ldots, v_k \in V$ and any permutation $\sigma \in S_k$. We denote by $\Lambda^k V^*$ the (vector) space of all such linear k-forms on V.

Exercise 4.24. Show that ω is skew-symmetric if and only if the expression $\omega(v_1, \ldots, v_k)$ changes sign whenever two entries v_i and v_j are interchanged.

Although our interest is $\Lambda^k V^*$, it will be useful to consider more general k-multilinear maps (without being skew-symmetric); we denote the space of such by $T^k V^*$. Note that, while

$$\Lambda^k V^* \subset T^k V^*,$$

there is also a skew-symmetrization map going the other way,

$$\text{Alt} : T^k V^* \to \Lambda^k V^*$$

where for $\omega \in T^k V^*$, $\text{Alt}(\omega)$ is given by

$$\text{Alt}(\omega)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Exercise 4.25. Show that, for $\omega \in T^k V^*$, one has:

$$\omega \in \Lambda^k V^* \Longleftrightarrow \text{Alt}(\omega) = \omega.$$

The main operation on forms is that of wedge-product. Before giving the precise definition/formula, note first that there a pretty obvious operation involving the spaces $T^k V^*$, namely:

$$T^k V^* \times T^l V^* \to T^{k+l} V^*, \quad (\omega, \eta) \mapsto \omega \bullet \eta,$$

where $\omega \bullet \eta$ is given by

$$(\omega \bullet \eta)(v_1, \ldots, v_{k+l}) := \omega(v_1, \ldots, v_k) \cdot \eta(v_{k+1}, \ldots, v_{k+l}).$$

Exercise 4.26. Do the following:

1. If $\omega \in T^2V^*$ is symmetric and $\eta \in T^lV^*$ is arbitrary, show that $\text{Alt}(\omega \bullet \eta) = 0$.
2. Prove the same but now assuming that $\omega \in T^kV^*$ satisfies $\text{Alt}(\omega) = 0$.
3. Deduce that for any $\omega \in T^kV^*$ and $\eta \in T^lV^*$ one has $\text{Alt}(\omega \bullet \eta) = \text{Alt}(\text{Alt}(\omega) \bullet \eta)$.

However, if ω and η are skew-symmetric, $\omega \bullet \eta$ may fail to be skew-symmetric. Therefore, to obtain a skew-symmetric element, we will apply the skew-symmetrization map Alt. To avoid un-necessary coefficients in the final formula we will do a small rescaling (see also below):

Definition 4.27. Given $\omega \in \Lambda^kV^*$ and $\eta \in \Lambda^lV^*$, define the element

$$\omega \wedge \eta \in \Lambda^{k+l}V^*,$$

called the **wedge product of ω and η**, by

$$\omega \wedge \eta = \frac{(k+l)!}{k! \cdot l!} \text{Alt}(\omega \bullet \eta).$$

Remark 4.28. When computing $\text{Alt}(\omega \cdot \eta)$, the fact that ω and η are skew-symmetric will be reflected in the fact that many of the terms will repeat. When we avoid repetitions, we get sums not over all permutations $\sigma \in S_{k+l}$ but only over those with the property that

$$\sigma(1) < \ldots < \sigma(k), \quad \sigma(k+1) < \ldots < \sigma(k+l).$$

Such permutations will be called (k,l)**-shuffles** and we denote by $S_{k,l} \subset S_{k+l}$ the resulting set. All together, we find the following explicit formula for the wedge product:

$$(\omega \wedge \eta)(v_1, \ldots, v_{k+l}) := \sum_{\sigma \in S_{k,l}} \text{sign}(\sigma) \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}).$$

For instance, for $\omega, \eta \in V^*$ one obtains $\omega \wedge \eta \in \Lambda^2V^*$ given by

$$(\omega \wedge \eta)(v,w) = \omega(v)\eta(w) - \omega(w)\eta(v).$$

Or, for $\omega, \in \Lambda^2V^*, \eta \in V^*$,

$$(\omega \wedge \eta)(u,v,w) = \omega(u,v)\eta(w) + \omega(v,w)\eta(u) - \omega(u,w)\eta(v).$$

The reason for the coefficients $k!$ used in the definition of the wedge was precisely obtain such formulas without coefficients (incidentally, let us point out that the resulting formula now makes send over any field, not necessarily of characteristic zero). Actually, replacing $k!$ by any coefficients c_k would still produce (modified) wedge-products with similar properties (see the next proposition) but, in some (precise) sense, they are all ”isomorphic”. However, simple choices of c_ks, such as $c_k = 1$, would produce uglier final formulas.

In this way we obtain operations

$$\cdot \wedge \cdot : \Lambda^kV^* \times \Lambda^lV^* \to \Lambda^{k+l}V^*.$$

Here are the main properties of these operations:

Proposition 4.29. *The wedge operation is bilinear, and:*

1. *graded-commutativity: for any* $\omega \in \Lambda^k V^*$ *and* $\eta \in \Lambda^l V^*$,

$$\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta .$$

2. *associativity: for any linear forms* ω, η *and* ζ *one has*

$$(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta).$$

Proof. For graded-commutativity, we look at $\eta \wedge \omega(v_1, \ldots, v_{k+l})$. The sum defining it has expressions of type

$$\mathrm{sign}(\sigma)\eta(v_{\sigma(1)}, \ldots, v_{\sigma(l)}) \cdot \omega(v_{\sigma(l+1)}, \ldots, v_{\sigma(l+k)})$$

where σ is an (l,k)-shuffle. Now, any such σ defines another permutation $\hat{\sigma}$ by

$$\hat{\sigma}(1) = \sigma(l+1), \ldots, \hat{\sigma}(k) = \sigma(l+k), \quad \hat{\sigma}(k+1) = \sigma(1), \ldots, \hat{\sigma}(k+l) = \sigma(l)$$

which is a (k,l)-shuffle. A moment of reflection shows that

$$\mathrm{sign}(\hat{\sigma}) = (-1)^{kl} \mathrm{sign}(\sigma).$$

All these clearly imply the graded-commutativity.

For associativity, up to a constant that is the same on both sides, we have to show that

$$\mathrm{Alt}(\mathrm{Alt}(\omega \bullet \theta) \bullet \zeta) = \mathrm{Alt}(\omega \bullet \mathrm{Alt}(\theta \bullet \zeta)) .$$

Using the last part of Exercise 4.26 to rewrite the left hand side, and the similar argument for the right hand side, what we have to prove becomes:

$$\mathrm{Alt}((\omega \bullet \theta) \bullet \zeta) = \mathrm{Alt}(\omega \bullet (\theta \bullet \zeta)) .$$

This follows right away since the operation $\bullet$ is clearly associative. 😄

Example 4.30. For any linear 1-forms $\xi_1, \ldots, \xi_k \in V^*$, one obtains a k-form

$$\xi_1 \wedge \ldots \wedge \xi_k \in \Lambda^k V^*.$$

As we shall soon see, any linear k-form can be written as a sum of expressions of this type. Even more, this operation allows us to produce an explicit basis for the vector space $\Lambda^k V^*$ by starting with a basis

$$e_1, \ldots, e_m \in V$$

of V. First of all, we consider the dual basis

$$e^1, \ldots, e^m \in V^*.$$

Then we start taking wedges of such elements; to obtain k-forms, we need to use k such (co)vectors. Given the graded commutativity of the wedge-operations, we can always re-order the indices in an increasing fashion. Therefore, we will consider sets of increasing indices

$$I = (i_1, \ldots, i_k), \quad \text{with} 1 \leq i_1 < \ldots < i_k \leq m;$$

the set of such Is will be denoted by $\mathrm{Ord}_k(m)$. And, for $I \in \mathrm{Ord}_k(m)$, consider

$$e^I := e^{i_1} \wedge \ldots \wedge e^{i_k} \in \Lambda^k V^*.$$

Lemma 4.31. *The vectors e^I, with $I \in Ord_k(m)$, form a basis of $\Lambda^k V^*$. In particular, for $k \leq m$ (where m is the dimension of V) one has*

$$dim(\Lambda^k V^*) = Card(Ord_k(m)) = \frac{m!}{k!(m-k)!},$$

while for $k > m$ one has $\Lambda^k V^* = 0$.

Proof. Note that, for any I as above and any other $J = (j_1, \ldots, j_k) \in \mathrm{ord}_k(M)$ one has

$$e^I(e_{j_1}, \ldots, e_{j_k}) = \delta_J^I$$

(i.e. 1 when $I = J$ and zero otherwise). From this we deduce right away that the e^Is are linearly independent: if $\sum_I \lambda_I e^I = 0$ with $\lambda_I \in \mathbb{R}$, evaluating on $(e_{j_1}, \ldots, e_{j_k})$ we find that $\lambda_J = 0$ for all J.

To see that our family spans the entire $\Lambda^k V^*$, it suffices to show that, for any $\omega \in \Lambda^k V^*$),

$$\omega = \sum_{I \in \mathrm{Ord}_k(m)} \omega(e_{i_1}, \ldots, e_{i_k}) \cdot e^I.$$

To check this equality, we first note that the difference of the two sides, call it $\eta \in \Lambda^k V^*$, is zero when evaluated on all elements of type $(e_{i_1}, \ldots, e_{i_k})$. But this is easily seen to imply that $\eta = 0$: for an arbitrary expression $\eta(v_1, \ldots, v_k)$, decomposing each v_i with respect to the basis $\{e_1, \ldots, e_m\}$ and using the multilinearity of η and then the antisymmetry, we obtain a sum of expressions of type $\eta(e_{i_1}, \ldots, e_{i_k})$. 😀

4.2.2 Differential forms

We now pass to differential forms on an arbitrary m-dimensional manifold M. As for 1-forms, one first talks about **set-theoretical *k*-forms**, by which we mean maps

$$\omega : M \ni p \mapsto \omega_p \in \Lambda^k T_p^* M. \tag{4.2.1}$$

And, still as for 1-forms, one makes sense of their smoothness using charts (U, χ): any such chart gives rise to the basis

$$\left(\frac{\partial}{\partial \chi_1}\right)_p, \ldots, \left(\frac{\partial}{\partial \chi_m}\right)_p$$

of $T_p M$, the dual basis

$$(d\chi_1)_p, \ldots, (d\chi_m)_p$$

of $T_p^* M$ and then, using Lemma 4.31, to the basis of $\Lambda^k T_p^* M$ given by

$$(d\chi)_p^I = (d\chi_{i_1})_p \wedge \ldots \wedge (d\chi_{i_k})_p$$

with $I \in \mathrm{Ord}_k(1, \ldots, m)$. Therefore, we can write our ω as

$$\omega_p = \sum_{I \in \mathrm{Ord}_k(m)} \omega_\chi^I(\chi(p))(d\chi)_p^I,$$

where each coefficient ω_χ^I is a function on the codomain of the chart.

Definition 4.32. A **differential *k*-form** on M, or a **differential form of degree** k on M, is any set-theoretical k-form (4.2.1) that is smooth in the sense that its coordinate functions ω_χ^I w.r.t. any chart χ are smooth.

We denote by $\Omega^k(M)$ the resulting space of all such k-forms.

It should be clear now that $\Omega^k(M)$ is a vector space, and it comes with a structure of $\mathscr{C}^\infty(M)$-module,

$$\mathscr{C}^\infty(M) \times \Omega^k(M) \to \Omega^k(M), \quad (f, \omega) \mapsto f \cdot \omega.$$

While, point-wise (at some $p \in M$), ω takes k tangent vectors (at p) to produce numbers, globally (when varying p), it takes k vector fields and produces a function. More explicitly, for $X^1, \ldots, X^k \in \mathfrak{X}(M)$, one obtains the function

$$\omega(X^1, \ldots, X^k) : M \to \mathbb{R}, \quad p \mapsto \omega_p(X^1_p, \ldots, X^k_p).$$

In other words, ω can be re-interpreted as a map

$$\omega : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k-\text{times}} \to \mathscr{C}^\infty(M).$$

Exercise 4.33. Check that the smoothness of ω is equivalent to the fact that $\omega(X^1, \ldots, X^k)$ is smooth for any $X^1, \ldots, X^k \in \mathfrak{X}(M)$.

Of course, with the case $k = 1$ already discussed in Theorem 4.20, the following does not come as a surprise.

Theorem 4.34. *The previous interpretation of k-forms gives a 1-1 correspondence between k-forms on M and multilinear maps*

$$\omega : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k-times} \to \mathscr{C}^\infty(M)$$

with the following two properties:

- *ω is skew-symmetric.*
- *ω is $\mathscr{C}^\infty(M)$-linear in each argument X_i:*

$$\begin{aligned} \omega(X_1, \ldots, X_{i_1}, X_i + X_i', X_{i+1}, \ldots, X_k) = \omega(X_1, \ldots, X_{i_1}, X_i, X_{i+1}, \ldots, X_k) +& \\ + \omega(X_1, \ldots, X_{i_1}, X_i', X_{i+1}, \ldots, X_k),& \end{aligned}$$

$$\omega(X_1, \ldots, X_{i_1}, f \cdot X_i, X_{i+1}, \ldots, X_k) = f \cdot \omega(X_1, \ldots, X_k).$$

Proof. For the moment, let denoted by $\hat{\omega}$ the version of ω that acts on vector fields. It is clear that it is skew-symmetric and $\mathscr{C}^\infty(M)$-multilinear. For the converse, let

$$A : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k-\text{times}} \to \mathscr{C}^\infty(M)$$

a map with the same properties and we want to show that $A = \hat{\omega}$ for some differential k-form ω. As in the previous sections, for $p \in M$ we define ω_p on tangent vectors $v^1_p, \ldots, v^k_p \in T_pM$ by:

$$\omega_p(v^1_p, \ldots, v^k_p) := A(X^1, \ldots, X^k)(p),$$

where $X^i \in \mathfrak{X}(M)$ are chosen so that, at p, they give back the vectors v^i_p. The main issue is, as for the similar discussion for 1-forms, to see that this definition does not depend on the choice of the X^is. But that follows exactly as for 1-forms (or one could even use that result inductively). The rest should be clear. 😀

Example 4.35. In Exercise 4.5 we have seen that, for a 1-form $\omega \in \Omega^1(M)$ to be exact one would need in particular that the expression

$$-\omega([X,Y]) + L_X(\omega(Y)) - L_Y(\omega(X))$$

to be zero for all $X, Y \in \mathfrak{X}(M)$. Looking at the expression above one can check right away that it is skew-symmetric and $\mathscr{C}^\infty(M)$-bilinear in X and Y. Therefore it defines a 2-form on M- that is usually denoted by

$$d\omega \in \Omega^2(M);$$

as we have seen, it arises as the obstruction to ω being exact.

Note that, in this way, we extended the differential of functions to 1-forms:

$$\mathscr{C}^\infty(M) = \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M).$$

This, and its further continuation will be discussed in more detail in the next section ("De Rham differential").

Next, the wedge-product discussed in the previous section, applied at each point $p \in M$, gives rise to similar operations

$$\cdot \wedge \cdot : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M), \quad (\omega, \eta) \mapsto \omega \wedge \eta,$$

i.e. defined by

$$(\omega \wedge \eta)_p := \omega_p \wedge \eta_p.$$

Or, using the interpretation of k-forms as in the previous proposition,

$$(\omega \wedge \eta)(X_1, \ldots, X_{k+l}) = \sum_{\sigma \in S_{k,l}} \text{sign}(\sigma)\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\eta(V_{\sigma(k+1)}, \ldots, V_{\sigma(k+l)}). \tag{4.2.2}$$

Of course, one still has:

Proposition 4.36. *The wedge operation is bilinear, and:*

1. *graded-commutativity: for any $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,*

$$\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta.$$

2. *associativity: for any differential forms ω, η and ζ on M one has*

$$(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta).$$

Proof. Just apply point-wise the similar linear algebra result.

And, still as for 1-forms, forms of arbitrary degree can be pulled-back via any smooth map $F : M \to N$, giving rise to operations:

$$F^* : \Omega^k(N) \to \Omega^k(M).$$

Explicitly, for $\omega \in \Omega^k(N)$, **the pull-back of ω by F**, $F^*\omega \in \Omega^k(M)$, is given by

$$F^*(\omega)_p(X_p^1, \ldots, X_p^k) := \omega_{F(p)}\left((dF)_p(X_p^1), \ldots, (dF)_p(X_p^k)\right).$$

Exercise 4.37. Show that the pull-back operation preserves the wedge-products: if $F : M \to N$ is smooth then, for any ω and η differential forms on N, one has

$$F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta).$$

Again, when M is a submanifold of N and F is the inclusion map, the pull-back of ω via the inclusion will be denoted by

$$\omega|_M \in \Omega^k(M)$$

and will be called **the restriction of ω to M**. And of particular interest is the case when M is a submanifold of some Euclidean space $\mathbb{R}^d$; then:

- k-forms on $\mathbb{R}^d$ are always of type

$$\sum_{1\leq i_1<\ldots<i_k\leq d} f_{i_1,\ldots,i_k}\cdot dx_{i_1}\wedge\ldots\wedge dx_{i_k}$$

  with $f_{i_1,\ldots,i_k}$ smooth functions on $\mathbb{R}^d$.
- one obtains k-forms on M by restricting such k-forms on $\mathbb{R}^d$ to M.
- with the same warning as for 1-forms: *different forms on $\mathbb{R}^k$ may give the same restriction to M.*

Exercise 4.38. On the following subspace of the 2-sphere

$$N := \{(x,y,z)\in S^2 : z\neq 0\}$$

consider the restriction of $\frac{1}{z}\cdot dx\wedge dy$ to N:

$$\sigma_0 := \frac{1}{z}\cdot dx\wedge dy\Big|_N .$$

Show that σ_0 can be extended to the entire S^2, i.e. there exists a (smooth) 2-form $\sigma\in\Omega^2(S^2)$ such that $\sigma|_N = \sigma_0$.

Exercise 4.39. Consider the so-called volume form of S^2,

$$\sigma := (xdy\wedge dz + ydz\wedge dx + zdx\wedge dy)\,|_{S^2} \in \Omega^2(S^2).$$

Since $xdx+ydy+zdz=0$ on S^2, on the part $z\neq 0$ of S^2 the 1-forms dz can be expressed in terms of dx and dy and, therefore, σ can be written as $f\cdot dx\wedge dy$ for some function f. Compute f.

Exercise 4.40. Show that Exercise 4.19 works exactly the same for forms of arbitrary degree, so that k-forms on $\mathbb{P}^n$ are, via the pull-back to S^n, in 1-1 correspondence to τ-invariant k-forms on S^n:

$$\Omega^k(S^n)^\tau := \left\{\eta\in\Omega^k(S^n) : \tau^*\eta = \eta\right\}.$$

Remark 4.41 (The graded world). It is customary to formally put together all the spaces $\Omega^k(M)$ and consider

$$\Omega^\bullet(M) := \bigoplus_k \Omega^k(M).$$

The notation "$\bullet$" is to indicate that we deal with a graded object; its elements are (finite) sums of homogeneous ones (the ones that live in one single degree). With this, the wedge product becomes now (after extending bi-linearly) a product

$$\cdot\wedge\cdot : \Omega^\bullet(M)\times\Omega^\bullet(M)\to\Omega^\bullet(M)$$

making $\Omega(M)$ into a (graded) algebra. Note the similarity with the algebra of polynomials in several variables, where each polynomial is a sum of homogeneous ones. And, as there, on sums of homogeneous elements, the wedge product is:

$$(\omega_0+\omega_1+\ldots)\wedge(\eta_0+\eta_1+\ldots) = (\omega_0\wedge\eta_0) + (\omega_0\wedge\eta_1+\omega_1\wedge\eta_0) + (\omega_0\wedge\eta_2+\omega_1\wedge\eta_1+\omega_2\wedge\eta_0)+\ldots$$

And here is one very useful slogan/principle when working with graded objects:

The sign rule: *looking at an expression involving elements that have a degree, when interchanging two such elements, say of degrees k and l, the expression changes sign by $(-1)^{kl}$.*

E.g., graded commutativity becomes precisely the one from Proposition 4.36.

4.2.3 The exterior (DeRham) differential

In Example 4.3 we mentioned the differential of functions (0-forms) as 1-forms, while in Example 4.35 we indicated a similar operation on from 1-forms to 2-forms. We now extend those to arbitrary degrees, i.e. a sequence of operators d:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \dots$$

or, equivalently, an operator on the full algebra of differential forms (cf. Remark 4.41),

$$\boxed{d : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)}$$

which rises the degrees by 1 (as indicated by the notation $\bullet + 1$"). From the graded point of view (Remark 4.41) one should think of the symbol d as having degree 1 (and this will dictate some signs later on).

There are quite a few different ways to proceed to introduce d. But, before anything, let us point out the properties that we may want to be satisfied by d; we will see that some of them already force what d should be, or at least its uniqueness. Already at the start, we have to decide whether we want to discuss d (properties, definitions, existence, etc) on a fixed manifold M, or for all manifolds at once.

Of course, first of all we want:

DeRham-0: *d is $\mathbb{R}$-linear and, on 0-forms (functions), it is the usual differential of functions.*

Next, by thinking about the behaviour of d on the product of functions, we would like a similar Leibniz identity w.r.t. the wedge product $\omega \wedge \eta$:

DeRham-1: *d satisfies the (graded) Leibniz identity:*

$$\boxed{d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^k \omega \wedge d(\eta).}$$

for all $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$.

For an explanation of the sign, please see the sign rule from Remark 4.41 and remember that the symbol d is assigned degree 1.

One way to come across d is by applying inductively the reasoning described in Example 4.35: if d was defined on forms up to degree $k-1$, and we wonder when a k-form $\omega \in \Omega^k(M)$ is exact (i.e. of the type $d\eta$ for some $\eta \in \Omega^{k-1}(M)$), looking for "the natural conditions" that ω must satisfy, one discovers an expression $d\omega$ whose non-vanishing is an obstruction to ω being exact; this will be d on the k-forms. Of course, underlying the entire idea is one very simple property that d will have:

DeRham-2: $\boxed{d \circ d = 0}$.

Example 4.42. On $M = \mathbb{R}^m$, if we assume these properties for d, we already know how to compute d on arbitrary forms. E.g., for

$$\omega = (x^2 + yz)dx \wedge dy,$$

applying the Leibniz rule we find

$$d\omega = d(x^2 + yz) \wedge dx \wedge dy + (x^2 + yz)d(dx \wedge dz).$$

For the term $d(x^2 + yz)$ we get, using "DeRham-0", $2xdx + ydz + zdy$. The term $d(dx \wedge dz)$ is, by Leibniz and $d^2 = 0$, zero. We obtain

$$d\omega = 2xdx\wedge dx\wedge dy + ydz\wedge dx\wedge dy + zdy\wedge dx\wedge dy.$$

Remembering that $dx\wedge dx = 0$ (by graded commutativity of the wedge), and similarly $zdy\wedge dx\wedge dy = 0$, we are left with

$$d\omega = ydz\wedge dx\wedge dy,$$

or, to keep the natural order of the variables,

$$d\omega = ydx\wedge dy\wedge dz.$$

Actually, a similar reasoning forces the definition of d on all k-forms on $\mathbb{R}^m$: any k-form is a sum of forms of type

$$fdx_{i_1}\wedge\ldots\wedge dx_{i_k}$$

and, by the same arguments as above,

$$d(fdx_{i_1}\wedge\ldots\wedge dx_{i_k}) = \sum_i \frac{\partial f}{\partial x_i} dx_i\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_k}.$$

The fact that the d defined in this way does indeed satisfy (DeRham-1) and (DeRham-2) is now a simple check forms of this type.

Of course, exactly the same applies to any open subset $U\subset\mathbb{R}^m$ or to any domain U of a coordinate chart (U,χ) in any manifold M (we would use then the 1-forms $d\chi_i$ instead of dx_i); given the uniqueness of d satisfying the axioms, it follows that d doesn't even depend on the actual χ that we use to write down the formulas.

Corollary 4.43. *If M is a manifold, then for any $U\subset M$ that is the domain of a coordinate chart, there is and is unique an operator*

$$d_U : \Omega^\bullet(U)\to\Omega^{\bullet+1}(U)$$

satisfying (DeRham-0), (DeRham-1) and (DeRham-2). It is given by

$$d_U(fd\chi_{i_1}\wedge\ldots\wedge d\chi_{i_k}) = \sum_i \frac{\partial f}{\partial \chi_i} d\chi_i\wedge d\chi_{i_1}\wedge\ldots\wedge d\chi_{i_k}$$

where $\chi : U\to\Omega\subset\mathbb{R}^m$ is any coordinate chart defined on U (but independent of the actual χ).

Therefore, an obvious way to proceed to define d on a general M is to "proceed locally". That would mean that we would be looking for operators d that are local in the following sense:

DeRham-3: *Locality: if $U\subset M$ is open and $\omega\in\Omega^\bullet(M)$ is supported inside U, then so is $d\omega$.*

As for functions, the support of ω is the closure of the set of points $p\in M$ at which $\omega_p\neq 0$.

Exercise 4.44. Show that a linear operator $d:\Omega^\bullet(M)\to\Omega^{\bullet+1}(M)$ is local if and only if for any open $U\subset M$ there is a linear operator $d|_U : \Omega^\bullet(U)\to\Omega^{\bullet+1}(U)$ with the property that

$$(d\,\omega)|_U = d|_U(\omega|_U) \quad \text{for all } \omega\in\Omega^\bullet(U).$$

$$\begin{array}{ccc} \Omega^\bullet(M) & \xrightarrow{d} & \Omega^{\bullet+1}(M) \\ \big\downarrow{\scriptstyle \text{restr}} & & \big\downarrow{\scriptstyle \text{restr}} \\ \Omega^\bullet(U) & \xrightarrow{d|_U} & \Omega^{\bullet+1}(U) \end{array}.$$

Then show that, given two local operators d and d', $d = d'$ if and only if M can be covered by opens U on which $d|_U = d'|_U$.

Corollary 4.45. *On any manifold M there exists and is unique an operator $d : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)$ which is local and so that for any domain U of a coordinate chart, $d|_U$ coincides with d_U from the previous corollary. Moreover, d will automatically satisfy (DeRham-0), (DeRham-1) and (DeRham-2).*

Proof. The definition is forced: for $p \in M$, choosing a coordinate chart (U, χ) around p we must have

$$(d\omega)_p = (d_U \omega|_U)_p.$$

Given the uniqueness property of the d_Us it follows that the right hand side does not depend on the coordinate chart that we use, hence it can be used to define $d\omega$ unambiguously. Since the conditions (DeRham-1) and (DeRham-2) are local and they hold for the d_Us, it follows that they hold for d as well.

It is interesting to note that locality can be obtained from the rest of the axioms:

Lemma 4.46. *On a manifold M, if a linear map $D : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)$ satisfies the condition that*

$$D(f \cdot \omega) = fD(\omega) + df \wedge \omega \quad \text{for all } f \in \mathscr{C}^{\infty}(M), \omega \in \Omega^{\bullet}(M)$$

(in particular if D satisfies (DeRham-0) and (DeRham-1)) then D is local.

Proof. We fix ω, we denote by A its support and we claim that the support of $d\omega$ is inside A. Note that A is the largest closed subset of M outside which ω vanishes. Therefore suffices to show that $d(\omega)$ vanishes outside A. Let $p \in M \setminus A$; since A is closed, $M \setminus A$ is open, hence we find a smooth function $\eta \in \mathscr{C}^{\infty}(M)$ that is zero on A and is 1 in a neighborhood of p. But then $\omega = (1 - \eta) \cdot \omega$ and, using the condition from the statement we obtain

$$d\omega = (1 - \eta) d\omega - d\eta \wedge d\omega.$$

Evaluating at the point p we obtain that $d\omega$ vanishes at p.

In particular,

Corollary 4.47. *On any manifold M there exists and is unique an operator $d : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)$ which satisfies (DeRham-0) , (DeRham-1) and (DeRham-2).*

Proof. One just needs to remark that, if d satisfies (DeRham-1) and (DeRham-2) then so do all its restrictions $d|_U$; for domains U of coordinate charts, due to the uniqueness of d_U, we obtain that $d|_U = d_U$.

Definition 4.48. The resulting operator $d : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)$ is called the **DeRham operator on** M (or the exterior differential).

While so far we concentrated on building $d = d_M$ on a given manifold M, one can proceed slightly differently and handle all the manifolds at once; then the natural axiom to impose is the behaviour of d_M when we change the manifold, i.e. its compatibility with pull-backs. In low degree this was seen in Exercise 4.18. It means that, for any smooth map $F : M \to N$ one has a commutative diagram

$$\begin{array}{ccc} \Omega^{\bullet}(N) & \xrightarrow{d_N} & \Omega^{\bullet+1}(N) \\ \downarrow F^* & & \downarrow F^* \\ \Omega^{\bullet}(M) & \xrightarrow{d_M} & \Omega^{\bullet+1}(M) \end{array},$$

i.e. DeRham differentials commute with pull-backs:

$$F^* \circ d_N = d_M \circ F^*.$$

Note that this condition forces automatically that each d is local. Therefore:

Corollary 4.49. *If we are looking for operators $d_M : \Omega^{\bullet}(M) \to \Omega^{\bullet+1}(M)$, one for each manifold M, such that they are compatible with pull-backs, we have uniqueness and existence under any of the following requirements:*

1. *on $\mathbb{R}^m$, $d_{\mathbb{R}^m}$ is the operator previously discussed.*
2. *all d_M satisfy (DeRham-0) and (DeRham-1).*

Of course, for any manifold M, d_M will be precisely the DeRham operator.

And here is another way to introduce the DeRham operator: just right away, by explicit formulas. One way to find those formulas would be by returning to the previous idea of proceeding like in Example 4.35 inductively, by looking for the natural conditions for a form to be exact.

Proposition 4.50. *One has the following explicit formula for the DeRham operator on any manifold M: for $\omega \in \Omega^k(M)$, $d\omega \in \Omega^{k+1}(M)$ is given by*

$$\begin{aligned} d\omega(X_0,\ldots,X_k) = & \sum_{i=0}^{k}(-1)^i L_{X_i}(\omega(X_0,\ldots,\hat{X}_i,\ldots,X_k)) \\ & + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_k)) \end{aligned} \tag{4.2.3}$$

where the elements under the hat are deleted.

Proof. Note that the previous formula is skew-symmetric and $\mathscr{C}^{\infty}(M)$-multilinear (check that!). Hence it defines an operator d_M for each M. Looking back at the previous corollary, we see that all that is used is the compatibility with pull-backs by diffeomorphisms and by inclusions of opens inside manifolds. That is clearly true for the d_Ms. Hence we are left with checking that, for $\mathbb{R}^m$, this formula gives back the standard DeRham operator. Moreover, a simple check shows that $d(f\cdot\omega) = f\cdot d\omega + df\wedge\omega$ for any function f, hence it suffices to restrict our attention to forms on $\mathbb{R}^m$ of type $\omega = dx_{i_1}\wedge\ldots\wedge dx_{i_k}$ and we are left with showing that

$$d_{\mathbb{R}^m}(dx_{i_1}\wedge\ldots\wedge dx_{i_k}) = 0$$

(where recall that $d_{\mathbb{R}^m}$ now denotes the operator from the statement). This can be done in several ways:

- direct computation (straightforward but not so nice).
- check it first for $k=1$ (obvious) and then check the Leibniz identity on expressions of form $\omega\wedge\eta$ where ω is a 1-form (and this computation is not that ugly and can easily be carried out on a general manifold M).

😄

Exercise 4.51. Check by direct computation that, defining d by the formula (4.2.3), then it satisfies the Leibniz identity on expressions of form $\omega\wedge\eta$ where ω is a 1-form.

Exercise 4.52 (from the 2019/2020 exam). Let $\theta \in \Omega^1(M)$ be a closed 1-form and define

$$d_\theta : \Omega^k(M) \to \Omega^{k+1}(M), \quad d_\theta(\omega) := d\omega + \theta\wedge\omega.$$

Show that $d_\theta \circ d_\theta = 0$ if and only if θ is closed.

Exercise 4.53 (from the 2018/2019 exam). Let $\theta_1,\ldots,\theta_n \in \Omega^1(M)$ be $n = \dim M$ one-forms on M which are dual to a set $\{X^1,\ldots,X^n\}$ of vector fields on M, in the sense that

$$\theta^i(X_j) = \delta_i^j \quad (1 \text{ if } i=j \text{ and } 0 \text{ otherwise}).$$

(a) show that all the 1-forms $\theta_1,\ldots,\theta_n$ are closed if and only if the vector fields $X^1,\ldots,X^n$ are pairwise commuting, i.e. $[X_i,X_j]=0$ for all i and j.

(b*) show that for any family of vector fields $X^1, \ldots, X^n$ that are pairwise commuting and which are linearly independent at some point $x_0 \in M$, there exists a chart (U, χ) near x_0 such that

$$X^1_x = \left(\frac{\partial}{\partial \chi_1}\right)_x, \quad \ldots \quad, X^n_x = \left(\frac{\partial}{\partial \chi_n}\right)_x \quad \text{for all } x \in U.$$

(c) Deduce that if $\{\theta_1, \ldots, \theta_n\}$ is a set of closed 1-forms on the n-dimensional manifold M, that are linearly independent at some point $x_0 \in M$, then there exists a chart (U, χ) of M around x_0 such that $\theta_i|_U = d\chi_i$ on U, for all $i \in \{1, \ldots, n\}$.

4.2.4 Cartan's magic formula $di_X + i_X d = \mathscr{L}_X$

We first discuss the Lie derivative of forms $\omega \in \Omega^\bullet(M)$ along vector fields $V \in \mathfrak{X}(M)$, giving rise to operations

$$\mathscr{L}_V : \Omega^\bullet(M) \to \Omega^\bullet(M).$$

These Lie derivatives fit in the general philosophy explained and illustrated in Section 3.6.4 of the previous chapter:

Definition 4.54. For $\omega \in \Omega^k(M)$ and $V \in \mathfrak{X}(M)$ define **the Lie derivative of ω along** V as

$$\mathscr{L}_V(\omega) := \left.\frac{d}{dt}\right|_{t=0} (\phi^t_V)^* \omega \in \Omega^k(M).$$

As before, when V is not complete, one has to look at each $p \in M$ and note that the resulting formula

$$\mathscr{L}_V(\omega)(X^1_p, \ldots, X^k_p) = \left.\frac{d}{dt}\right|_{t=0} \omega_{\phi^t_V(p)} \left((d\phi^t_V)_p(X^1_p), \ldots, (d\phi^t_V)_p(X^k_p)\right) \tag{4.2.4}$$

makes sense independent of whether V is complete or not.

Not also that, from the graded point of view, the operation $\mathscr{L}_V$ has degree 0 (because it does not change the degree of the forms).

Proposition 4.55. *For any vector field $V \in \mathfrak{X}(M)$:*

1. *$\mathscr{L}_V : \Omega^\bullet \to \Omega^\bullet(M)$ is a degree zero derivation, i.e. it is linear and satisfies*

$$\mathscr{L}_V(\omega \wedge \eta) = \mathscr{L}_V(\omega) \wedge \eta + \omega \wedge \mathscr{L}_V(\eta)$$

   for any two differential forms ω and η.
2. *$\mathscr{L}_V$ commutes with d: $\mathscr{L}_V \circ d = d \circ \mathscr{L}_V$.*
3. *$\mathscr{L}_V$ is the only degree zero derivation on $\Omega^\bullet(M)$ which commutes with d and which, on functions, it is the usual derivative L_V.*

Moreover, it can be described by the explicit formula:

$$\mathscr{L}_V(\omega)(X_1, \ldots, X_k) = L_V(\omega(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \ldots, X_{i-1}, [V, X_i], X_{i+1}, \ldots, X_k).$$

Proof. The derivation identity follows from the standard one and the fact that $(\phi_V^t)^*$ is compatible with the wedge operation:

$$\frac{d}{dt}\bigg|_{t=0}(\phi_V^t)^*(\omega\wedge\eta) = \frac{d}{dt}\bigg|_{t=0}\left((\phi_V^t)^*\omega\wedge(\phi_V^t)^*\eta\right) = \left(\frac{d}{dt}\bigg|_{t=0}(\phi_V^t)^*\omega\right)\wedge\eta + \omega\wedge\left(\frac{d}{dt}\bigg|_{t=0}(\phi_V^t)^*\eta\right).$$

That $\mathscr{L}_V$ commutes with d follows from the similar property of the pull-backs $(\phi_V^t)^*$.

For the uniqueness in (3), assume that D is another operators with the same properties. It follows that $\mathscr{L}_V(df) = dL_V(f)$, and we obtain that $D(\omega) = \mathscr{L}_V(\omega)$ on differential forms of type

$$\omega = \sum_i f_i \wedge dg_i^1 \wedge \ldots \wedge dg_i^k.$$

It would suffices to show that any k-form can be written in this way. This is true and it would follow for instance if we proved that M can be embedded in some Euclidean space (why would that be enough?). To avoid using such a result, one can first restrict to manifolds on which it is certainly true: domain of coordinate charts. In more detail: due to the Leibniz identity it follows (as in the discussions about d) that D is local, we can talk about $D|_U$ and, on domains of coordinate charts, $D|_U$ still has the same properties as D. Hence $D|_U = \mathscr{L}_V|_U$, and then $D = \mathscr{L}_V$.

For the final formula, we start from the explicit formula (4.2.4) and we use that $\frac{d}{dt}|_{t=0}(\phi_V^t)^*(X) = [V,X]$ (Proposition 3.95 in the previous chapter). Or, equivalently,

$$\frac{d}{dt}\bigg|_{t=0}(d\phi_V^{-t})_{\phi_V^t(p)}(X_{\phi_V^t(p)}) = [V,X]_p$$

for all X. Similar to (3.6.6) (and for the same reasons)

$$(d\phi_V^{-t})_{\phi_V^t(p)}(X_{\phi_V^t(p)}) = X_p + t[V,X]_p + o(t^2)$$

for t near 0, and then

$$(d\phi_V^t)_p(X_p) = X_{\phi_V^t(p)} - t[V,X]_{\phi_V^t(p)} + o(t^2).$$

Inserting this in (4.2.4) and using also Proposition 3.94 from the previous chapter (applied to V and $f = \omega(X_1,\ldots,X_k)$ we find the formula from the statement. 😀

Another interesting operators induced by a vector field $V \in \mathfrak{X}(M)$, but simpler this time, is the interior product:

Definition 4.56. For $\omega \in \Omega^k(M)$ and $V \in \mathfrak{X}(M)$ define **the interior product ω by V**, denoted $i_V(\omega)$ as the differential form of degree $k-1$ given by

$$i_V(\omega)(X_1,\ldots,X_{k-1}) = \omega(V,X_1,\ldots,X_{k_1}).$$

This defines an operator

$$i_V : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M);$$

since it lowers the degree by 1, we say it is of degree -1.

Proposition 4.57. *For any vector field $V \in \mathfrak{X}(M)$,i_V is a degree -1 derivation, i.e. it is linear and satisfies*

$$i_V(\omega\wedge\eta) = i_V(\omega)\wedge\eta + (-1)^k\omega\wedge i_V(\eta)$$

for any two differential forms $\omega \in \Omega^k(M)$ and η. Actually, it is the only degree -1 derivation on $\Omega^\bullet(M)$ which, on 1-forms, is the evaluation on V.

Proof. The derivation formula is a completely straightforward computation. For the uniqueness, one can just imitate the arguments used for the uniqueness of $\mathscr{L}_V$, just that it is a bit simpler now (being degree -1, it kills of the function, and the condition from the statement prescribes i_V on 1-forms hence, by the Leibniz identity, on all the sums of wedges of 1-forms. One can either show that any differential form can be written as such a sum, or just provide a local argument completely similar to that for $\mathscr{L}_V$.

Next: the Cartan magic formula, which says that $\mathscr{L}_V$ is the graded commutator of d and i_V; given that d has degree 1 and i_V has degree -1, that means:

Theorem 4.58. *For any vector field $V \in \mathfrak{X}(M)$ one has*

$$\mathscr{L}_V = d \circ i_V + i_V \circ d.$$

Proof. A simple computation using the Leibinz identities for d and i_V (with special care at the signs involved) shows that $d \circ i_V + i_V \circ d$ is a derivation of degree zero. Since $d \circ d = 0$, we see that it also commutes with d. Moreover, on functions f it gives $i_V d(f) = L_V(f)$. Hence, by the uniqueness from Proposition 4.55, it must coincide with $\mathscr{L}_V$.

Exercise 4.59 (from the 2018/2019 exam). Let M be a manifold, and $X, Y \in \mathfrak{X}(M)$ two vector fields. We look at

$$L_X \circ i_Y - i_Y \circ L_X : \Omega^k(M) \to \Omega^{k-1}(M) \quad (\text{for all } k \geq 0).$$

Show that:

(a) on 1-forms $\theta \in \Omega^1(M)$, it coincides with $i_{[X,Y]}$.
(b) it satisfies a Leibniz-type identity that you have to write down yourself.
(c) $L_X \circ i_Y - i_Y \circ L_X = i_{[X,Y]}$.

Exercise 4.60. Consider the sphere $S^2 \subset \mathbb{R}^3$ and we use (x,y,z) to denote the standard coordinates in $\mathbb{R}^3$. We consider the following vector field tangent to the sphere

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(S^2)$$

as well as the volume form on the sphere:

$$\sigma = x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy \in \Omega^2(S^2)$$

(as before, while the previous formula defines a 2-form on $\mathbb{R}^3$, σ is the restriction to S^2).

(a) Compute $i_X(\sigma)$ and $d(i_X(\sigma))$.
(b) Compute $d\sigma$ and $i_X(d\sigma))$.
(c) Compute $L_X(\sigma)$ in two ways: one using the Cartan formula, and one using the properties of L_X (being a derivation, and commuting with d).
(d) Compute the flow ϕ^t of X.
(e) Show that $(\phi^t)^*\sigma = \sigma$ for all $t \in \mathbb{R}$.

Exercise 4.61. Consider the following 1-form on $\mathbb{R}^3$:

$$\theta = (z^2 + 2xy) \cdot dx + (x^2 + 2yz) \cdot dy + (y^2 + 2zx) \cdot dz.$$

Do the following:

1. compute $\theta \wedge dy \wedge dz$ (i.e. write it as $? \cdot dx \wedge dy \wedge dz$, with ? to be computed).
2. show that $d\theta = 0$.
3. find a smooth function $f \in C^\infty(\mathbb{R}^3)$ such that $\theta = df$.

4. for $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, $\phi(x,y,z) = (y,z,x)$ show that $\phi^*(\theta) = \theta$.
5. show that for any smooth map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$, the 2-form $\theta \wedge \psi^*(\theta)$ is closed.
6. For the vector field $X = x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} + z \cdot \frac{\partial}{\partial z}$ compute $i_X(\theta)$ and $L_X(\theta)$.
7. Compute the flow ϕ_X^t of X.
8. With the formula for the ϕ_X^t and $L_X(\theta)$ that you found at the previous two points check directly that, indeed,
$$L_X(\theta) = \frac{d}{dt}\Big|_{t=0} (\phi_X^t)^*(\theta).$$
9. show that $\theta|_{S^2} \neq 0$.
10. Find/describe a 2-dimensional submanifold $M \subset \mathbb{R}^3$ such that $\theta|_M = 0$.

Exercise 4.62. As a continuation of Exercise 3.103:

8. Show that there exist, and they are unique, two 1-forms $d\theta_1$ and $d\theta_2$ on $\mathbb{T}^2$ with the property that
$$d\theta_i\left(\frac{\partial}{\partial \theta_j}\right) = \delta_{i,j} \text{ (1 if } i = j \text{ and 0 otherwise).}$$
9. Show that for any smooth function f on $\mathbb{T}^2$ one has
$$df = \frac{\partial f}{\partial \theta_1} \cdot d\theta_1 + \frac{\partial f}{\partial \theta_2} \cdot d\theta_2.$$
10. In contrast with what the notation may suggest, $d\theta_1$ and $d\theta_2$ are not exact forms. To see this, you are asked to prove something a bit more general: on any compact manifold M, any 1-form that is exact must vanish at at least one point in M.
11. However, show that $d\theta_1$ and $d\theta_2$ are both closed.

Exercise 4.63. Consider the following three 1-forms on the sphere S^3:
$$\theta^1 := 2(-ydx + xdy + tdz - zdt),$$
$$\theta^2 := 2(-zdx - tdy + xdz + ydt),$$
$$\theta^3 := 2(-tdx + zdy - ydz + xdt).$$
Show that:

(a) the differentials of these 1-forms satisfy:
$$d\theta_1 = -\theta_2 \wedge \theta_3, \quad d\theta_2 = -\theta_3 \wedge \theta_1, \quad d\theta_3 = -\theta_1 \wedge \theta_2$$
(b) that, for all $f \in \mathscr{C}^\infty(S^3)$, one has:
$$df = L_{V^1}(f) \cdot \theta_1 + L_{V^2}(f) \cdot \theta_2 + L_{V_3}(f) \cdot \theta_3,$$
where V^1, V^2 and V^3 are the vector fields from Exercise 3.49.
(c) For the vector field V from Exercise 3.38 one has
$$L_V \theta_1 = L_V \theta_2 = L_V \theta_3 = 0.$$

Exercise 4.64. Let us first show that, in the previous exercise, all that matters are the main properties of the vector fields involved (one doesn't even know that we are working on the sphere). More precisely, assume that M is a connected 3-dimensional manifold, and $V, V^1, V^2, V^3 \in \mathfrak{X}(M)$ are vector fields on M with the property that V_p^1, V_p^2, V_p^3 form a basis of T_pM for all $p \in M$ and
$$[V^1, V^2] = V^3, \quad [V^2, V^3] = V^1, \quad [V^3, V^1] = V^2.$$

$$[V,V_1] = [V,V_2] = [V,V_3] = 0.$$

Let $\theta_1, \theta_2, \theta_3 \in \Omega^1(M)$ be the 1-forms that are dual to V^1, V^2, V^3, i.e. satisfying

$$\theta_i(V^j) = \delta^i_j \quad \text{(1 if } i = j \text{ and 0 otherwise).}$$

Show that:

(a) for any $f \in \mathscr{C}^\infty(M)$ one has

$$df = L_{V^1}(f) \cdot \theta_1 + L_{V^2}(f) \cdot \theta_2 + L_{V^3}(f) \cdot \theta_3.$$

(b) the 1-forms θ_i satisfy

$$d\theta_1 = -\theta_2 \wedge \theta_3, \quad d\theta_2 = -\theta_3 \wedge \theta_1, \quad d\theta_3 = -\theta_1 \wedge \theta_2.$$

$$L_V \theta_1 = L_V \theta_2 = L_V \theta_3 = 0.$$

On the other hand, despite the apparent greater generality of the setting, we are actually getting closer to ... the Hopf fibration. More precisely, defining

$$h_1 = i_V(\theta_1), \quad h_2 = i_V(\theta_2), \quad h_3 = i_V(\theta_3),$$

show that:

(c) the differentials of these functions satisfy:

$$dh_1 = h_2 \cdot \theta_3 - h_3 \cdot \theta_2,$$
$$dh_2 = h_3 \cdot \theta_1 - h_1 \cdot \theta_3,$$
$$dh_3 = h_1 \cdot \theta_2 - h_2 \cdot \theta_1$$

(d) the function

$$h = (h_1, h_2, h_3) : M \to \mathbb{R}^3$$

takes values in a sphere S^2_r (of some radius $r > 0$).

(e) furthermore, h must be a submersion.

(f) What happens when $M = S^3$ and V, V^1, V^2 and V^3 are the vector fields from Exercise 3.38 and Exercise 3.49?