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Contents

Chapter 1
Reminders on Topology and Analysis

1.1 Reminder 1: Topology; topological manifolds

Here is a very brief reminder on the basic notions from Topology. For those which are not so familiar with these basics, one may skip the later parts of this section (most notably the part on partitions of unity) and return to it later on, when necessary.

1.1.1 The objects of Topology

First of all, the main objects of Topology: **a topological space** is a set X endowed with a **topology**, i.e. a collection $\mathscr{T}$ of subsets of X (called **the opens of the topological space**, or simply **opens in** X) such that $\emptyset$ and X are open in X, arbitrary unions of opens are open and finite intersections of opens are open. We usually omit $\mathscr{T}$ from the notations, and we simply say that X is a topological space; hence that means that X is a set and we can talk about the subsets of X that are open (in X).

A topology on X allows us to make sense of the central phenomena of Topology: "two points being close to each other". First of all we can make sense of neighborhoods in a topological space X: given $x \in X$, a **neighborhood** (in X) of x is any subset $V \subset X$ that contains at least an open neighborhood of x, i.e. an open U with $x \in U$. In turn, this allows us to talk about convergence: a sequence $(x_n)_{n\geq 1}$ of elements of X **converges** (in the topological space X) to $x \in X$ if for any neighborhood V of x there exists an integer n_V such that $x_n \in V$ for all $n \geq n_V$.

The notion of neighborhoods also allows to talk also about an important property one requires on topological spaces in order to exclude pathological examples- Hausdorffness: a topological space X called **Hausdorff** if for any $x, y \in X$ distinct, there are neighborhoods U of x and V of y such that $U \cap V = \emptyset$. Hence, intuitively, this means that "if two are distinct, then they cannot be too close to each other" (yes, not having this sounds pathological but, since this condition is not automatic, it is often imposed precisely to avoid "strange/pathological spaces).

1.1.2 The morphisms/isomomorphisms of Topology

The relevant maps (the only ones that really matter) in Topology are the continuous ones: a map $f : X \to Y$ between topological spaces is called **continuous** if for any U-open in Y, its pre-image $f^{-1}(U)$ is open in X. "Isomorphism" between topological spaces are known under the name of **homeomorphisms**: they are the bijections $f : X \to Y$ with the property that both f as well as f^{-1} are continuous.

In the language of "Category Theory", Topology is the category whose objects are topological spaces, and whose morphisms (between objects) are the continuous maps.

Remark 1.1. Note that, while proving that two topological spaces are homeomorphic (i.e there exists a homeomorphism between them) is relatively easy in principle (one just has to produce ONE single homeomorphism between them- and for that it is often enough to follow ones intuition), proving that two spaces are not homeomorphic is much harder. One way to proceed is by understanding the specific "topological properties" of the spaces under discussion (such as Hausdorffness, compactness, etc); if one of them has such a topological property and the other one does not, then they cannot be homeomorphic. A more advanced approach consists of constructing topological invariants of algebraic nature (such as numbers, groups, etc)- and that is what Algebraic Topology is about.

1.1.3 Metric topologies; bases

One of the largest class of topological spaces are metric spaces (X,d): any metric $d : X \times X \to \mathbb{R}$ induces a topology $\mathscr{T}_d$ on X: a subset $U \subset X$ is open iff for any $x \in U$ there exists $r > 0$ such that U contains the d-ball of center x and radius r:

$$B_d(x,r) := \{y \in X : d(x,y) < r\}. \tag{1.1.1}$$

In general, a topological space X is called **metrizable** if there is a metric d on X such that the original topology on X coincides with $\mathscr{T}_d$ (note also that, if such a d exists, in general it is far from being unique; e.g. already $2d$, $3d$, $\frac{d}{d+1}$ would do the same job). One of the most interesting questions about topological spaces is to decide whether they are metrizable or not; **metrizability theorems** aim at finding simple topological conditions that imply metrizability.

Considering the Euclidean metric d on $\mathbb{R}^m$, or on any subset $A \subset \mathbb{R}^m$, we see that A is endowed with a canonical topology- called **the Euclidean topology** on the subset $A \subset \mathbb{R}^m$ (exercise: show that also the square metric induces the same topology). Note that, in this case, the resulting notion of convergence (and continuity) coincides with the one from Analysis.

Remark 1.2. Continuing the previous remark, let us point out that showing that two Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ of different dimensions $m \neq n$ are not homeomorphic is non-trivial. When $m = 1$, this can be done using the notion of connectedness but, for $m,n \geq 2$, one has to appeal to tools from Algebraic Topology.

We now return to general metric spaces. A metric d on X allows us to talk about the open balls $B_d(x,r)$ for $x \in X$, $r \in \mathbb{R}_+$ (see (1.1.1)), giving rise to the collection of open balls induced by d:

$$\mathscr{B}_d = \{B_d(x,r) : x \in X, r \in \mathbb{R}_+\}.$$

This is not a topology on X, but $\mathscr{T}_d$ is the smallest topology containing $\mathscr{B}_d$.

Recall also (simple exercise) that any family $\mathscr{B}$ of subsets of X gives rise to a topology $\mathscr{T}(\mathscr{B})$ on X, defined as the smallest one containing $\mathscr{B}$. It is called **the topology generated by** $\mathscr{B}$. In general, the members of $\mathscr{T}(\mathscr{B})$ are arbitrary unions of finite intersections of members of $\mathscr{T}$.

Depending on the properties of $\mathscr{B}$, the members of $\mathscr{T}(\mathscr{B})$ may have simpler descriptions. The most common case is when $\mathscr{B}$ is a **topology basis**, i.e. satisfies the following axioms: any $x \in X$ is contained in at least one member B of $\mathscr{B}$ and, for any $B_1, B_2 \in \mathscr{B}$ and any $x \in B_1 \cap B_2$, there exists $B \in \mathscr{B}$ containing x with $B \subset B_1 \cap B_2$. In this case, for $U \subset X$, the following are equivalent:

(0) U belongs to $\mathscr{T}(\mathscr{B})$.
(1) for any $x \in U$ there exists $B \in \mathscr{B}$ s.t. $x \in B \subset U$.
(2) U is a union of members of $\mathscr{B}$.

For instance, for any metric d on X, the collection $\mathscr{B}_d$ is a topology basis, and (1) is precisely the original definition of $\mathscr{T}_d$.

One can change a bit the point of view and, starting with a topology $\mathscr{T}$ on X, look for collections $\mathscr{B}$ generating $\mathscr{T}$, i.e. such that $\mathscr{T} = \mathscr{T}(\mathscr{B})$. Of course, one possibility is to take $\mathscr{B} = \mathscr{T}$, but this is the least interesting one. The more interesting choices are the ones for which $\mathscr{B}$ is smaller- e.g. countable. And here is the precise terminology: given a topological space X, a **basis for the topological space** X is any collection $\mathscr{B}$ of subsets of X with the property it is a topology basis and $\mathscr{T} = \mathscr{T}(\mathscr{B})$. As above, for a collection $\mathscr{B}$ of subsets of X, the following are equivalent:

(0) $\mathscr{B}$ is a basis for the space X.
(1) for any open U in X and any $x \in U$ there exists $B \in \mathscr{B}$ s.t. $x \in B \subset U$.
(2) any open in X is a union of members of $\mathscr{B}$.

(in particular, each of the conditions (1) and (2) imply that $\mathscr{B}$ is a topology basis).

Repeating what we said before, but with a slightly different wording, we have that for any metric d on X, the metric topology admits $\mathscr{B}_d$ as basis. Another possible basis for the space X (endowed with the topology $\mathscr{T}_d$), slightly smaller, is

$$\mathscr{B}_d = \{B_d(x, \frac{1}{n}) : x \in X, n \in \mathbb{N}\}.$$

For the Euclidean metric d_{Eucl} on $\mathbb{R}^m$ we can do even better:

$$\mathscr{B}_{\mathbb{Q}} := \{B_{d_{\text{Eucl}}}(q, \frac{1}{n}) : q \in \mathbb{Q}^m, n \in \mathbb{N}\}$$

is still a basis for the Euclidean topology on $\mathbb{R}^m$, but it it "much smaller": it is countable.

In general, one says that a topological space X is **second countable** if it admits a basis $\mathscr{B}$ which is countable.

1.1.4 Topological manifolds

The second countability condition is a very subtle one and turns out to be of capital importance in establishing some central results in Topology and Geometry- such as metrizability and embedding theorems. In particular, it is part of the basic axioms for the notion of manifolds. For now:

Definition 1.3. A **topological m-dimensional manifold** is a topological space X satisfying the following:

(TM0): any point $x \in X$ admits a neighborhood X which is homeomorphic to an open subset of $\mathbb{R}^m$.
(TM1): it is Hausdorff.
(TM2): it is second countable.

A homeomorphism

$$\chi : U \to \Omega \subset \mathbb{R}^m$$

from an open subset U of X to an open subset Ω in $\mathbb{R}^m$ is called a **m-dimensional topological chart for** X, and U is called the domain of the chart- so that axiom (TM0) can also be read as:

(TM0): X can be covered by (domains) of m-dimensional topological charts.

You should convince yourself (or remember) why some of the usual examples of topological spaces such as spheres, tori, Moebius band, etc are topological manifolds. Also, in all these examples, one should concentrate first on the condition (TM0) (... as the labelling indicates). Note however that, while the notion of dimension is intuitively clear (at least in all examples), handling it theoretically is not such a piece of cake; see Remark 1.2. This is due to the fact that there is no obvious topological characterization of the (intuitive notion) of dimension. This will be much less of a problem as soon as we move to (differentiable) manifolds.

Remark 1.4. Since the notion of "topological space" is built on the notion of "open", so are most of the basic definitions in Topology- such as continuity, Hausdorffness, compactness, etc etc. However, under rather mild assumptions, such definitions can be rephrased more intuitively, using sequences. The main "mild assumption" that we have in mind here is that of "first countability"; please see the basic course on Topology. This condition is weaker even than the second countability condition. For instance, metric topologies are always first countable but may fail to be second countable. For our purpose, it is enough to know that either of the conditions (TM0) or (TM2) implies 1st countability (and, if you look at the definitions, you will see that this statement is completely trivial).

What is interesting to know here is that, when restricting to spaces X which are first countable, many of the basic notions can be reformulated in terms of sequences. E.g.:

- X is Hausdorff iff any convergent sequence in X has at most one limit.
- $f: X \to Y$ is continuous iff it is sequential continuous i.e.: if $(x_n)_{n\geq 1}$ is a sequence converging in X to $x \in X$, then $(f(x_n))_{n\geq 1}$ converges in Y to $f(x)$.

1.1.5 Inside a topological space

Recall that, given a space X, a subset $A \subset X$ is said to be **closed in** X if its complement $X \setminus A$ is open. Of course, knowing the closed subsets of X is equivalent to knowing the open ones- hence one could have introduced the notion of topology completely in terms of closed subsets (which would then be the axioms?). Opens are preferred because some of the the most important properties can be described more directly in terms of opens (and perhaps also because they are closer in spirit to the notion of ”ball” in a metric space). However, closed subsets often have some very nice properties- e.g. when talking about compactness.

Given the axioms of a topology (namely the fact that arbitrary unions of opens is open or, equivalently, that arbitrary intersections of closeds is closed), it follows that for any subset A of a topological space X one can talk about:

- the largest open contained in A- and this is called **the interior of** A (in the space X), and denoted $\mathrm{Int}(A)$.
- the smallest closed containing A- and this is called **the closure of** A (in the space X), and denoted $\mathrm{Cl}(A)$,

Recall also that, under the first countability axiom (in particular, for topological manifolds), the closure has a particularly nice description in terms of sequences:

$$\mathrm{Cl}(A) = \{x \in X : \exists \text{ a sequence in } A \text{ converging to } x\}.$$

1.1.6 Construction of topological spaces

We have already seen two (related) ways of constructing topologies on a set X: the metric topology $\mathscr{T}_d$ induced by any metric d on X, and the topology $\mathscr{T}(\mathscr{B})$ generated by any family $\mathscr{B}$ of subsets of X (with the particularly nice situation when $\mathscr{B}$ is a topology basis).

There are various other important constructions of topologies out of the old ones. For instance, given any two topological spaces X and Y, the Cartesian product

$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$

carries a canonical topology, called **the product topology**. There is a slight complication: while we would like that the products of opens is open,

$$\mathscr{B}_{X\times Y} := \{U \times V : U - \text{open in } X, V - \text{open in } Y\}$$

is not a topology on $X \times Y$; instead, it is a topology basis, and the product topology is defined as the topology generated by $\mathscr{B}_{X\times Y}$. Equivalently, and more conceptually, it is the smallest topology on $X \times Y$ with the property that the projections

$$\mathrm{pr}_X : X \times Y \to X, \mathrm{pr}_Y : X \times Y \to Y$$

are continuous.

The last description is more conceptual because it follows a general philosophy that one should apply when looking for topologies: require that the most interesting maps that you have around to be continuous, and look for ”the best (least boring)” topology that does that (usually ”the best” means ”the largest” or ”the smallest”).

Another example of this philosophy is **the induced topology**: given a topological space X, any subset $A \subset X$ carries a canonical, induced, topology: it is the smallest topology with the property that the canonical inclusion

$$i : A \to X, \quad i(a) = a$$

is continuous (why would looking for the largest topology with this property be "boring"?). Explicitly, the opens in A (endowed with the topology induced from X) are the intersections $A \cap U$ of A with opens U of X.

Yet another example is that of **quotient topology**. In some sense, it is the other extreme compared to the previous example. While before we started with an inclusion $I : A \to X$, we now start with a surjection

$$\pi : X \to Y,$$

where X is a topological space and Y is just a set (on which we would like to induce a topology). This time, looking for "the most interesting" topology on Y, we are lead to looking at the largest topology on Y with the property that π is continuous (why?). We obtain the quotient topology on Y: a subset $U \subset Y$ is an open of this topology if an donly if $\pi^{-1}(U)$ is open in X (check that this is, indeed, a topology on Y).

The terminology "quotient" comes from the fact that, typically, the situation of having a surjection $\pi : X \to Y$ arises when starting with X and an equivalence relation R on X. Then, with the intuition that we want to glue the points of X that are equivalent (w.r.t. the equivalence relation R), we obtain the quotient space

$$Y = X/R$$

(abstractly made of R-equivalence classes $[x]_R = \{y \in X : (x,y) \in R\}$ of points $x \in X$) together with the canonical projection

$$\pi_R : X \to X/R, \quad \pi_R(x) = [x]_R.$$

Therefore, starting with an equivalence relation R on a topological space X, we see that the resulting quotient X/R carries a canonical (quotient) topology.

One of the most interesting examples of quotient topologies is the canonical topology on the projective space

$$\mathbb{P}^m := \{l : l - \text{line through the origin in } \mathbb{R}^{m+1}\}$$

(i.e. the set of all 1-dimensional vector subspaces of $\mathbb{R}^{m+1}$). We can put ourselves in the previous situation by considering

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{P}^m, x \mapsto l_x,$$

where l_x is the line through the origin and x (i.e. the vector subspace $\mathbb{R} \cdot x$ spanned by x). In terms of equivalence relations, we deal with the equivalence relation on $\mathbb{R}^{m+1} \setminus \{0\}$ given by:

$$x \sim y \iff l_x = l_y \iff y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}.$$

Using the Euclidean topology on $\mathbb{R}^{m+1} \setminus \{0\}$ we obtain a natural topology on $\mathbb{P}^m$; endowed with this topology, $\mathbb{P}^m$ is called **the projective space** (of dimension m). You should convince yourself that convergence in this topology corresponds to the intuitive idea of "lines getting close to each other".

Finally, given the notion of induced topology, one can make use of that of embeddings: an **embedding** of a topological space X into a topological space Y is any map $i : X \to Y$ that is continuous, injective and, when interpreted as a continuous map $i : X \to i(X)$ and we endow $i(X) \subset Y$ with the topology induced from Y, it is a homemorphism (note that the last map is automatically continuous and bijective, but that does not imply that its inverse is continuous as well!). Next to metrizability theorems (see above), one of the most interesting problems in Topology/Geometry is that of deciding whether a space X can be embedded in a Euclidean space; results in this direction are usually labelled as **embedding theorems**. Looking at the notion of topological manifold, it is worth pointing out that, due also to the axioms (TM1) and (TM2), it follows that any topological manifold is metrizable and can be embedded in some Euclidean space!

1.1.7 Topological properties

As we have pointed out in Remark 1.1, to distinguish topological spaces from each other (or to understand better each specific one), it is useful to isolate the various topological properties that spaces may have. By a topological property we mean any property that can be described by only using the notion of opens or, equivalently, any property that is preserved via homeomorphisms. We have already mentioned several such properties: Hausdorffness and second countability. Here we recall a few more.

The first one is that of connectedness: a space X is called **connected** if it cannot be written as $X = U \cup V$ with U,V-disjoint non-empty opens in X. Or, equivalently, if the only subsets of X that are both open and closed are $\emptyset$ and X. In general, if X is not connected, it can be "broken" into connected pieces; more precisely, recall that a **connected component** of a space X is any connected subset $C \subset X$ which (when endowed with the induced topology) is connected, and which is maximal (w.r.t. the inclusion) with this property. Then the set of connected components defines a partition of X by closed subspace. In examples, the partition into connected components is usually easy to guess intuitively; here is a simple exercise that can be used as a recipe to confirm such guesses: assume that we manage to write X as

$$X = X_1 \cup \ldots \cup X_k, \quad \text{with } X_i \cap X_j = \emptyset \text{ for } i \neq j.$$

Assume also that all the X_is are open or, equivalently (why?), that all the X_is are closed. Then $\{X_1, \ldots, X_k\}$ must coincide with the partition into connected components.

Remark 1.5. Of course, the number of connected components may sometimes be infinite (even non-countable). Note however that, for topological manifolds M, due to the second countability axiom, the number of connected components if always at most countable (and finite if M is compact). Actually, one often restricts the attention to connected manifolds.

Another important topological property is that of compactness. While this is a property that one usually encounters in the first courses in Analysis (compacts in $\mathbb{R}^m$ being the subsets $A \subset \mathbb{R}^m$ that are closed and bounded), the fact that this is a topological property (i.e. can be described by appealing only to the notion of opens in A, without any reference to the Euclidean metric or to the way that A sits inside $\mathbb{R}^m$) is not at all obvious. That makes the resulting general definition less intuitive and a bit hard to digest at first: a topological space X is said to be **compact** if for any open cover

$$\mathscr{U} = \{U_i : i \in I\}$$

of X (i.e. each U_i is open in X, their union is X, and I is an indexing set), one can extract a finite subcover, i.e. there exists $i_1, \ldots, i_k \in I$ such that $\{U_{i_1}, \ldots, U_{i_k}\}$ is still a cover of X- i.e.

$$X = U_{i_1} \cup \ldots \cup U_{i_k}.$$

Here is the list of the most important properties of compactness:

1. Compact inside Hausdorff is closed: if X is a topological space, $A \subset X$ is endowed with the induced topology (see above) then:
$$A - \text{compact}, X - \text{Hausdorff} \Longrightarrow A - \text{is closed in} X.$$
2. Closed inside compact is compact: if X is a topological space, $A \subset X$ is endowed with the induced topology (see above) then:
$$A - \text{is closed in } X, X - \text{compact} \Longrightarrow A - \text{is compact}$$
3. Any compact Hausdorff space is automatically normal:
$$X - \text{compact} \Longrightarrow X - \text{normal}.$$
Recall here that a topological space X is said to be **normal** if for any $A,B \subset X$ closed disjoint subsets, one can find opens in X, U containing A and V containing B, such that $U \cap V = \emptyset$.
4. Product of compacts is compact: if X and Y are compact spaces then $X \times Y$, endowed with the product topology (see above), is compact:

$$X, Y - \text{compact} \Longrightarrow X \times Y - \text{compact}.$$

5. Continuous applied to compact is compact: if $f : X \to Y$ is continuous and $A \subset X$ (with the induced topology) is compact, then so is $f(A) \subset Y$:

$$f : X \to Y \text{ continuous}, A - \text{compact inside } X \Longrightarrow f(A) - \text{compact}.$$

6. In particular: quotients of compacts are compacts.
7. A continuous bijection from a compact space to a Hausdorff one is automatically a homeomorphism:

$$(\text{continuous } f) : (\text{compact space } X) \to \ (\text{Hausdorff space } Y) \Longrightarrow f \text{ is a homeomorphism}.$$

More generally: a continuous injection from compact to Hausdorff is automatically an embedding (see above).

Exercise 1.6. Assuming that you already know that the unit interval $[0,1]$ (endowed with the Euclidean topology) is compact, use the properties listed above to deduce that: for subsets of $\mathbb{R}^m$ endowed with the Euclidean topology:

$$A \subset \mathbb{R}^m \text{ is compact} \Longleftrightarrow A \text{ is closed and bounded in } \mathbb{R}^m.$$

A related topological property is the local version of compactness: one says that a space X is **locally compact** if any point $x \in X$ admits a compact neighborhood. If X is also Hausdorff, it follows that any point in X admits "arbitrarily small compact neighborhoods": for any neighborhood U of x in X there exists a compact neighbhborhood of x, contained in U. In general, Hausdorff locally compact spaces can be compactified by adding one extra-point. More on the 1-point compactification can be found in the lecture note on Topology.

For topological manifolds, axiom (MT0) ensures that they are automatically locally compact. But also axioms (MT1), (MT2) interact nicely with local compactness: they ensure the existence of "exhaustions". This is Theorem 4.37 in the notes on Topology:

Theorem 1.7. *Any locally compact, Hausdorff, 2nd countable space X admits an exhaustion, i.e. a family $\{K_n : n \in \mathbb{Z}_+\}$ of compact subsets of X such that $X = \cup_n K_n$ and $K_n \subset \overset{\circ}{K}_{n+1}$ for all n.*

Proof. Let $\mathscr{B}$ be a countable basis and consider $\mathscr{V} = \{B \in \mathscr{B} : \overline{B} - \text{compact}\}$. Then $\mathscr{V}$ is a basis: for any open U and $x \in X$ we choose a compact neighborhood N inside U; since $\mathscr{B}$ is a basis, we find $B \in \mathscr{B}$ s.t. $x \in B \subset N$; this implies $\overline{B} \subset N$ and then $\overline{B}$ must be compact; hence we found $B \in \mathscr{V}$ s.t. $x \in B \subset U$. In conclusion, we may assume that we have a basis $\mathscr{V} = \{V_n : n \in \mathbb{Z}_+\}$ where $\overline{V}_n$ is compact for each n. We define the exhaustion $\{K_n\}$ inductively, as follows. We put $K_1 = \overline{V}_1$. Since $\mathscr{V}$ covers the compact K_1, we find i_1 such that

$$K_1 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_1}.$$

Denoting by D_1 the right hand side of the inclusion above, we put

$$K_2 = \overline{D}_1 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_1}.$$

This is compact because it is a finite union of compacts. Since $D_1 \subset K_2$ and D_1 is open, we must have $D_1 \subset \overset{\circ}{K}_2$; since $K_1 \subset D_1$, we have $K_1 \subset \overset{\circ}{K}_2$. Next, we choose $i_2 > i_1$ such that

$$K_2 \subset V_1 \cup V_2 \cup \ldots \cup V_{i_2},$$

we denote by D_2 the right hand side of this inclusion, and we put

$$K_3 = \overline{D}_2 = \overline{V}_1 \cup \overline{V}_2 \cup \ldots \cup \overline{V}_{i_2}.$$

As before, K_3 is compact, its interior contains D_2, hence also K_2. Continuing this process, we construct the family K_n, which clearly covers X. 😀

1.1.8 The algebra of continuous functions

Given a topological space X, an "observable on X" has a precise meaning: it is a continuous function

$$f : X \to \mathbb{R}.$$

The set of all such continuous functions is denoted by

$$\mathscr{C}(X).$$

One of the simplest but most fundamental ideas in various parts of Geometry is that of understanding a space X via the associated "object" $\mathscr{C}(X)$. This will allow one to consider "more relevant observables": e.g. for subspaces $X \subset \mathbb{R}^m$, one can consider only fs that are smooth, or polynomials. Or even to handle "spaces" which, although are quite intuitive, are not topological spaces in the strict sense of the word. All together, this point of view gives rise to several directions in Geometry: Differential Geometry (where the key-word is "smooth" instead of "continuous"), Algebraic Geometry (where the key-word is "polynomial", or "complex analytic"), Noncommutative Geometry (where X does not even make sense, but $\mathscr{C}(X)$ does).

Of course, what makes these work is the rich structure that $\mathscr{C}(X)$ posses- making the "object" $\mathscr{C}(X)$ (a priory just a set) into a more interesting mathematical object. We recall here the most important part of the algebraic structure present on $\mathscr{C}(X)$: it is an algebra. Recall here:

Definition 1.8. A (real) **algebra** is a vector space A over $\mathbb{R}$ together with an operation

$$A \times A \to A, \ \ (a,b) \mapsto a \cdot b$$

which is unital in the sense that there exists an element $1 \in A$ such that

$$1 \cdot a = a \cdot 1 = a \quad \forall\, a \in A,$$

and which is $\mathbb{R}$-bilinear and associative, i.e., for all $a, a', b, b', c \in A$, $\lambda \in \mathbb{R}$,

$$(a+a') \cdot b = a \cdot b + a' \cdot b, \ a \cdot (b+b') = a \cdot b + a \cdot b',$$

$$(\lambda a) \cdot b = \lambda (a \cdot b) = a \cdot (\lambda b),$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c\ .$$

We say that A is commutative if $a \cdot b = b \cdot a$ for all $a, b \in A$.

Similarly one talks about complex algebras: then A is a vector space over $\mathbb{C}$ and $\lambda \in \mathbb{C}$.

For a topological space X, the algebra structure on $\mathscr{C}(X)$ is defined simply by pointwise addition and multiplication: for $f, g \in \mathscr{C}(X)$ and $\lambda \in \mathbb{R}$, $f+g, f \cdot g, \lambda \cdot f \in \mathscr{C}(X)$ are given by:

$$(f+g)(x) = f(x) + g(x), \quad (f \cdot g)(x) = f(x)g(x), \quad (\lambda \cdot f)(x) = \lambda f(x).$$

And, considering the space $\mathscr{C}(X, \mathbb{C})$ of $\mathbb{C}$-valued continuous functions on X, one obtains a complex algebra.

The fact that, under certain assumptions, a topological space X can be recovered from the algebra $\mathscr{C}(X)$, is the content of the Gelfand-Naimark theorem. While we refer to the basic course on Topology for the full statement and details, here is the very brief summary:

Theorem 1.9 (informative version of Gelfand Naimark theorem). *There is a way to associate to any algebra A a topological space $X(A)$ (called the spectrum of A) so that, when applied to $A = \mathscr{C}(X)$- the algebra of continuous functions on a compact Hausdorff space X, one recovers X (i.e. $X(\mathscr{C}(X))$ is homeomorphic to X).*

Remark 1.10 (Some details). The spectrum $X(A)$ of an algebra A is defined as the set of characters on A, i.e. maps

$$\chi : A \to \mathbb{R}$$

which preserve the algebra structure, i.e. which are linear, multiplicative ($\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in A$) and send the unit of A to $1 \in \mathbb{R}$. The topology on $X(A)$ is "the best" for which all the evaluation maps

$$\mathrm{ev}_a : X(A) \to \mathbb{R}, \quad \chi \mapsto \chi(a) \quad \text{(one for each } a \in A)$$

are continuous.

For instance, when $A = \mathscr{C}(X)$ for a compact Hausdorff space X, then any point $x \in M$ gives rise to a character χ_x on $\mathscr{C}(X)$, namely the evaluation at x, and the resulting map

$$X \to X(A), \quad x \mapsto \chi_x$$

is the one the realises the desired homeomorphism (of course, there are things to prove along the way). Let us give here a direct argument showing that, if X is a compact space, then any character on $\mathscr{C}(X)$,

$$\chi : \mathscr{C}(X) \to \mathbb{R},$$

is necessarily of type χ_x for some $x \in M$ (proving that the previous map is surjective). □

And, with the mind at the fact that we may want to consider more restrictive conditions than continuity (e.g. smoothness), here is the resulting relevant abstract notion:

Definition 1.11. Given an algebra A (over the base field $\mathbb{R}$ or $\mathbb{C}$), a **subalgebra** of A is any vector subspace $B \subset A$, containing the unit 1 of A and such that

$$b \cdot b' \in B \quad \forall\, b, b' \in B.$$

When we want to be more specific about the base field, we talk about real or complex subalgebras.

For instance, for $X \subset \mathbb{R}^m$, when looking at smooth or polynomial functions, we obtain a sequence of subalgebras:

$$\mathscr{C}^{\mathrm{polyn}}(X) \subset \mathscr{C}^{\infty}(X) \subset \mathscr{C}(X).$$

Finally, when looking at a subset

$$\mathscr{A} \subset \mathscr{C}(X),$$

(subalgebra or not), there are several interesting properties that turn out to be interesting- and we say that:

(1) $\mathscr{A}$ is **point separating** if for any $x, y \in X$ distinct there exists $f \in \mathscr{A}$ such that $f(x) \neq f(y)$ or, equivalently, if there exists $f \in \mathscr{A}$ such that $f(x) = 0$ and $f(y) = 1$.
(2) $\mathscr{A}$ is **normal** if for any two disjoint closed subset $A, B \subset X$, there exists $f \in \mathscr{A}$ such that $f|_A = 0$, $f|_B = 1$.
(3) $\mathscr{A}$ is **closed under sums** if $f + g \in \mathscr{A}$ whenever $f, g \in \mathscr{A}$.
(3) $\mathscr{A}$ is **closed under quotients** if $f/g \in \mathscr{A}$ whenever $f, g \in \mathscr{A}$ and g is nowhere vanishing.

For instance, the Stone-Weierstrass theorem (which will not be used in the rest of the course) says that, if X is a compact Hausdorff space, then any point-separating sub-algebra $\mathscr{A} \subset \mathscr{C}(X)$ is dense in $\mathscr{C}(X)$; with particular cases of the type: real valued continuous functions on $[0,1]$ (or other similar spaces) can be approximated by polynomial functions.

Note that, for a general topological space X, even the entire $\mathscr{A} = \mathscr{C}(X)$ need not be point separating or normal. Actually, it is a rather simple exercise to check that the point separation of $\mathscr{C}(X)$ implies that X must be Hausdorff, while the normality of $\mathscr{C}(X)$ implies that the topological space X must be normal (i.e., as recalled above: any two disjoint closed subsets $A, B \subset X$ can be separated topologically: there exist opens $U, V \subset X$ containing A and B, respectively, with $U \cap V = \emptyset$). What is far less obvious (actually one of the most non-trivial basic results in Topology) is the converse, known as the Urysohn lemma: if a topological space X is Hausdorff and normal then $\mathscr{C}(X)$ is normal; more precisely, for any two disjoint closed subsets $A, B \subset X$ there exists

$$f : X \to [0,1] \text{ continuous and such that } f|_A = 0, f|_B = 1.$$

This will not be used later in the course; we mention it here just for completeness.

1.1.9 Partitions of unity

Finally, one more basic topic from Topology- but this time one that is difficult to appreciate (and perhaps even to digest) without entering the realm of Differential Geometry and/or Analysis: partitions of unity. To be able to talk about partitions of unity that are not just continuous (as we will be interested only on smooth functions), we can place ourselves in the following setting: X is a topological space and

$$\mathscr{A} \subset \mathscr{C}(X)$$

is a given vector subspace; we will be looking at partitions of unity that belong to $\mathscr{A}$. For the main definition, we first need to recall the notion of support: given $\eta : X \to \mathbb{R}$ continuous, **the support of η in X**, denoted $\text{supp}_X(\eta)$ or simply $\text{supp}(\eta)$ is the closure in X of the set $\eta \neq 0$ of points of X on which η does not vanish:

$$\text{supp}_X(\eta) := \overline{\{\eta \neq 0\}} = \overline{\{x \in X : \eta(x) \neq 0\}}^X.$$

Given an open $U \subset X$, we say that η **is supported in** U if $\text{supp}(\eta) \subset U$. This condition allows one to promote functions that are defined only on U, $f : U \to \mathbb{R}$, to functions on X, at least after multiplying by η; namely, $\eta \cdot f$, a priory defined only on U, if extended to X by declaring it to be zero outside U, the resulting function

$$\eta \cdot f : X \to \mathbb{R}$$

will be continuous (check this and, by looking at examples, convince yourselves that this does not work if the condition $\text{supp}(\eta) \subset U$ is replaced by the weaker one that $\{\eta \neq 0\} \subset U$).

We now move to partitions of unity; we start with the finite ones.

Definition 1.12. Let X be a topological space, $\mathscr{U} = \{U_1, \ldots, U_n\}$ a finite open cover of X. A continuous partition of unity subordinated to $\mathscr{U}$ is a family of continuous functions $\eta_i : X \to [0,1]$ satisfying:

$$\eta_1 + \ldots + \eta_k = 1, \ \ \text{supp}(\eta_i) \subset U_i.$$

Given $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\{\eta_i\}$ is an $\mathscr{A}$-partition of unity if $\eta_i \in \mathscr{A}$ for all i.

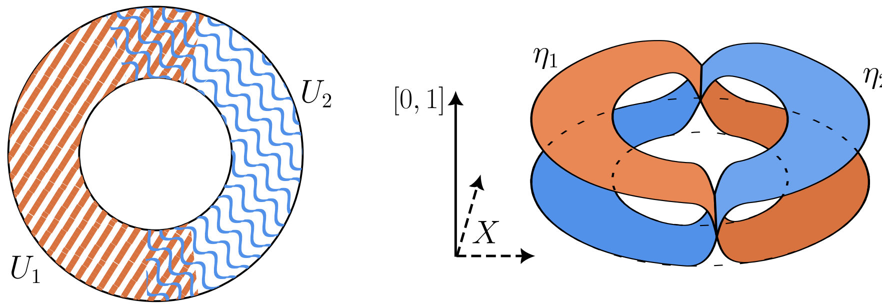


Fig. 1.1 On the left, an annulus X is covered by two open sets U_1 and U_2. The graph on the right shows two functions $\eta_i : X \to [0,1]$ that form a partition of unity subordinate to this cover.

Theorem 1.13. *Let X be a topological space and assume that $\mathscr{A} \subset \mathscr{C}(X)$ is normal and is closed under sums and quotients. Then, for any finite open cover $\mathscr{U}$, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.*
In particular if X is Hausdorff and normal, by the Uryshon Lemma (to ensure that $\mathscr{A} := \mathscr{C}(X)$ is normal), any finite open cover $\mathscr{U}$ admits a continuous partition of unity subordinated to $\mathscr{U}$.

Proof (sketch; for more details, see the lecture notes on Topology). The main ingredients are:

(St1) the remark made above that the normality of $\mathscr{A}$ implies that X is a normal space (simple exercise).
(St2) the fact that, in a normal space X, whenever we have $A \subset U$ with A-closed in X and U-open in X, one can find a smaller open V such that

$$A \subset V \subset \overline{V} \subset U$$

(short proof, but a bit tricky).
(St3) the shrinking lemma: for any finite open cover $\mathscr{U} = \{U_1, \ldots, U_k\}$ of a normal space X one can find another cover $\mathscr{V} = \{V_1, \ldots, V_n\}$ such that

$$\overline{V}_i \subset U_i \quad \forall\, i \in \{1, \ldots, k\}.$$

(this follows by applying the previous step inductively, starting with $U = U_1$ $A = X \setminus (U_2 \cup \ldots \cup U_k)$.

Now the proof of the theorem. Apply the shrinking lemma twice and choose open covers $\mathscr{V} = \{V_i\}$, $\mathscr{W} = \{W_i\}$, with $\overline{V}_i \subset U_i$, $\overline{W}_i \subset V_i$. For each i, we use the separation property of $\mathscr{A}$ for the disjoint closed sets $(\overline{W}_i, X - V_i)$. We find $f_i : X \to [0,1]$ that belongs to $\mathscr{A}$, with $f_i = 1$ on $\overline{W}_i$ and $f_i = 0$ outside V_i. Note that

$$f := f_1 + \ldots + f_k$$

is nowhere zero. Indeed, if $f(x) = 0$, we must have $f_i(x) = 0$ for all i, hence, for all i, $x \notin W_i$. But this contradicts the fact that $\mathscr{W}$ is a cover of X. From the properties of $\mathscr{A}$, each

$$\eta_i := \frac{f_i}{f_1 + \ldots + f_k} : X \to [0,1]$$

is continuous. Clearly, their sum is 1. Finally, $\mathrm{supp}(\eta_i) \subset U_i$ because $\overline{V}_i \subset U_i$ and $\{x : \eta_i(x) \neq 0\} = \{x : f_i(x) \neq 0\} \subset V_i$. 😀

And here is a nice application of the existence of (finite) partitions of unity:

Theorem 1.14. *Any compact topological manifold M can be embedded in some Euclidean space $\mathbb{R}^m$.*

Proof. Cover M by opens that are homeomorphic to $\mathbb{R}^d$, where d is the dimension of M. Using that M is compact, we find an open cover $\mathscr{U} = \{U_1, \dots, U_n\}$ together with homeomorphisms $\chi_i : U_i \to \mathbb{R}^d$. Since M is compact it is also normal hence we find a partition of unity $\{\eta_1, \dots, \eta_n\}$ subordinated to $\mathscr{U}$. Each of the functions $\eta_i \cdot \chi_i : U_i \to \mathbb{R}^d$ is extended to M by declaring it to be zero outside U_i; by the previous comments, the resulting functions $\tilde{\chi}_i : M \to \mathbb{R}^d$ are continuous. Consider now

$$i = (\eta_1, \dots, \eta_k, \tilde{\chi}_1, \dots, \tilde{\chi}_k) : M \to \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{k \text{ times}} \times \underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{k \text{ times}} = \mathbb{R}^{k(d+1)}.$$

One check directly that i is injective; since M is compact and $\mathbb{R}^{k(d+1)}$ is Hausdorff, by the properties recalled on compactness, i will be an embedding. 😀

Finite partitions of unity are useful mainly when working over compacts (so that one can ensure finite open covers). For the more general case one first has to make precise sense of "infinite sums $\sum_i \eta_i$". For that first recall that, given a topological space X, a family $\mathscr{S}$ of subsets of X is said to be **locally finite** (in X) if for any $x \in X$ there exists a neighborhood V of x which intersect only a finite number of members of $\mathscr{S}$. Given a family $\{\eta_i\}_{i \in I}$ (I some indexing set) of continuous functions $\eta_i : X \to \mathbb{R}$, we say that $\{\eta_i\}_{i \in I}$ **is locally finite** in X if their supports (in X) $\text{supp}(\eta_i)$ form a locally finite family of subsets of X. Note that in this case the sum

$$\sum_{i \in I} \eta_i : X \to \mathbb{R}$$

can be defined pointwise (at any $x \in X$ only a finite number of terms do not vanish), and the resulting function is continuous. For a subset $\mathscr{A} \subset \mathscr{C}(X)$, we say that is **closed under locally finite sums** if for any locally finite family $\{\eta_i\}$ with $\eta_i \in \mathscr{A}$, $\sum_i \eta_i$ is again in $\mathscr{A}$.

With these, we can now talk about infinite partitions of unity:

Definition 1.15. Let X be a topological space, $\mathscr{U} = \{U_i : i \in I\}$ an open cover of X. A (continuous) partition of unity subordinated to $\mathscr{U}$ is a locally finite family of continuous functions $\eta_i : X \to [0,1]$ satisfying:

$$\sum_{i \in I} \eta_i = 1, \;\; \text{supp}(\eta_i) \subset U_i.$$

Given $\mathscr{A} \subset \mathscr{C}(X)$, we say that $\{\eta_i\}$ is an $\mathscr{A}$-partition of unity if $\eta_i \in \mathscr{A}$ for all i.

If we are pragmatic and we only care about what is directly applicable later on in this course, the result to have in mind is:

Theorem 1.16. *Let X be a Hausdorff, locally compact and 2nd countable space, $\mathscr{A} \subset \mathscr{C}(X)$ and assume that:*

- *$\mathscr{A}$ is closed under locally finite sums and under quotients and*
- *$\mathscr{A}$ satisfies: for any $x \in M$ and any open neighborhood U of x, there exists $f \in \mathscr{A}$ supported in U with $f(x) > 0$.*

Then, for any open cover $\mathscr{U}$ of X, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.

For the curious student, here is the more detailed discussion to which the previous theorem belongs (with the explanation of how the proof goes). The existence of partitions of unity subordinated to arbitrary open covers forces a topological property of X called paracompactness: we say that a topological space X is **paracompact** if for any open cover $\mathscr{U}$ of X, there exists a locally finite open cover $\mathscr{V}$ that is a refinement of $\mathscr{U}$ in the sense that any $V \in \mathscr{V}$ is included inside some $U \in \mathscr{U}$. The existence of arbitrary partitions of unity is ensured by the following:

Theorem 1.17. *Let X be a paracompact Hausdorff space and assume that $\mathscr{A} \subset \mathscr{C}(X)$ is normal, closed under locally finite sums and closed under quotients.*
Then, for any open cover $\mathscr{U}$ of X, there exists an $\mathscr{A}$-partition of unity subordinated to $\mathscr{U}$.

Since paracompact spaces are automatically normal, hence we can use Uryshon's lemma, it follows that in a paracompact Hausdorff space a for any open cover there exists a continuous partition of unity subordinated to the cover.

The proof of the previous theorem is almost identical with the one from the finite case- just that one now has to establish an infinite version of the shrinking lemma (and that is where paracompactness enters),

To apply the previous theorem, there are two points that may be difficult to check: the paracompactness of X and, when working with arbitrary $\mathscr{A}$, that $\mathscr{A}$ is normal. For the first one, the following comes in handy:

Theorem 1.18. *Any Hausdorff, locally compact and 2nd countable space is paracompact.*

In particular, topological manifolds are automatically paracompact. One can actually show that, under the axioms (TM0) and (TM1), the axiom (TM2) on second countability is equivalent to the fact that M is paracompact and has a countable number of connected components.

Proof. We use an exhaustion $\{K_n\}$ of X (Theorem 1.7). Let $\mathscr{U}$ be an open cover of X. For each $n \in \mathbb{Z}_+$ there is a finite family $\mathscr{V}_n$ which covers $K_n - \mathrm{Int}(K_{n-1})$, consisting of opens V with the properties: $V \subset \mathrm{Int}(K_{n+1}) - K_{n-1}$, $V \subset U$ for some $U \in \mathscr{U}$. Indeed, for any $x \in K_n - \mathrm{Int}(K_{n-1})$ let V_x be the intersection of $\mathrm{Int}(K_{n+1}) - K_{n-1}$ with any member of $\mathscr{U}$ containing x; since $K_n - \mathrm{Int}(K_{n-1})$ is compact, just take a finite subcollection $\mathscr{V}_n$ of $\{V_x\}$, covering $K_n - \mathrm{Int}(K_{n-1})$. Set $\mathscr{V} = \cup_n \mathscr{V}_n$; it covers X since each $K_n - K_{n-1} \subset K_n - \mathrm{Int}(K_{n-1})$ is covered by $\mathscr{V}_n$. Finally, it is locally finite: if $x \in X$, choosing n and V such that $V \in \mathscr{V}_n$, $x \in V$, we have $V \subset \mathrm{Int}(K_{n+1}) - K_{n-1}$, hence V can only intersect members of $\mathscr{V}_m$ with $m \leq n+1$ (a finite number of them!). 😀

Finally, to check the normality of $\mathscr{A}$ needed in Theorem 1.17, the following comes in handy:

Theorem 1.19. *Let X be a Hausdorff paracompact space and $\mathscr{A} \subset \mathscr{C}(X)$ closed under locally finite sums and under quotients. If X is also locally compact, then the following are equivalent:*

1. *$\mathscr{A}$ is normal.*
2. *for any $x \in M$ and any open neighborhood U of x, there exists $f \in \mathscr{A}$ supported in U with $f(x) > 0$.*

In particular, for a topological manifold M, checking that a subset $\mathscr{A} \subset \mathscr{C}(M)$ is normal is a local matter- and that is very useful since, locally, topological manifolds look just like Euclidean spaces.

Proof. That 1 implies 2 is clear: apply the separation property to $\{x\}$ and $X - V$. Assume 2. We claim that for any $C \subset X$ compact and any open U such that $C \subset U$, there exists $f \in \mathscr{A}$ supported in U, such that $f|_C > 0$. Indeed, by hypothesis, for any $c \in C$ we can find an open neighborhood V_c of c and $f_c \in \mathscr{A}$ positive such that $f_c(c) > 0$; then $\{f_c \neq 0\}_{c \in C}$ is an open cover of C in X, hence we can find a finite subcollection (corresponding to some points $c_1, \ldots, c_k \in C$) which still covers C; finally, set $f = f_{c_1} + \ldots + f_{c_k}$.

To prove 1, let $A, B \subset X$ be two closed disjoint subsets. As terminology, $D \subset X$ is called relatively compact if $\overline{D}$ is compact. Since X is locally compact, any point has arbitrarily small relatively compact open neighborhoods. For each $y \in X - A$, we choose such a neighborhood $D_y \subset X - A$. For each $a \in A$, since $a \in X - B$, applying step (St2) from the proof of Theorem 1.13, we find an open D_a such that $a \in D_a \subset X - B$. Again, we may assume that $\overline{D}_a$ is relatively compact. Then $\{D_x : x \in X\}$ is an open cover of X; let $\mathscr{U} = \{U_i : i \in I\}$ be a locally finite refinement. We split the set of indices as $I = I_1 \cup I_2$, where I_1 contains those i for which $U_i \cap A \neq \emptyset$, while I_2 those for which $U_i \subset X - A$. Using the shrinking lemma (the infinite version of the one described in (St3) of he proof of Theorem 1.13) we can also choose an open cover of X, $\mathscr{V} = \{V_i : i \in I\}$, with $\overline{V}_i \subset U_i$. Note that, by construction, each U_i (hence also each V_i) is relatively compact. Hence, by the claim above, we can find $\eta_i \in \mathscr{A}$ such that

$$\eta_i|_{\overline{V}_i} > 0, \ \ \mathrm{supp}(\eta_i) \subset U_i.$$

Finally, we define

$$f(x) = \frac{\sum_{i \in I_1} \eta_i(x)}{\sum_{i \in I} \eta_i(x)}$$

From the properties of $\mathscr{A}$, $f \in \mathscr{A}$. Also, $f|_A = 1$. Indeed, for $a \in A$, a cannot belong to the U_i's with $i \in I_2$ (i.e. those $\subset X - A$); hence $\eta_i(a) = 0$ for all $i \in I_2$, hence $f(a) = 1$. Finally, $f|_B = 0$. To see this, we show that $\eta_i(b) = 0$ for all $i \in I_1$, $b \in B$. Assume the contrary. We find $i \in I_1$ and $b \in B \cap U_i$. Now, from the construction of $\mathscr{U}$, $U_i \subset D_x$ for some $x \in X$. There are two cases. If $x = a \in A$, then the defining property for D_a, namely $D_a \cap B = \emptyset$, is in contradiction with our assumption ($b \in B \cap U_i$). If $x = y \in X - A$, then the defining property for D_y, i.e. $D_y \subset X - A$, is in contradiction with the fact that $i \in I_1$ (i.e. $U_i \cap A \neq \emptyset$).

1.2 Reminder 2: Analysis

The relationship between Analysis and Differential Geometry is subtle. On one hand, Differential Geometry relies on the very basics of Analysis. On the other hand, various notions/results from Analysis become much more transparent/intuitive once the geometric perspective/intuition is brought into picture. In some sense, in many cases, the geometric point of view indicates the (expected) results while analysis provides the tools to prove them.

1.2.1 $\mathbb{R}^n$

The basic playground for multivariate analysis is the standard Euclidean space

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n\ times}.$$

Despite its simplicity, this "space" has many different (but related) structures- and often the problem with handling $\mathbb{R}^n$ comes from the fact that it may not be completely clear which of the structures present on $\mathbb{R}^n$ is relevant for the specific discussions. Here are some of the many interesting structures present on $\mathbb{R}^n$:

- it is a vector space. When we want to emphasize this structure, we will denoted by

  $$v = (v_1, \ldots, v_n) \in \mathbb{R}^n$$

  its elements and we will think of them as "vectors"/"directions". Intrinsic in this notation is the presence of yet another piece of structure: it is not just a vector space- it comes with a preferred (canonical) basis:

  $$e_1, \ldots, e_m \in \mathbb{R}^n;$$

  in coordinates, e_i has 1 on the i-th position and 0 everywhere else.
- it is a vector space endowed with an inner product:

  $$\langle v, w \rangle = \sum_{i=1}^{n} v_i \cdot w_i,$$

  hence it is also a normed vector space, with the norm:

  $$||v|| = \sqrt{\langle v, v \rangle}.$$

- it is a topological space- endowed with the standard Euclidean topology. When we want to emphasize this structure, we will denote by

  $$x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

its elements and we will think of them as ”points”. For instance, when looking at a circle in $\mathbb{R}^2$, the vector space space structure on $\mathbb{R}^2$ is not so relevant, and we think of the circle as made by points rather then vectors. Also, when talking about the continuity of a function $f : \mathbb{R}^n \to \mathbb{R}$, the vector space structure of $\mathbb{R}^n$ is not relevant (though it may be useful).
Recall also that the topology on $\mathbb{R}^n$ is a shadow of yet another structure: $\mathbb{R}^m$ is also a metric space, with the standard Euclidean metric:

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

We say a ”shadow” because uses part of what the metric allows us top talks about: points being ”close to each other” (or, more precisely: convergence and continuity). In particular, there are several other natural metrics on $\mathbb{R}^n$ that induce the same Euclidean topology- e.g. the so called square metric

$$d'(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

- even when thinking about $\mathbb{R}^n$ as a topological space, so of its elements as points, each point $x \in \mathbb{R}^n$ can be represented using ”canonical coordinates”- used already above. Again, the coordinates are not so relevant/important: they are useful and can be used, but they are not intrinsic to the structure. For instance, a circle in $\mathbb{R}^2$ can be described using coordinates by the equation $x^2 + y^2 = 1$, but the circle itself can be drawn without any coordinate axes at our disposal.
Note that the standard coordinates we mentioned are the simplest illustration of the notion of ”chart”- to be discussed in a bit more detail below, and essential in defining the notion of manifold.
- it is a topological space ”on which analysis can be performed” (... i.e. a manifold).
- etc.

Of course, all these are inter-related but, in each situation, it is important to realize which of these structures really matter. In particular, whenever one encounters a definition or result, it is instructive to figure out whether the elements in $\mathbb{R}^m$ that show up play the role of points and which ones of vectors, and how much the definition/result depends on the coordinates. This is the first step towards a geometric understanding of Analysis.

1.2.2 The differential and the inverse function theorem

One can talk about various notions of derivatives of a function f at a point

$$x \in \mathbb{R}^n$$

whenever we have a function f defined on a neighborhood of x- so that the expressions $f(y)$ used below makes sense for all y near x or, equivalently, $f(x+v)$ is defined for small vectors v.

Typically one assumes that f is defined on an open subset $\Omega \subset \mathbb{R}^n$ and takes values in some other Euclidean space $\mathbb{R}^k$,

$$f : \Omega \to \mathbb{R}^k,$$

so that it makes sense to talk about derivatives of f at any point in its domain, $x \in \Omega$.

The most intrinsic notion of derivative is that of ”**total derivative**”, also called **the differential of** f (at the given point $x \in \Omega$). This notion arises when trying to approximate f, near x, by simpler (linear-like) functions. Understanding f near x is about understanding

$$v \mapsto f(x+v)$$

for $v \in \mathbb{R}^n$ near 0. It sends $v = 0$ to $f(x)$, hence the best one can hope for is to approximate

$$v \mapsto f(x+v) - f(x)$$

by functions that are linear in v. Think that we try to write the last expression as a function linear in v, plus one that is qudratic in v, etc (plus eventually an "error term"), but we are interested only in the linear term A. We see we are looking for a linear map

$$A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k)$$

with the property that

$$\lim_{v \to 0} \frac{f(x+v) - f(x) - A(v)}{||v||} = 0. \tag{1.2.1}$$

It is easy to see that, if such a map A exists, then it is unique. And that is what the differential of f at x is.

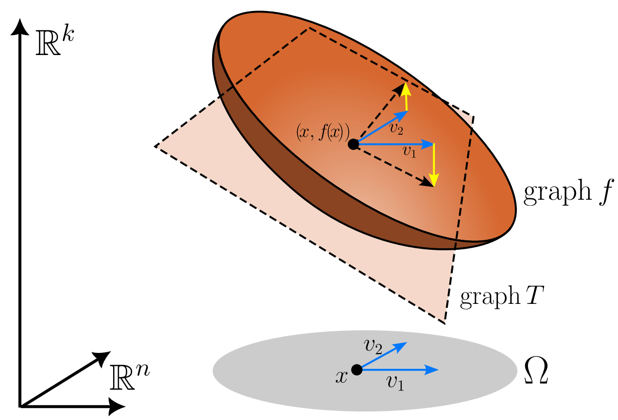


Fig. 1.2 Visualization of the total derivative for a function $f : \Omega \to \mathbb{R}^k$ defined over an open $\Omega \subseteq \mathbb{R}^n$, in this case with $n = 2$ and $k = 1$. Infinitesimal movements through a point $x \in \Omega$ are represented by the blue horizontal vectors v_i and the resulting infinitesimal movement in the codomain $\mathbb{R}^k$ is represented by the yellow vertical vectors $D_x f(v_i)$. They sum up to dashed vectors of the form $(v_i, D_x f(v_i)) \in \mathbb{R}^n \times \mathbb{R}^k$ that are tangent to the graph of f in the point $(x, f(x))$. The graph of the best affine approximation T of f in x (which sends any $\tilde{x} \in \mathbb{R}^n$ to $T(\tilde{x}) = f(x) + D_x f(\tilde{x} - x)$) is an affine plane spanned by the dashed vectors.

Definition 1.20. We say that $f : \Omega \to \mathbb{R}^k$ is **differentiable at** x if there exists a linear map $A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k)$ satisfying (1.2.1). The linear map A (necessarily unique) is called **the differential of** f **at the point** x (or the total derivative of f at x) and is denoted

$$D_x f = (Df)_x : \mathbb{R}^n \to \mathbb{R}^k.$$

We say that f is **differentiable** if it differentiable at all points x in its domain Ω. We say that f is **of class** C^1 if it is differentiable and the resulting map

$$Df : \Omega \to \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k), \quad x \mapsto (Df)_x$$

is continuous (recall here that, via the matrix representation of linear maps, $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^k)$ can be interpreted as the Euclidean space $\mathbb{R}^{n \cdot k}$).

We say that f is of class C^2 if it differentiable and df is of class C^1; proceeding inductively, we can talk about f being of class C^l for any $l \in \mathbb{N}$. We say that f is **smooth** if it is of class C^l for all l.

Recall here also the chain rule that allows one to compute the differential of a composition of two functions:

Proposition 1.21 (the chain rule). *Given opens $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^k$ and functions*

$$\Omega \xrightarrow{f} \Omega' \xrightarrow{g} \mathbb{R}^l,$$

if f is differentiable at $x \in \Omega$ and g is differentiable at $f(x) \in \Omega'$, then $g \circ f$ is differentiable at x and

$$(D(g \circ f))_x = (Dg)_{f(x)} \circ (Df)_x.$$

Despite the fact that the differential $(Df)_x$ arises as "the linear approximation" of f near x, it contains a great deal of information of f near x- and that makes it extremely useful. Probably the best and most fundamental illustration is the inverse function theorem. Recall here that

Definition 1.22. A map $f : \Omega \to \Omega'$ between two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$ is said to be a **diffeomorphism** if it is bijective and both f and f^{-1} are smooth.

We say that f is a **local diffeomorphism** around $x \in \Omega$ if there exist opens $\Omega_x \subset \Omega$ and $\Omega'_{f(x)} \subset \Omega'$ with $x \in \Omega_x$, such that $f|_{\Omega_x} : \Omega_x \to \Omega'_{f(x)}$ is a diffeomorphism.

It is interesting to draw an analogy with Topology, where the main objects are topological spaces, the relevant maps are the continuous ones and two spaces are "isomorphic in Topology" (homeomorphic) if there exists a bijection f between them such that both f as well as f^{-1} are continuous. However, in topology is it usually very hard to prove that two given spaces are not homeomorphic (and one often has to appeal to methods from Algebraic Topology); for instance, just the simple the fact that $\mathbb{R}^n$ and $\mathbb{R}^k$ are homeomorphic only when $n = k$ is very hard to prove. In contrast, the similar statements for diffeomorphisms are much easier to prove thanks to the notion of differential. Indeed, using the chain rule, the following should be a rather easy exercise:

Exercise 1.23. Show that if a map $f : \Omega \to \Omega'$ between two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$ is a local diffeomorphism around $x \in \Omega$, then

$$(Df)_x : \mathbb{R}^n \to \mathbb{R}^k$$

is a linear isomorphism. Deduce that if two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$ are diffeomorphic, then $n = k$.

Although this clearly shows the usefulness of the differential, its great power is due to the inverse function theorem (and its immediate consequences, such as the implicit function theorem -see below). Indeed, we see that a condition on the differential of f at a single (given) point x tells us information about f around x:

Theorem 1.24 (The inverse function theorem). *Given a smooth map $f : \Omega \to \Omega'$ between two opens $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^k$, if f is differentiable at a point $x \in \Omega$ and $(Df)_x$ is an isomorphism, then f is a local diffeomorphism around x.*

1.2.3 *Directional/partial derivatives; the implicit function theorem*

Note that, when talking about the differential $(Df)_x(v)$ (hence $f : \Omega \to \mathbb{R}^k$, with $\Omega \subset \mathbb{R}^n$ open), $x \in \Omega$ should be thought of as a point, while $v \in \mathbb{R}^n$ as a direction (vector). This becomes more apparent if we reformulate the total derivative as a directional derivative.

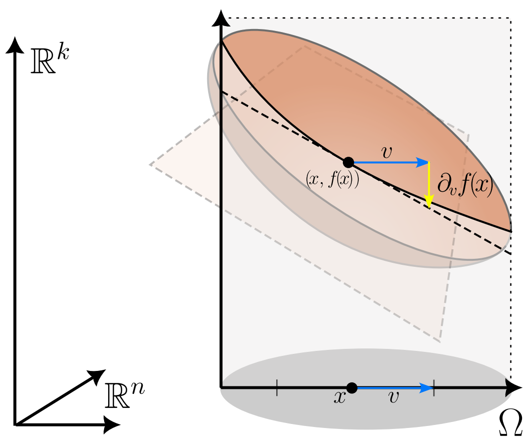


Fig. 1.3 The directional derivative $\partial_v f(x)$ of the function f from Fig. 1.2 can be seen to arise geometrically in the following way: The map $t \mapsto x + vt$ traces a line through Ω. The slice of the graph of f over this line is exactly the graph of the function $t \mapsto f(x+tv)$ if we label the horizontal axis by integer multiples of v. The directional derivative can now be defined as the normal derivative of this function since we have a one-dimensional domain.

Definition 1.25. With $f : \Omega \to \mathbb{R}^k$ and $x \in \Omega$ as above, and an arbitrary vector $v \in \mathbb{R}^n$, **the derivative of** f **at** x **in the direction** v is defined as the vector

$$\partial_v(f)(x) = \frac{\partial f}{\partial v}(x) := \left.\frac{d}{dt}\right|_{t=0} f(x+tv) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \in \mathbb{R}^k.$$

When this derivative exists, we say that f is differentiable at x in the v-direction.

The relationship with the total differential is immediate: just replace in (1.2.1) v (small enough) by tv with $v \in \mathbb{R}^n$ fixed (but arbitrary) and $t \in \mathbb{R}$ approaching 0; using that A is linear, we find that:

$$(Df)_x(v) = \frac{\partial f}{\partial v}(x).$$

This relationship is visualized in Fig. 1.3. In particular, if f is differentiable at x then it is differentiable in all directions. The converse is not true; however, one can show that if f is of class C^1 if and only if all the directional derivatives $\frac{\partial f}{\partial v}$ exist and are continuous (see also the discussion below on partial derivatives).

Applying the previous definition to $v \in \{e_1, \ldots, e_n\}$, a vector in the standard basis of $\mathbb{R}^n$, we obtain the partial derivatives

$$\frac{\partial f}{\partial x_i}(x) := \frac{\partial f}{\partial e_i}(x) = \left.\frac{d}{dy}\right|_{y=x_i} f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \in \mathbb{R}^k.$$

Its components are the partial derivatives of the components f_i of f:

$$\frac{\partial f}{\partial x_i}(x) = \left(\frac{\partial f_1}{\partial x_j}(x), \ldots, \frac{\partial f_k}{\partial x_j}(x)\right) \quad \text{(where } f = (f_1, \ldots, f_k)\text{)}.$$

These partial derivatives contain the same information as $(Df)_x$, just in a less intrinsic way; however, they allow one to handle $(Df)_x$ more concretely, via matrices. For that recall that, due to the fact that the standard Euclidean spaces come with a preferred basis, linear maps

$$A : \mathbb{R}^n \to \mathbb{R}^k$$

can be represented as matrices

$$A = (A^i_j)_{1 \le i \le k, 1 \le j \le n} = \begin{pmatrix} A^1_1 & \ldots & A^1_n \\ \ldots & \ldots & \ldots \\ A^k_1 & \ldots & A^k_n \end{pmatrix}$$

To make a distinction between the matrix A and the linear map A, one may want to denote by $\hat{A}$ the linear map, at least for a while. Then the relationship between the two is (by definition):

$$\hat{A}(e_j) = \sum_{i=1}^{k} A^i_j e_i$$

or, on a general vector $v = v^1 e_1 + \ldots + v^n e_n \in \mathbb{R}^n$, one has

$$\hat{A}(v) = \sum_{i=1}^{k} (\sum_{j=1}^{n} A^i_j v^j) e_i.$$

To write also this formula in terms of matrix multiplication, we interpret any $v \in \mathbb{R}^n$ as a row matrix and we denote by v^T its transpose (column matrix):

$$v^T = \begin{pmatrix} v^1 \\ \ldots \\ v^n \end{pmatrix}$$

With this, the previous formula becomes

$$\hat{A}(v)^T = A \cdot v^T$$

It follows immediately that the standard multiplication of matrices,

$$(A \cdot B)^i_j = \sum_k A^i_k B^k_j,$$

corresponds to the composition of linear maps:

$$\widehat{AB} = \hat{A} \circ \hat{B}.$$

All together, there should be no confusion in identifying A with $\hat{A}$ even notationally.

In the case of the differential $(Df)_x$, to see the matrix representing it we write

$$(Df)_x(e_j) = \frac{\partial f}{\partial x_j}(x) = \left(\frac{\partial f_1}{\partial x_j}(x), \ldots, \frac{\partial f_k}{\partial x_j}(x) \right) = \sum_{i=1}^{k} \frac{\partial f_i}{\partial x_j}(x) e_j$$

i.e., in the matrix notation,

$$(Df)_x = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \ldots & \ldots & \ldots \\ \frac{\partial f_k}{\partial x_1}(x) & \ldots & \frac{\partial f_k}{\partial x_n}(x) \end{pmatrix}.$$

Note that, with this, the fact that $(Df)_x$ is an isomorphism is equivalent to the fact that the matrix above is invertible. More generally, the rank of $(Df)_x$ as a linear map coincides with the rank as a matrix.

With these:

Proposition 1.26. *A function $f : \Omega \to \mathbb{R}^k$ is of class C^1 if and only if all the partial derivatives $\frac{\partial f}{\partial x_i}$ exist and are continuous functions on Ω.*

In this case we can further look at partials derivatives of order two etc. Hence the higher, order l, partial derivatives are defined inductively:

$$\frac{\partial^l f}{\partial x_{i_1} \ldots \partial x_{i_l}} = \frac{\partial}{\partial x_{i_1}} \left(\frac{\partial}{\partial x_{i_2}} \left(\ldots \left(\frac{\partial f}{\partial x_{i_l}} \right) \right) \right).$$

The previous proposition that extends to a characterization of f being of class C^l; in particular, f is smooth if and only if all its higher partial derivatives exist. We will denote by

$$\mathscr{C}^\infty(\mathbb{R}^n)$$

the space (algebra!) of smooth functions on $\mathbb{R}^n$. A **smooth partitions of unity** on $\mathbb{R}^n$ (subordinated to an open cover) is any partition of unity whose members η_i are smooth- i.e. Definition 1.12 (finite case) and Definition 1.15 (general case) applied at $\mathscr{A} := \mathscr{C}^\infty(\mathbb{R}^n) \subset \mathscr{C}(\mathbb{R}^n)$.

Theorem 1.27. *Any open cover of $\mathbb{R}^n$ admits a smooth partition subordinated to it.*

Proof. We want to use Theorem 1.17 for $\mathscr{A} := \mathscr{C}^\infty(\mathbb{R}^n)$. This is clearly closed under quotients and, for the same reason that locally finite sums of continuous functions are continuous, it is closed under locally finite sums. We still have to check the last condition on $\mathscr{A}$ or, equivalently: for any $x \in \mathbb{R}^n$ and any ball centered at x, $B(x, \varepsilon)$, there exists a smooth functions $f : \mathbb{R}^n \to [0,1]$ such that $f(x) > 0$ and f is supported in the ball. It is clear that we may assume that $x = 0$. Also, by rescaling the argument of f (i.e. multiply it by a constant) we may assume that $\varepsilon = 1$. Then set $f(x) = g(x_1^2 + \ldots + x_n^2)$ where $g : \mathbb{R} \to [0,1]$ is any smooth function with $g(0) > 0$ and $g = 0$ outside $[-\frac{1}{2}, \frac{1}{2}]$. That such a function exists should be clear by thinking of its graph. The following exercise provides and explicit formula. 😀

Exercise 1.28. Show that

$$g_0 : \mathbb{R} \to \mathbb{R}, \quad g_0(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is a smooth function. Then show that

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = g_0(x + \frac{1}{2}) g_0(\frac{1}{2} - x).$$

is a smooth function with the properties required at the end of the previous proof.

We now recall the implicit function theorem- which is one the important and rather immediate consequences of the inverse function theorems. The importance is rather geometric, as it arises when looking at curves, surfaces (or higher dimensional ... submanifolds) in $\mathbb{R}^n$. While such subspaces are usually given by equations of type $f(x_1, \ldots, x_n) = 0$ (think e.g. of $x^2 + y^2 = 1$, defining the unit circle in the plane), one would like to express some of the coordinates x_i in terms of the others (or, equivalently, describe our subspace as a graph).

Theorem 1.29. *Let $f : \Omega \to R^k$ be a smooth map defined on an open $\Omega \subset \mathbb{R}^m \times \mathbb{R}^k$ whose elements we label as $(x,y) = (x_1, ..., x_m, y_1, ..., y_k)$. Furthermore let $(\tilde{x}, \tilde{y}) \in \Omega$ be a point where $f(\tilde{x}, \tilde{y}) = 0$ and the matrix*

$$\left(\frac{\partial f_i}{\partial y_j}(\tilde{x}, \tilde{y}) \right)_{1 \le i,j \le k}$$

is non-singular. Then there exists a function $par : U \to \mathbb{R}^k$ defined in a neighborhood U of $\tilde{x}$ such that for all (x,y) near $(\tilde{x}, \tilde{y})$, one has

$$f(x,y) = 0 \Longleftrightarrow y = par(x).$$

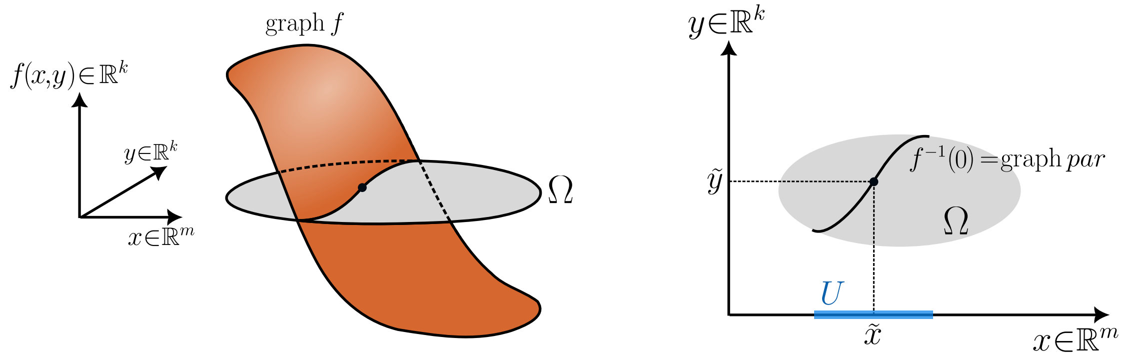


Fig. 1.4 The implicit function Theorem 1.29 illustrated for a concrete choice of $f : \Omega \to \mathbb{R}^k$. The theorem establishes that the preimage $f^{-1}(0)$ (under appropriate assumptions) can locally be written as a graph of a function $par : U \to \mathbb{R}^k$ over a subset of the variables. In other words, the condition that f vanishes *implicitly* defines the function par.

The matrix appearing in this theorem is exactly the Jacobian matrix of the map $y \mapsto f(\tilde{x}, y)$ at $y = \tilde{y}$. You can convince yourself using Fig. 1.4 that its non-singularity is a neccessary condition to find a smooth function par: In the depicted situation, it corresponds exactly to a tangency of $f^{-1}(0)$ in the y-direction at $(\tilde{x}, \tilde{y})$.

Remark 1.30. Note that this theorem is not completely canonical: It gives preference to the last k components of the arguments of f, i.e. depends on how we split the components of elements of Ω into x and y-components. For example, there is an obvious modification in which the starting assumption is that the Jacobian with respect to the *first* k components is regular, instead of the last ones. Such modifications are necessary even when looking at the simplest examples: E.g., for the unit circle where $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + y^2 - 1$, the condition

$$\frac{\partial f}{\partial y}(x,y) \neq 0$$

(necessary for the theorem) is valid at *almost* all the points (x,y) in the circle (and, indeed, we can always solve $y = \pm\sqrt{1-x^2}$), except for the points $(1,0)$ and $(-1,0)$ (and, indeed, there is a problem there: we would need a function two take two values simultaneously close to $y = 0$ in order for it's graph to match $f^{-1}(0)$). However, at those points one can switch the roles of x and y- and, indeed, around those points which are problematic for $\pm\sqrt{1-x^2}$, one can write $x = \pm\sqrt{1-y^2}$.

A more intrinsic version of the theorem (and with exactly the same proof) can be obtained by requiring that $(Df)_x$ has maximal rank k, without specifying which minor is non-singular. the conclusion will be that there exists a permutation $\sigma \in S_{m+k}$ such that $f(p) = 0$ near $\tilde{p} \in \mathbb{R}^{m+k}$ is equivalent to

$$\pi_y(\sigma \cdot p) = par(\pi_x(\sigma \cdot p)),$$

where π_x and π_y are the projections of $\mathbb{R}^{m+k}$ onto the first m and last k components, respectively, and we write $\sigma \cdot p$ for the result of permuting p by σ. But perhaps the most geometric formulation is what is know as the submersion theorem- see below.

Proof. Consider the map

$$F : \Omega \to \mathbb{R}^{m+k}, \quad F(x,y) := (x, f(x,y)).$$

Then the non-singularity condition in the statement precisely means that $(DF)_{(\tilde{x},\tilde{y})}$ is non-singular. Hence, by the inverse function theorem, we find a smooth inverse G of F, defined near $F(\tilde{x},\tilde{y})$. Given the form of F, it follows that G is of a similar form:

$$G(x,z) = (x, g(x,z)).$$

That $G \circ F$ and $F \circ G$ are the identity maps (near $(\tilde{x},\tilde{y})$, and $F(\tilde{x},\tilde{y})$, respectively) translates into

$$g(x, f(x,y)) = y \qquad \text{and} \qquad f(x, g(x,z)) = z. \tag{1.2.2}$$

The first equation shows that

$$f(x,y) = 0 \Longrightarrow g(x,0) = y,$$

hence we have an obvious candidate $par(x) := g(x,0)$. Note that the assumption $f(\tilde{x},\tilde{y}) = 0$ guarantees that g is defined for (x,z) near $(\tilde{x},0)$. The fact that, indeed, $f(x,par(x)) = 0$ for x close to $\tilde{x}$ is just the second equation in (1.2.2) applied when $z = 0$. 😀

1.2.4 Local coordinates/charts

The standard coordinates in $\mathbb{R}^n$, despite being ”obvious”, are often not the best ones to use in specific problems. E.g.: often when dealing with (algebraic or differential) equations or computing integrals, one proceeds to a change of variables (i.e. passing to more convenient coordinates). Baby example: looking at the curve in $\mathbb{R}^2$ defined by

$$5x^2 + 2xy + 2y^2 = 1,$$

a change of coordinates of type

$$x = \frac{u+v}{3}, \quad y = \frac{u-2v}{3} \tag{1.2.3}$$

brings us to the simpler looking equation $u^2 + v^2 = 1$. A very common change of coordinates in $\mathbb{R}^2$ is the passing to polar coordinates:

$$x = r\cos(\theta), \quad y = r\sin(\theta). \tag{1.2.4}$$

To formalise such changes of coordinates, one talks about charts:

Definition 1.31. A **smooth chart of** $\mathbb{R}^n$ is a diffeomorphism

$$\chi = (\chi_1, \ldots, \chi_n) : U \to \Omega \subset \mathbb{R}^n$$

between an open $U \subset \mathbb{R}^n$ and an open $\Omega \subset \mathbb{R}^n$. The open U is called the **domain of the chart** and, for $p \in U$,

$$(\chi_1(p), \ldots, \chi_n(p))$$

are called the **coordinates of** p **w.r.t. the chart** (U,χ) and we also say that (U,χ) is a smooth chart around p.

For instance the change of coordinates (1.2.3) is about the chart

$$\chi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \chi(x,y) = (2x+y, x-y) \tag{1.2.5}$$

so that, in the new coordinates, a point $p = (x,y)$ will have the coordinates (w.r.t. χ)

$$u(x,y) = 2x+y, \quad v(x,y) = x-y.$$

Similarly for the polar coordinates where, computing the inverse of $(r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$, one finds the chart

$$\chi(x,y) = \left(\sqrt{x^2+y^2}, arctg(\frac{y}{x})\right).$$

1.2.5 Changing coordinates to make functions simpler (the immersion/submersion theorem)

In general, given a smooth function

$$f : \mathbb{R}^n \to \mathbb{R}^k$$

and a point $p \in \mathbb{R}^n$, whenever we have two new charts χ and χ' around p and $f(p)$, respectively, one can represent the function f using the new resulting coordinates: **the representation of f w.r.t. the charts χ and χ'** is

$$f_{\chi}^{\chi'} = \chi' \circ f \circ \chi^{-1}.$$

Of course, for the standard charts (the identity maps) one obtains back f. If just χ', or just χ, is the standard chart then we use the notations f_{χ} and $f^{\chi'}$, respectively.

For instance, for the function

$$f(x,y) = 5x^2 + 2xy + 2y^2,$$

with respect to the new chart (1.2.5) one obtains $f_{\chi}(u,v) = u^2 + v^2$.

In general, it is interesting to try to write smooth functions in the simplest possible way, modulo change of coordinates. The simplest types of functions for which this is possible are the most "non-singular" ones. More precisely, given

$$f : U \to \mathbb{R}^k$$

a smooth map defined on an open $U \subset \mathbb{R}^n$ and given $x \in U$, the "non-singular behaviour" that we require is that

$$(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$$

has maximal rank. It is interesting to consider the cases $n \geq k$ and $n \leq k$ separately. The first case brings us to the more canonical version of the implicit function theorem:

Theorem 1.32 (the submersion theorem). *Assume that f is a **submersion** at a given point $p \in U$ in the sense that $(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is surjective. Then there exists a smooth chart χ of $\mathbb{R}^n$ around p such that, around $\chi(p)$, $f_{\chi} = f \circ \chi^{-1}$ is given by*

$$f_{\chi}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k).$$

Proof. Since the matrix representing $(Df)_p$ is of maximal rank, one of its maximal minors (an $k \times k$ matrix) is invertible; we may assume that the invertible minor is precisely the one made of the last k rows (why?)- which is also the hypothesis of the implicit function theorem (Theorem 1.29). Looking at the proof of the theorem, one remarks that the desired chart is $\chi = \tilde{f}$. 😀

A similar argument gives rise to the following:

Theorem 1.33 (the immersion theorem). *Assume that f is an immersion at a given point $p \in U$ in the sense that $(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is injective. Then there exists a smooth chart χ' of $\mathbb{R}^k$ around $f(p)$ such that, in a neighborhood p, $f^{\chi'} = \chi' \circ f$ is given by*

$$f^{\chi'}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \underbrace{0, \ldots, 0}_{k-n \text{ zeros}}). \tag{1.2.6}$$

More precisely, denoting $q = f(p)$, there exist:

- *a smooth chart $\chi' : U'_q \to \Omega'_q$ of $\mathbb{R}^k$ around q,*
- *a neighborhood Ω_p of p in $\mathbb{R}^n$, inside the domain of f*

such that (1.2.6) holds on Ω_p. Furthermore, one may choose χ' and Ω_p so that:

$$f(\Omega_p) = \{u \in U'_q : \chi'_{L+1}(u) = \ldots = \chi'_k(u) = 0\}.$$

Proof. Let us give a proof that makes reference to $(Df)_p$ as a linear map and not as a matrix. Since $(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$ is injective, we find a second linear map $B : \mathbb{R}^{k-n} \to \mathbb{R}^k$ such that

$$((Df)_p, B) : \mathbb{R}^n \times \mathbb{R}^{k-n} \to \mathbb{R}^k$$

is an isomorphism. Consider then

$$h : U \times \mathbb{R}^{k-n} \to \mathbb{R}^k, \quad h(x_1, x_2) = f(x_1) + B(x_2).$$

We see that h satisfies the hypothesis of the inverse function theorem at the point $(x, 0)$. Hence it is a diffeomorphism around a neighborhood of $(x, 0)$. We denote by

$$\chi' : U' \to \Omega'$$

its inverse. Note that:

1. U' is an open neighborhood of $f(p)$ in $\mathbb{R}^k$.
2. Ω' is an open neighborhood of $(p, 0)$ in $\mathbb{R}^k$, contained in $U \times \mathbb{R}^{k-n}$.
3. the intersection of $\Omega' \subset \mathbb{R}^k$ with $\mathbb{R}^n \times \{0\}$,

   $$\Omega := \{u \in \mathbb{R}^n : (u, 0) \in \Omega'\},$$

   is an open neighborhood of p included in the domain of f.

Note that, since $\chi'(h(x_1, x_2) = 0$ for all $(x_1, x_2) \in \Omega'$ and $h(x, 0) = f(x)$, we have $\chi'(f(x)) = (x, 0)$ for all $x \in \Omega$. This proves the main part of the theorem; for the last part, note that we have, by the first part, that $f(\Omega)$ is inside the zero set of $\chi'_2 : U' \to \mathbb{R}^{k-n}$ (the second component of the chart χ' w.r.t. the decomposition $\mathbb{R}^k = \mathbb{R}^n \times \mathbb{R}^{k-n}$). For the reverse inclusion, let $x \in U'$ with $\chi'_2(x) = 0$; since $h \circ \chi' = \mathrm{id}$ on U', we obtain

$$x = h(\chi'(x)) = h(\chi'_1(x), 0) = f(\chi'_1(x))$$

where, for the last equality, we used the explicit formula for h. Moreover, since $\chi'(x) \in \Omega'$ and since $\chi'(x) = (\chi'_1(x), 0)$, by the definition of Ω, we have $\chi'_1(x) \in \Omega$. With the previous equality in mind, we obtain $x \in f(\Omega)$. 😀

For later use let us introduce the notion of smoothness defined on arbitrary subsets $M \subset \mathbb{R}^n$.

Definition 1.34. Given $M \subset \mathbb{R}^n$ and a function $f : M \to \mathbb{R}^k$, we say that f **is smooth around** $p \in M$ if, in a neighborhood U of p in M, $f|_U$ admits a smooth extension to an open inside $\mathbb{R}^n$ containing U. When this happens around all $p \in M$, we say that f is **smooth**.

A **diffeomorphism** between $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ is any bijection $f : M \to N$ with both f and f^{-1} smooth.

And here is a nice application of the existence of smooth partitions of unity.

Exercise 1.35. Show that if $M \subset \mathbb{R}^n$ is a closed subset that any smooth function $f : M \to \mathbb{R}^k$ admits a smooth extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^k$. (Hint: $\mathbb{R}^n \setminus M$ is open; get an open cover of $\mathbb{R}^n$ out of one of M.)

1.2.6 Embedded submanifolds of $\mathbb{R}^L$

We now move to the notion of (smooth) embedded submanifolds of $\mathbb{R}^L$ [1]. In low dimensions, these are curves (1-dimensional) and surfaces (2-dimensional); for an arbitrary dimension m we will be talking about m-dimensional submanifolds of $\mathbb{R}^L$. For instance, the standard sphere

$$S^m = \{(x_0, \ldots, x_m) \in \mathbb{R}^{m+1} : \sum_i (x_i)^2 = 1\} \quad (L = m+1)$$

will be such a smooth m-dimensional submanifold. Already when looking at the simplest examples one sees that such subspaces may (naturally) be described in several different (but equivalent) ways. E.g., already for the unit circle in the plane, one has the standard descriptions:

- implicit (by equations): $x^2 + y^2 = 1$.
- parametric: $x = \cos(t), y = \sin(t)$ with $t \in \mathbb{R}$.

Accordingly, the notion of submanifold of $\mathbb{R}^L$ can be introduced in several ways that look differently (but which turn out to be equivalent).

We start with the definition that can be seen as just a small variation on the notion of topological manifold from Definition 1.3 just that, for $M \subset \mathbb{R}^L$ the axioms (TM1), (TM2) are automatically satisfied, and one can can further take advantage of the Euclidean space to talk about *smoothness* of charts- as in Definition 1.34.

Definition 1.36. An m-dimensional embedded submanifold of $\mathbb{R}^L$ is any subset $M \subset \mathbb{R}^L$ which, for each $p \in M$, satisfies **the** (m-dimensional) **manifold condition at** p in the following sense: there exists a topological chart of M (Definition 1.3)

$$\chi : U \to \Omega$$

($U \subset M$ open neighborhood of p, Ω open in $\mathbb{R}^m$) which is also a diffeomorphism (i.e. χ and χ^{-1} are smooth in the sense of Definition 1.34). These will also be called **smooth (*m*-dimensional) charts** for M.

Of course, when $M = \mathbb{R}^L$, the resulting notion of "smooth chart for $\mathbb{R}^L$" coincides with the one already introduced in Definition 1.31. For general M, a particularly nice class of smooth charts of M are the ones that can be obtained by restricting such charts of $\mathbb{R}^L$. More precisely, given a subset $M \subset \mathbb{R}^L$, a smooth chart of $\mathbb{R}^L$

$$\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^L \tag{1.2.7}$$

said to be **adapted to** M if it takes $U := M \cap \widetilde{U}$ into $\Omega := \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\})$:

[1] here L is an integer, possibly large, that will denote the dimension of the Euclidean space inside which our manifolds $M \subset \mathbb{R}^L$; we use here the letter L not only to suggest that L may be possibly large w.r.t. the dimension of M, but also to emphasise that the role of the dimension L is very different than that of the dimension m of M

$$\widetilde{\chi}|_U : U \to \Omega. \tag{1.2.8}$$

Equivalently: inside $\widetilde{U} \subset \mathbb{R}^L$, the points that belong to M are characterised by the equations $\widetilde{\chi}_i = 0$ for $i > m$:

$$M \cap \widetilde{U} = \{q \in \widetilde{U} : \widetilde{\chi}_{m+1}(q) = \ldots = \widetilde{\chi}_L(q) = 0.\}$$

Note also that Ω may be, and will be, interpreted as an open in $\mathbb{R}^m$; in this way, any smooth chart (1.2.7) of $\mathbb{R}^L$ that is adapted to M induces a smooth chart (1.2.8) of M.

Not every smooth chart χ of M is induced by an adapted smooth chart $\widetilde{\chi}$ of $\mathbb{R}^L$. However:

Proposition 1.37. *For $M \subset \mathbb{R}^L$ and $p \in M$, the manifold condition for M at p is equivalent to the existence of a smooth chart of $\mathbb{R}^L$ around p, that is adapted to M.*

The proof will be done together with the proof of the following theorem. This theorem describes submanifolds parametrically (think of $x = \cos(t), y = \sin(t)$ for the circle) and by equations (think of $x^2 + y^2 = 1$ for the circle), taking care of the precise conditions.

Theorem 1.38. *Given a subset $M \subset \mathbb{R}^L$, $p \in M$, the following are equivalent:*

1. *M satisfies the m-dimensional manifold condition at p.*
2. *M admits an m-dimensional **parametrization around** p- by which we mean a homeomorphism*

$$par : \Omega \to U \subset M$$

between an open $\Omega \subset \mathbb{R}^m$ and an open neighborhood U of p in M satisfying the regularity condition that, as a map from Ω to $\mathbb{R}^L$, par is an immersion.

3. *M can be described by an m-dimensional **implicit equation around** p- by which we mean a submersion*

$$eq : \widetilde{U} \to \mathbb{R}^{L-m}$$

defined on an open neighborhood $\widetilde{U}$ of p in $\mathbb{R}^L$ and which describes M near p by the equation $eq = 0$:

$$M \cap \widetilde{U} = \{q \in \widetilde{U} : eq(q) = 0\}.$$

Proof. For keeping track of notations note that, throughout the proof, we look around the given point $p \in M \subset \mathbb{R}^L$ and around the corresponding point

$$x = \chi(p) = par^{-1}(p) \in \Omega \subset \mathbb{R}^m.$$

Therefore, we will deal with

- neighborhoods U_p of p in M, and $\widetilde{U}_p$ of p in $\mathbb{R}^L$.
- neighborhoods Ω_x of x in $\mathbb{R}^m$.

The points in the neighborhoods of p will be denoted by q, while the ones in the neighborhoods of x by y; for them, we may be looking at similar neighborhoods U_q, $\widetilde{U}_q$ and Ω_y.

We first prove that (1) implies (2). We start with the the chart $\chi : U \to \Omega \subset \mathbb{R}^m$ defined in a neighborhood U of p in M. Setting $par = \chi^{-1}$ we have to check that par is a homeomorphism -which is clear by construction (it has the continuous χ as inverse)- and that, as a map $\Omega \to \mathbb{R}^L$, it is an immersion. For the last part use that the composition

$$\Omega \xrightarrow{par} U \xrightarrow{\chi} \Omega$$

is the identity on Ω and then apply the chain rule to deduce that, for each point $y \in \Omega$, $(D\chi)_{par(y)} \circ (D\,par)_y$ is the identity- hence, in particular, $(D\,par)_y$ will be injective.

We now prove that (2) implies both (1) as well as (3). Hence we start with a parametrization $par : \Omega \to U \subset M$; as above, we set $\chi = par^{-1} : U \to \Omega$. To get (1), we still have to check that χ is smooth in the sense of Definition

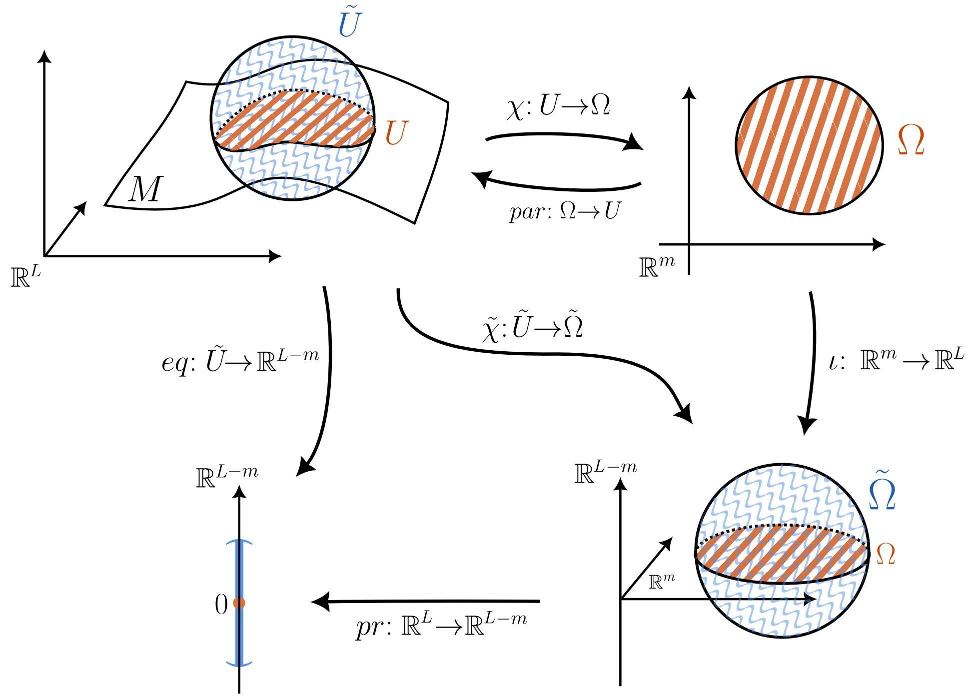


Fig. 1.5 The equivalent ways of phrasing the manifold condition in Theorem 1.38 involve (adapted) charts, parametrizations or implicit equations that are related as shown in this figure. Note that $U = \tilde{U}\cap M$ and $\Omega = \tilde{\Omega}\cap\mathbb{R}^m$. ι and pr are canonical inclusions and projections of the product space $\mathbb{R}^L = \mathbb{R}^m\times\mathbb{R}^{L-m}$ in the lower right. All these maps commute where they can be evaluated.

1.34: i.e., around any point $q\in U$, it is obtained by restricting a smooth map defined on an open $\widetilde{U}_q\subset\mathbb{R}^L$. For that we use the immersion theorem (Theorem 1.33) applied to $par:\Omega\to\mathbb{R}^L$ around

$$y=\chi(q)\in\Omega.$$

We find:

- an open neighborhood Ω_y of y in $\Omega\subset\mathbb{R}^m$
- a diffeomorphism $\widetilde{\chi}:\widetilde{U}_q\to\widetilde{\Omega}_q$ from an open neighborhood $\widetilde{U}_q\subset\mathbb{R}^L$ of p' to an open $\widetilde{\Omega}_q\subset\mathbb{R}^L$,

so that, on Ω_y, $\widetilde{\chi}\circ par$ becomes the inclusion on the first factors. We now write $\widetilde{\chi}=(\widetilde{\chi}_1,\widetilde{\chi}_2)$ where we use again the decomposition $\mathbb{R}^L=\mathbb{R}^m\times\mathbb{R}^{L-m}$. We deduce that $\widetilde{\chi}_1(par(z))=z$ for all $z\in\Omega_y$. Since $z=\chi(par(z))$ for all $z\in\Omega$, we deduce that $\widetilde{\chi}_1(r)=\chi(r')$ for all $r\in par(\Omega_q)$. In this way, on the neighborhood $par(\Omega_q)$ of q in U, χ is now the restriction of a smooth function defined on an open neighborhood of q in $\mathbb{R}^L$- namely $\widetilde{\chi}_1:\widetilde{U}_q\to\mathbb{R}^m$.

To prove (3) (still assuming (2)), we use $q=p$ in the previous reasoning and the resulting diffeomorphism $\widetilde{\chi}:\widetilde{U}_p\to\widetilde{\Omega}_p$; in principle, the desired function f will be $\widetilde{\chi}_2$, but we have to choose the domain of definition carefully. For that we use the last part of Theorem 1.33 which says that we may assume that

$$par(\Omega_x)=\{y\in\widetilde{U}_p:\widetilde{\chi}_{m+1}(y)=0,\ldots,\widetilde{\chi}_L(y)=0\}.$$

Since this is open in M, we can write it as $M\cap W_p$ for some open $W_p\subset\mathbb{R}^L$. Considering now

$$\widetilde{U}:=\widetilde{U}_p\cap W_p,\quad eq=\widetilde{\chi}_2|_{\widetilde{U}}:\widetilde{U}\to\mathbb{R}^{L-m}$$

one checks right away that $M \cap \widetilde{U}$ is the zero set of eq (why is eq a submersion?).

Exercise 1.39. Conclude now that $\widetilde{\chi}$ is actually an adapted chart.

We are now left with proving that (3) implies (1). Let $eq : \widetilde{U} \to \mathbb{R}^{L-m}$ satisfying the conditions from the hypothesis. Note that if we replace $\widetilde{U}$ by a smaller open neighborhood of p in $\mathbb{R}^L$ (and eq by its restriction), those conditions will still be satisfied. Therefore, using the submersion theorem applied to eq, we may assume that we also find a diffeomorphism $\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega}$ into an open subset of $\mathbb{R}^L$, such that $eq = \widetilde{\chi}_2$. This chart will then take the zero set of eq into the zero set of the second projection $\mathrm{pr}_2 : \widetilde{\Omega} \to \mathbb{R}^{L-m}$, i.e. into

$$\Omega := \{u \in \mathbb{R}^m : (u,0) \in \widetilde{\Omega}\}.$$

We deduce that the restriction of $\widetilde{\chi}$ to $U = M \cap \widetilde{U}$,

$$\chi := \widetilde{\chi}|_U : U \to \Omega \subset \mathbb{R}^m,$$

is a smooth chart of M (around p). 😀

Example 1.40. Returning to the circle S^1,

- $h(x,y) = x^2 + y^2 - 1$ serves as a (1-dimensional) implicit equation (around any point!)
- $p(t) = (\cos(t), \sin(t))$, when considered on sufficiently small intervals (on which it is injective) serves as paramatrization of S^1 around any point in S^1.
- as smooth (1-dimensional) charts one could use two projections $pr_1, pr_2 : S^1 \to \mathbb{R}$, restricted to the appropriate domains (so that they become homeomorphisms). Another possible choice of charts is given by the stereographic projections (see the lecture notes on Topology).

Exercise 1.41. Generalize this discussion to the spheres S^m of arbitrary dimension.

1.2.7 From directional derivatives to tangent spaces

The point of view provided by the directional derivatives brings us closer to the intrinsic nature of (p,v) when talking about $(Df)_p(v)$: that of tangent vector. The key point is that $(Df)_p(v)$ depends only on the behaviour of f near p, in "the direction of v"- and how we realize that "direction" is less important. This is best seen by looking at arbitrary paths through p with the original speed v, i.e. any smooth map

$$\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$$

(with $\varepsilon > 0$) satisfying

$$\gamma(0) = p, \quad \frac{d\gamma}{dt}(0) = v. \tag{1.2.9}$$

For instance, one could take $\gamma(t) = p + tv$, but the point is that the variation of $f(p+tv)$ at $t = 0$ does not depend on this specific choice of γ.

Lemma 1.42. *If f is differentiable at p then, for any path γ satisfying (1.2.9), one has*

$$(Df)_p(v) = \frac{\partial f}{\partial v}(p) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)).$$

In particular, if f is constant along such a path γ, then $(Df)_p(v) = 0$.

This point of view becomes extremely useful when looking at more general subspaces

$$M \subset \mathbb{R}^n.$$

Definition 1.43. Let $M \subset \mathbb{R}^n$ and consider a point $p \in M$. A **smooth curve in** M is any smooth map $\gamma : I \to \mathbb{R}^n$ defined on some interval $I \subset \mathbb{R}$, which takes values in M.

A **vector tangent to** M **at** p is any vector $v \in \mathbb{R}^n$ which can be realized as the speed at $t = 0$ of a smooth curve in M that passes through p at $t = 0$ (i.e. for which $0 \in I$ and $\gamma(0) = p$):

$$v = \frac{d\gamma}{dt}(0).$$

The set of such vectors is denoted by $T_p^{\text{geom}}M$; hence

$$T_p^{\text{geom}}M \subset \mathbb{R}^n.$$

Although we use the name "tangent *space*", in general (for completely random Ms inside $\mathbb{R}^n$), $T_p^{\text{geom}}M$ is just a subset of $\mathbb{R}^n$ (... but it is a vector subspace if M is "nice").

Exercise 1.44. Compute $T_p^{\text{geom}}M$ when:

1. $M \subset \mathbb{R}^2$ is the unit circle and $p = (1,0)$.
2. $M \subset \mathbb{R}^2$ is the union of the coordinate axes and $p = (0,0)$.

Exercise 1.45. Assume that $M \subset \mathbb{R}^n$ is defined by an equation $f(x) = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth function. For $p \in \mathbb{R}^n$ we denote by $\text{Ker}_p(Df)$ the kernel (= the zero set) of the differential $(Df)_p : \mathbb{R}^n \to \mathbb{R}^k$. Show that, in general,

$$T_p^{\text{geom}}(M_f) \subset \text{Ker}_p(Df),$$

but the inclusion may be strict. Then prove this inclusion becomes and equality when

$$f(x_1, \ldots, x_n) = (x_1)^2 + \ldots + (x_n)^2 - 1.$$

Exercise 1.46. With the notations from the previous exercise show that for all $p \in M_f$ at which f is a submersion

$$T_p^{\text{geom}}(M_f) = \text{Ker}_p(Df).$$

We now return to our discussion on differentials/directional derivatives, recast in terms of tangent spaces. Namely, Lemma 1.42 gives us right away:

Corollary 1.47. *Given $M \subset \mathbb{R}^n$, $p \in M$ and a function $\tilde{f} : \mathbb{R}^n \subset \mathbb{R}^k$ differentiable at p then, for any vector $v \in \mathbb{R}^n$ tangent to M at p, $(D\tilde{f})_p(v)$ depends only on $\tilde{f}|_M$.*

Of course, a similar conclusion holds slightly more generally, for any function $\tilde{f} : U \to \mathbb{R}^k$ defined on an open neighborhood $U \subset \mathbb{R}^n$ of p- the otcome being that $(D\tilde{f})_p(v)$ only depends on the values of $\tilde{f}|_M$ near p. This shows how to define the differential of a function $f : M \to \mathbb{R}^k$ which is differentiable at $p \in M$ in the sense of Definition 1.34: for $f : M \to \mathbb{R}^k$ that is differentiable at p, one has a well-defined differential

$$(Df)_p : T_p^{\text{geom}}M \to \mathbb{R}^k,$$

defined using an extension $\tilde{f}$ of f near p, but independent of the extension.

Exercise 1.48. Show that if $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^k$ and $f : M \to N$ is smooth at $p \in M$, then $(Df)_p$ takes values in $T_{f(p)}^{\text{geom}}N$. Then prove the chain rule in this context and deduce that, if f is a diffeomorphism, then $(Df)_p$ is a bijection between $T_p^{\text{geom}}M$ and $T_{f(p)}^{\text{geom}}N$.

Finally, let us look at tangent spaces of submanifolds of $\mathbb{R}^n$.[2]

[2] is this the right place?

Proposition 1.49. *If $M \subset \mathbb{R}^n$ is a m-dimensional embedded submanifold then, for any $p \in M$, the tangent space of M at p is an m-dimensional vector subspace of $\mathbb{R}^n$, which can also be described as follows:*

1. *as the kernel of $(D\,eq)_p : \mathbb{R}^n \to \mathbb{R}^{L-m}$, where $eq : \widetilde{U} \to \mathbb{R}^{L-m}$ is any implicit equation defining M around p.*
2. *as the image of $(D\,par)_p : T_p\Omega \to \mathbb{R}^n$, where $par : \Omega \to M$ is any parametrization of M around p.*

Proof. Exercise.

Exercise 1.50. Compute again the tangent spaces of the spheres, but applying now the previous proposition.

1.2.8 More exercises

Exercise 1.51. Consider two smooth functions

$$U \xrightarrow{f} U' \xrightarrow{g} \mathbb{R}^p,$$

defined on opens $U \subset \mathbb{R}^n$, $U' \subset \mathbb{R}^k$. Using the interpretation of linear maps as matrices (as made precise on page 23) show that the chain rule becomes:

$$\frac{\partial g \circ f}{\partial x_i}(x) = \sum_{j=1}^{n} \frac{\partial g}{\partial y_j}(f(p)) \frac{\partial f_j}{\partial x_i}(x)$$

for all $x \in U$ and $1 \leq i \leq n$.

Exercise 1.52. Show that for any function $g : \mathbb{R} \to \mathbb{R}$, the function

$$\tilde{g} : \mathbb{R}^n \to \mathbb{R}, \quad \tilde{g}(x_1, \ldots, x_n) = g((x_1)^2 + \ldots + (x_n)^2)$$

is not a submersion at $x = 0$.

Exercise 1.53. Assume that $f : U_0 \to \mathbb{R}^k$ is a smooth map, $U \subset \mathbb{R}^n$ open, $p \in U$. Let

$$\chi : U \to \Omega \subset \mathbb{R}^n, \quad \chi' : U' \to \Omega' \subset \mathbb{R}^k$$

be charts, of $\mathbb{R}^n$ around p and of $\mathbb{R}^k$ around $f(p)$, respectively. What is the (maximal) domain of definition of $f_{\chi}^{\chi'}$?

Exercise 1.54. Consider

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = 3 \cdot \sqrt[3]{x^2 + 2xy + 2y^2}$$

and look around $p = (1,0)$. Find a chart χ of $\mathbb{R}^2$ around p and a chart χ' of $\mathbb{R}$ around $f(p) = 3$ such that, w.r.t. these charts,

$$f_{\chi}^{\chi'}(u,v) = u^2 + v^2.$$

Exercise 1.55. Show that

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(t) = (\cos(t), \sin(t))$$

is an immersion at each point. Then, looking around $t = 0$, find a chart χ' of $\mathbb{R}^2$ around $f(0) = (1,0)$ such that, w.r.t. this chart,

$$f_{\chi'}(t) = (t, 0).$$

Exercise 1.56. Show that if $f : U \to \mathbb{R}^k$, $p \in U$ satisfy the conclusion of the submersion theorem, then f must be a submersion at p. Similarly for the immersion theorem.
(Hint: try it! If it really doesn't work, then look at the next exercise).

Exercise 1.57. Assume that $f : U \to \mathbb{R}^k$ is a smooth map, $U \subset \mathbb{R}^n$ open, $p \in U$. Let χ be a chart of $\mathbb{R}^n$ around p and let χ' be a chart of $\mathbb{R}^k$ around $f(p)$. Show that f is a submersion/immersion at p if and only if $f_{\chi}^{\chi'}$ is a submersion/immersion at $\chi(p)$.

Exercise 1.58. Consider the stereographic projection w.r.t. the north pole p_N, denoted

$$\chi_N : S^2 \setminus \{p_N\} \to \mathbb{R}^2$$

and similarly the one w.r.t. the south pole, denoted χ_S. Show that

$$\chi_S \circ \chi_N^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$$

is a diffeomorphism.

Exercise 1.59. For any $\varepsilon > 0$ describe a smooth function

$$f : \mathbb{R}^n \to [0, 1]$$

with the property that $f(0) > 0$ and whose support (in $\mathbb{R}^n$) is contained in the ball $\{x \in \mathbb{R}^n : ||x|| < \varepsilon\}$.

Exercise 1.60. Assume that $f : U \to U'$ and $g : U' \to U''$ are two smooth functions, with $U \subset \mathbb{R}^n$, $U' \subset \mathbb{R}^k$ and $U'' \subset \mathbb{R}^p$ opens. Show that if $g \circ f$ is a local diffeomorphism around a given point $x \in U$, then:

1. f is an immersion at x and g is a submersion at $f(x)$.
2. however, it may happen that f is not a submersion at x and g is not an immersion at $g(x)$ (describe an example!).
3. if, furthermore, f is a submersion at x or g is an immersion at $f(x)$, then both f and g are local diffeomorphisms (around x and $f(x)$, respectively).

Chapter 2
Smooth manifolds

2.1 Manifolds

2.1.1 Charts and smooth atlases

The difference between topological manifolds (see Definition 1.3) and smooth manifolds is, as the terminology suggests, that we assume smoothness for all the objects one considers (so that, on smooth manifolds, unlike for topological ones, we will be able to talk about speeds of curves, tangent vectors, differential forms, etc etc). For subspaces $M \subset \mathbb{R}^n$, making use of the ambient space $\mathbb{R}^n$, we managed to make sense of smoothness of various objects on M, such as charts- giving rise to the notion of smooth submanifold of $\mathbb{R}^n$ (as in Definition 1.36). However, in the general setting, there is no intrinsic way to make sense of smoothness just for a topological space M (not necessarily embedded into $\mathbb{R}^n$)- instead, we need extra-data on M that serves precisely that purpose. And that is the notion of smooth atlas that we start with here.

Recall from Definition 1.3 that, given a topological space M, an m-dimensional chart is a homeomorphism χ between an open U in M and an open subset $\chi(U)$ of $\mathbb{R}^m$,

$$\boxed{\chi : U \to \chi(U) \subset \mathbb{R}^m.} \qquad \text{(coordinate charts)}$$

We also say that (U, χ) is a chart for M, and we call U the domain of the chart. Given such a chart, each point $p \in U$ is determined/parametrized by its coordinates w.r.t. χ:

$$(\chi_1(p), \dots, \chi_m(p)) \in \mathbb{R}^m$$

(a more intuitive notation would be: $(x^1_\chi(p), \dots, x^m_\chi(p))$).

Given a second chart

$$\chi' : U' \to \chi'(U') \subset \mathbb{R}^m,$$

the map

$$\boxed{c^{\chi'}_{\chi} := \chi' \circ \chi^{-1} : \chi(U \cap U') \to \chi'(U \cap U')} \qquad \text{(coordinate changes)}$$

is a homeomorphism between two opens in $\mathbb{R}^m$. It will be called **the change of coordinates map** from the chart χ to the chart χ'. The terminology is motivated by the fact that, denoting

$$c^{\chi'}_{\chi} = (c_1, \dots, c_m),$$

the coordinates of a point $p \in U \cap U'$ w.r.t. χ' can be expressed in terms of those w.r.t. χ by:

$$\chi_i'(p) = c_i(\chi_1(p), \ldots, \chi_m(p)).$$

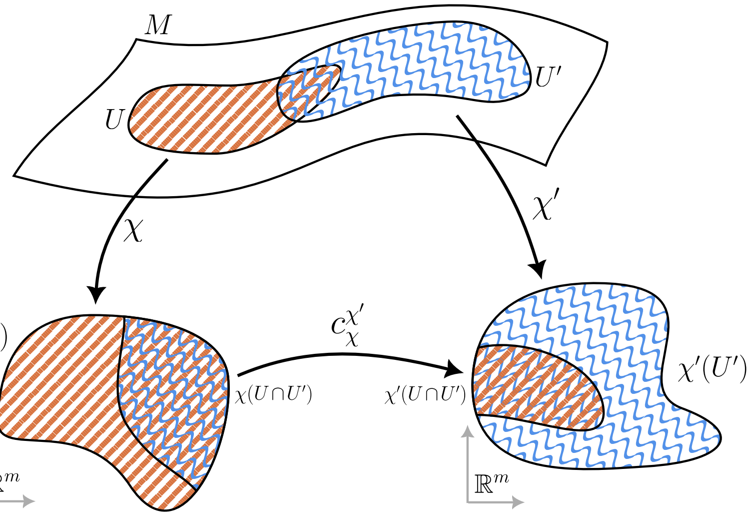


Fig. 2.1 Two charts $\chi : U \to \chi(U)$ and $\chi' : U' \to \chi'(U') \subseteq \mathbb{R}^m$ together with the change of coordinates map $c_\chi^{\chi'} : \chi(U \cap U') \to \chi'(U \cap U')$. These maps commute wherever they can be evaluated sensibly, i.e. $c_\chi^{\chi'} \circ \chi = \chi'$ holds on $U \cap U'$.

Definition 2.1. We say that two charts (U, χ) and (U', χ') are **smoothly compatible** if the change of coordinates map $c_\chi^{\chi'}$ (a map between two opens in $\mathbb{R}^m$) is a diffeomorphism.

Definition 2.2. A (m-dimensional) **smooth atlas** on a topological space M is a collection $\mathscr{A}$ of (m-dimensional) charts of M with the following properties:

1. the domains of the charts that belong to $\mathscr{A}$ cover M entirely.
2. each two charts from $\mathscr{A}$ are smoothly compatible.

Example 2.3. (Euclidean spaces) On $M = \mathbb{R}^m$ there are several interesting atlases. We mention here the extreme ones: the atlas $\mathscr{A}_{\mathbb{R}^m}$ consisting of only one chart, namely the identity chart $\mathrm{Id}_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m$, and $\mathscr{A}_{\mathbb{R}^m}^{\max}$ consisting of all smooth charts of $\mathbb{R}^m$ in the sense of Definition 1.31.

Example 2.4. (inside Euclidean spaces) For embedded m-dimensional submanifolds $M \subset \mathbb{R}^L$ (cf. Definition 1.36), there are two interesting atlases on M: the atlas $\mathscr{A}_M^{\max}$ consisting of all smooth m-dimensional charts of M in the sense of Definition 1.36, and the atlas $\mathscr{A}_M^{\text{adapt}}$ consisting of all charts that arise from smooth charts of $\mathbb{R}^L$ that are adapted to M (see the discussion following Definition 1.36).

The charts of an atlas are used to transfer notions and properties that involve smoothness from the Euclidean spaces (and opens inside) $\mathbb{R}^m$ to M; the compatibility of the charts ensures that the resulting notions (now on M) do not depend on the choice of the charts from the atlas. Hence one may say that an atlas on M allows us to

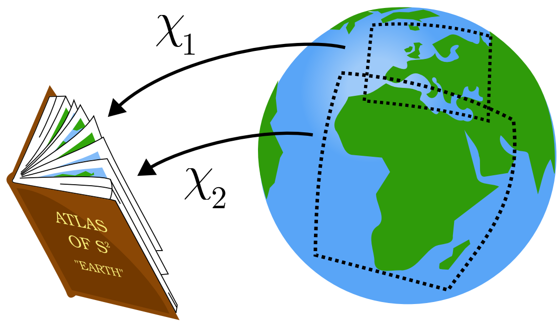


Fig. 2.2 An atlas of Earth is exactly a collection of charts that covers the surface of the globe. The same region might be included in several charts of a smooth atlas in a way that measured distances, angles and areas do not match - e.g. one chart might use the *Mercator projection* whereas another might faithfully depict areas. However, the smooth compatibility between charts does guarantee at least that a smooth path in one chart cannot develop any kinks in another.

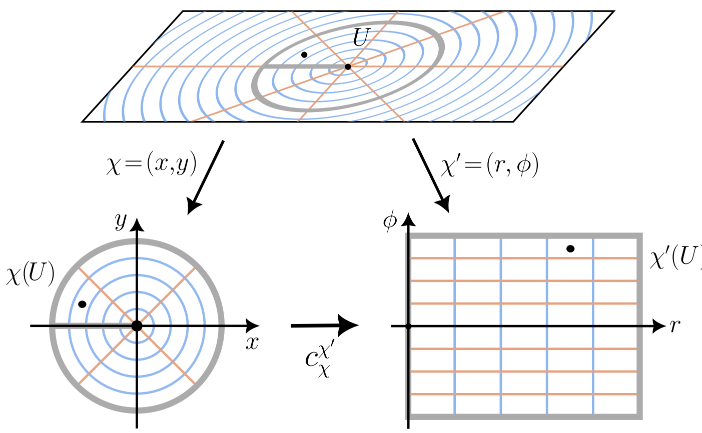


Fig. 2.3 Example of two smoothly compatible charts over the same open subset $U = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < 5\} \setminus (-\infty,0] \times \{0\}$ of the plane: The identity chart $\chi = (x,y)$ as well as polar coordinates $\chi' = (r,\phi)$. The change of coordinate map going from right to left, for example, can be visualized by first collapsing the left boundary of the rectangle into the origin and then glueing the two boundary components that touch the origin together on the negative x-axis.

put a "smooth structure" on M. For instance, given a smooth m-dimensional atlas $\mathscr{A}$ on the space M, a function $f : M \to \mathbb{R}$ is called **smooth w.r.t. the atlas** $\mathscr{A}$ if for any chart (U,χ) that belongs to $\mathscr{A}$,

$$f_\chi := f \circ \chi^{-1} : \chi(U) \to \mathbb{R}$$

is smooth in the usual sense ($\chi(U)$ is an open in $\mathbb{R}^m$!). We will temporarily denote by

$$\mathscr{C}^\infty(M,\mathscr{A}) \tag{2.1.1}$$

the set of such smooth functions. However, there is a little "problem": the fact that two different atlases may give rise to the same smooth functions.

Exercise 2.5. Returning to Example 2.3, show that

$$\mathscr{C}^\infty(\mathbb{R}^m,\mathscr{A}_{\mathbb{R}^m}) = \mathscr{C}^\infty(\mathbb{R}^m,\mathscr{A}^{\max}_{\mathbb{R}^m}).$$

2.1.2 Smooth structures

As we have pointed out, there is one aspect that requires a bit of attention: the fact that two different atlases may give rise to the same "smooth structure". One way to overcome this "problem" is by using smooth atlases that are maximal:

Definition 2.6. An m-dimensional **smooth structure** on a topological space M is an m-dimensional smooth atlas $\mathscr{A}$ on M which is maximal, i.e. with the property that there is no smooth atlas strictly containing $\mathscr{A}$.

Example 2.7. ((opens in) the Euclidean spaces) On $\mathbb{R}^m$ the collection of all its smooth charts in the sense of Definition 1.31,

$$\mathscr{A}^{\max}_{\mathbb{R}^m} := \{\chi : U \to \Omega_\chi \text{ diffeomorphisms between opens } U, \Omega_\chi \subset \mathbb{R}^m\},$$

is a maximal atlas and, therefore, it defines a smooth structure on $\mathbb{R}^m$. Similarly for opens $\Omega \subset \mathbb{R}^m$ (just restrict to charts with domain $U \subset \Omega$). These will be called

the standard smooth structure on the Euclidean space $\mathbb{R}^m$/ on the open $\Omega \subset \mathbb{R}^m$.

Unless otherwise stated, from now, on the Euclidean spaces $\mathbb{R}^m$ and opens inside them will always be endowed with this smooth structure.

Example 2.8. (submanifolds of the Euclidean spaces) Similarly, for an embedded submanifold $M \subset \mathbb{R}^L$ the collection of all smooth charts of M in the sense of Definition 1.36 form a smooth maxima atlas $\mathscr{A}^{\max}_M$ and therefore defines a smooth structure on M- called **the standard smooth structure on the embedded submanifold**.

Here is a more direct characterization of the maximality condition:

Exercise 2.9. Show that, given any smooth atlas $\mathscr{A}$ on any topological space M, one has:

$$(\mathscr{A} \text{ is maximal}) \iff \left(\begin{array}{c}\text{any topological chart of } M \text{ (see Def 1.3)}\\ \text{which is smoothly compatible with all the charts from } \mathscr{A}\\ \text{must belong to } \mathscr{A}\end{array}\right)$$

Actually, starting with an arbitrary (maximal or not) smooth atlas $\mathscr{A}$ on M, the collection of all charts of M that are compatible with all the charts that belong to $\mathscr{A}$,

$$\mathscr{A}^{\max} := \{\text{charts } \chi \text{ of } M : \chi \text{ is smoothly compatible with all } \chi' \in \mathscr{A}\},$$

is a new smooth atlas on M (exercise!), which is maximal (why?), and which contains $\mathscr{A}$ (why?). And the previous exercise says that $\mathscr{A}$ is maximal if and only of $\mathscr{A} = \mathscr{A}^{\max}$.

Definition 2.10. Given a smooth atlas $\mathscr{A}$ on M, **the smooth structure on M induced by the atlas $\mathscr{A}$** is the associated maximal atlas $\mathscr{A}^{\max}$.

Exercise 2.11. Show that for any smooth atlas $\mathscr{A}$ one has

$$\mathscr{C}^\infty(M,\mathscr{A}) = \mathscr{C}^\infty(M,\mathscr{A}^{\max}),$$

As we shall see a bit later (see Corollary 2.36 below), if $\mathscr{A}_1$ and $\mathscr{A}_2$ are two maximal smooth atlases, then

$$\mathscr{C}^\infty(M,\mathscr{A}_1) = \mathscr{C}^\infty(M,\mathscr{A}_2) \Longleftrightarrow \mathscr{A}_1 = \mathscr{A}_2. \tag{2.1.2}$$

One should keep in mind that very often a smooth structure is exhibited by describing a rather small atlas (ideally "the smallest possible one").

Example 2.12. (various atlases on Euclidean spaces) The standard smooth structure on $\mathbb{R}^m$ (see Example 2.7) can be induced by a very small atlas,

$$\mathscr{A}_{\mathbb{R}^m} := \{\mathrm{Id}_{\mathbb{R}^m}\},$$

i.e. the one consisting only of the identity chart

$$\chi = \mathrm{Id}_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m.$$

And a lot more is possible. E.g. here are two new smooth atlases:

$$\mathscr{A}_1 := \{\mathrm{Id}_U : U - \text{open in } \mathbb{R}^m\},$$

or $\mathscr{A}_2$ defined similarly, but using only open balls $B \subset \mathbb{R}^m$. Note that:

$$\mathscr{A}_{\mathbb{R}^m} \subset \mathscr{A}_1, \quad \mathscr{A}_{\mathbb{R}^m} \cap \mathscr{A}_2 = \{\emptyset\};$$

however, they all induce the same smooth structure on $\mathbb{R}^m$ (the standard one):

$$\mathscr{A}_1^{\max} = \mathscr{A}_2^{\max} = \mathscr{A}_{\mathbb{R}^m}^{\max}.$$

Example 2.13. Also for embedded submanifolds $M \subset \mathbb{R}^L$, there may be smaller and/or nicer atlases inducing the standard smooth structure on M described in Example 2.8. E.g., in full generality, one has the atlas $\mathscr{A}_M^{\mathrm{adapt}}$ arising from adapted charts of $\mathbb{R}^L$ (as in Example 2.4). But probably the nicest and most convincing example is provided by the stereographic projections for the spheres- see below.

A slightly different way of understanding smooth structures is via equivalence classes of atlases where, in principle, two atlases are equivalent if they induce the same smooth functions. Let us make this more precise.

Definition 2.14. We say that two smooth atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ are **smoothly equivalent** if any chart in $\mathscr{A}_1$ is smoothly compatible with any chart in $\mathscr{A}_2$ (or, shorter: if $\mathscr{A}_1 \cup \mathscr{A}_2$ is again a smooth atlas).

This defines an equivalence relation on the collection of all smooth atlases. From this point of view, the main property of maximal atlases is that: in each equivalence class one can find one, and only one, maximal atlas (so that maximal atlases are in 1-1 correspondence with equivalence classes of smooth atlases). This is made more precise in the following simple exercise:

Exercise 2.15. Show that, for any atlas $\mathscr{A}$, $\mathscr{A}$ is (smoothly) equivalent to $\mathscr{A}^{\max}$; then show that for two atlases $\mathscr{A}_1$ and $\mathscr{A}_2$, one has

$$\mathscr{A}_1 \text{ is smoothly equivalent to } \mathscr{A}_2 \Longleftrightarrow \mathscr{A}_1^{\max} = \mathscr{A}_2^{\max}.$$

We see that one obtains a 1-1 correspondence

$$\left\{\begin{array}{c}\text{smooth}\\ \text{structures}\\ \text{on } M\end{array}\right\} = \left\{\begin{array}{c}\text{maximal}\\ \text{atlases}\\ \text{on } M\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{c}\text{equivalence classes of}\\ \text{smooth atlases}\\ \text{on } M\end{array}\right\},$$

and, for this reason, some text-books introduce the notion of smooth structure as an equivalence class of smooth atlases. The bottom line is that

any atlas $\mathscr{A}$ on M induces a smooth structure on M

and, depending on the point of view on smooth structure that we adopt, the smooth structure associated to an atlas $\mathscr{A}$ is interpreted either as the maximal atlas $\mathscr{A}^{\max}$ associated to $\mathscr{A}$, or as the equivalence class $[\mathscr{A}]$, respectively. The use of maximal atlases is more "down to earth"- in the sense that it avoids the use of equivalence classes. However, as we have illustrated above (e.g. in Example 2.12), one should keep in mind that very often a smooth structure is exhibited by describing a rather small atlas (ideally "the smallest possible one").

2.1.3 Manifolds

We now come to the main objects of study of this course.

Definition 2.16. A **smooth m-dimensional manifold** is a Hausdorff, second countable topological space M together with am m-dimensional smooth structure on M.

Given a smooth m-dimensional manifold M, when saying that (U, χ) is a **chart of the smooth manifold** M we mean that (U, χ) belongs to the maximal atlas $\mathscr{A}$ defining the smooth structure on M.

Example 2.17. [Euclidean spaces and its embedded submanifolds] As a conclusion of the discussions from Example 2.7 and 2.8:

- $\mathbb{R}^m$ endowed with the standard smooth becomes an m-dimensional manifold; and its charts are precisely the classical smooth charts of $\mathbb{R}^m$ in the sense of Definition 1.31.
- similarly, any embedded submanifold $M \subset \mathbb{R}^L$ as above, endowed with its standard smooth structure, becomes an m-dimensional manifold whose charts are the smooth charts from Definition 1.36. [1].

Example 2.18. (Opens) Given an m-dimensional manifold M, any non-empty open $U \subset M$ carries a natural (induced) smooth structure that makes U itself into an m-dimensional manifold: the charts of U are, by definition, the charts of M whose domain are contained in U.

In particular, any open

$$\Omega \subset \mathbb{R}^m$$

comes with a standard smooth structure making it into an m-dimensional manifold. Note that, again, this smooth structure can be induced by a very small atlas, namely

$$\mathscr{A}_\Omega := \{\mathrm{Id}_\Omega\};$$

actually, these are all the possible manifolds for which the smooth structure can be induced by an atlas consisting of one chart only- see Exercise 2.48.

Exercise 2.19. Show that the unit circle S^1 (with the topology induced from $\mathbb{R}^2$) admits an atlas made of two charts, but does not admit an atlas made of a single chart.

Exercise 2.20. Describe a smooth structure on the torus such that the underlying topology is the usual (Euclidean) one (just intuitively, on the picture for now). How many charts do you need?

Can you do the same for the Moebius band?

---

[1] actually, one can prove that that is all there is- in the sense that any manifold M can be "embedded" in some Euclidean space. This is a very interesting result, but the conclusion may be a bit misleading and it is not so important as it may seem: the "general" theory frees the embedded submanifolds from the ambient spaces and describe the geometry of the space that is independent of the ambient space- shading light even on the notions that are described, originally, using those embeddings. Think e.g. of what happens in Topology, when compactness was originally described for subsets of $\mathbb{R}^L$ requiring them to be closed and bounded, and then turned out to be a completely topological property (with important consequences). Moreover, some manifolds just do not come with "natural embeddings" into some Euclidean spaces- see e.g. the projective spaces!

Exercise 2.21. (product of manifolds) Let M and N be two manifolds of dimensions m and n, respectively and we want to make $M \times N$ into a manifold of dimension $m+n$. For that, for any smooth charts $\chi : U \to \Omega \subset \mathbb{R}^m$ of M and $\chi' : U' \to \Omega' \subset \mathbb{R}^n$ of N we would like that their product

$$\chi \times \chi' : U \times U' \to \Omega \times \Omega' \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}, \quad (p,q) \mapsto (\chi(p), \chi'(q))$$

becomes a smooth chart for $M \times N$. Show that $M \times N$ carries a unique smooth structure satisfying this property. (this is called **the product smooth structure**).

Exercise 2.22. In the definition of smooth manifolds show that the condition that M is second countable is equivalent to the fact that the smooth structure on M can be defined by an atlas that is at most countable.

Remark 2.23 (Avoiding Topology?). In an attempt to avoid any reference to topology, one may give a definition of manifolds as a structure on a set M rather than a topological space M. One problem with such an approach is that its "directness" (avoiding topology) is only apparent, and the various topological notions that inevitably show up over and over again would look even more complicated and also rather mysterious (for instance, to hide the second countability axiom we would have to require countable atlases- as indicated by the previous exercise). Another problem with such an approach (even more important than the previous one) is the fact that the various manifolds that we encounter are, before anything, topological spaces in a very natural way. Not making any reference to such natural topologies would be artificial and less intuitive.

However, here is an exercise that indicates how one could (but should not) proceed.

Exercise 2.24. Let M be a set and let $\mathscr{A}$ be a collection of bijections

$$\chi : U \to \chi(U) \subset \mathbb{R}^m$$

between subsets $U \subset M$ and opens $\chi(U) \subset \mathbb{R}^m$. We look for topologies on M with the property that each $\chi \in \mathscr{A}$ becomes a homeomorphism. We call them topologies compatible with $\mathscr{A}$.

(i) If the domains of all the $\chi \in \mathscr{A}$ cover M, show that M admits at most one topology compatible with $\mathscr{A}$.
(ii) Assume that, furthermore, for any $\chi, \chi' \in \mathscr{A}$, $\chi' \circ \chi^{-1}$ (defined on $\chi(U \cap U')$, where U is the domain of χ and U' is of χ') is a homeomorphism between opens in $\mathbb{R}^m$. Show that M admits a topology compatible with $\mathscr{A}$.

(Hint: try to define a topology basis).

2.1.4 Variations

There are several rather obvious variations on the notion of smooth manifold. For instance, keeping in mind that

$$\text{smooth} = \text{of class } \mathscr{C}^\infty,$$

one can consider a $\mathscr{C}^k$-version of the previous definitions for any $1 \leq k \leq \infty$. E.g., instead of talking about smooth compatibility of two charts, one talks about $\mathscr{C}^k$-compatibility, which means that the change of coordinates is a $\mathscr{C}^k$-diffeomorphism. One arises at the notion of **manifold of class $\mathscr{C}^k$**, or **$\mathscr{C}^k$-manifold**. For $k = \infty$ we recover smooth manifolds, while for $k = 0$ we recover topological manifolds.

Yet another possibility is to require "more than smoothness"- e.g analyticity. That gives rise to the notion of **analytic manifold**. Looking at manifolds of dimension $m = 2n$, hence modeled by $\mathbb{R}^m = \mathscr{C}^n$, one can also restrict even further- to maps that are holomorphic. That gives rise to the notion of **complex manifold**. Etc.

Another possible variation is to change the "model space" $\mathbb{R}^m$. The simplest and most standard replacement is by the upper-half planes

$$\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m \geq 0\}.$$

This gives rise to the notion of **smooth manifold with boundary**: what changes in the previous definitions is the fact that the charts are homeomorphisms into opens inside $\mathbb{H}^m$.

For instance, the closed interval $[0,1]$ and the closed disk $D^2 \subset \mathbb{R}^2$ (with their standard Euclidean topology) can be made into manifolds with boundary.

Exercise 2.25. Go back to Exercise 2.20 and do again the second part.

Exercise 2.26. If M is an m-dimensional manifold with boundary:

1. while this may be clear intuitively, give a precise definition of "the boundary ∂M of M" (note that you cannot use the notion of boundary from Topology, as M is not part of a larger space).
2. prove that ∂M is a smooth $(m-1)$-dimensional manifold (without boundary).

Exercise 2.27. Return to products $M \times N$ of manifolds, as in Exercise 2.21. What if N is a manifold with boundary? And what if both M and N are manifolds with boundary?

2.2 Smooth maps

2.2.1 Smooth maps

Having introduced the main objects (manifolds), we now move to the maps between them. The idea will always be the same: use charts to move to Euclidean spaces, and use the standard notions there.

Definition 2.28. Let $f: M \to N$ be a map between two manifold M and N of dimensions m and n, respectively. Given charts (U, χ) and (U', χ') of M and N, respectively, the **representation of f with respect to χ and χ'** is

$$f_{\chi}^{\chi'} := \chi' \circ f \circ \chi^{-1}.$$

This map makes sense when applied to a point of type $\chi(p) \in \mathbb{R}^d$ with $p \in U$ with the property that $f(p) \in U'$, i.e. $p \in U$ and $p \in f^{-1}(U')$. Therefore, it is a map

$$f_{\chi}^{\chi'} : \mathrm{Domain}(f_{\chi}^{\chi'}) \to \mathbb{R}^n.$$

whose domain is the following open subset of $\mathbb{R}^m$:

$$\mathrm{Domain}(f_{\chi}^{\chi'}) = \chi(U \cap f^{-1}(U')) \subset \mathbb{R}^m.$$

Definition 2.29. Let M and N be two manifolds and

$$F: M \to N$$

a map between them. We say that F is **smooth** if its representation $F_{\chi}^{\chi'}$ with respect to any chart χ of M and χ' of N, is smooth (in the usual sense of Analysis).

Exercise 2.30. With the notation from the previous definitions, let $\mathscr{A}_M$ and $\mathscr{A}_N$ be two arbitrary (i.e. not necessarily maximal) atlases inducing the smooth structure on M and N, respectively. Show that, to check that F is smooth, it suffices to check that $F_{\chi}^{\chi'}$ is smooth for $\chi \in \mathscr{A}_M$ and $\chi' \in \mathscr{A}_N$.

Example 2.31. If $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ are embedded submanifolds endowed with their standard smooth structure (see Example 2.8) then the smoothness of a map $f: M \to M'$ in the sense discussed here is equivalent to the smoothness of f as a function between subsets of Euclidean spaces, i.e. in the sense of Section 1.2, Definition 1.34.

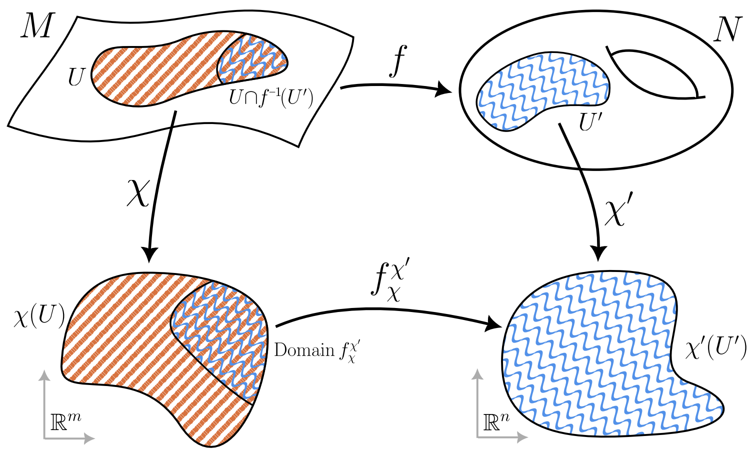


Fig. 2.4 A map $f : M \to N$ between an m- and n-dimensional manifold equipped with local charts χ and χ' defined on opens $U \subseteq M$ and $U' \subseteq N$, respectively, can locally be characterized by the representative $f^{\chi'}_{\chi}$: Domain $f^{\chi'}_{\chi} \to \mathbb{R}^n$. These maps commute wherever they can be evaluated sensibly, i.e. $\chi' \circ f = f^{\chi'}_{\chi} \circ \chi$ holds on $U \cap f^{-1}(U')$.

Proof. Let us call "A-smoothness" the notion from Section 1.2 of Chapter 1, and smoothness the one from this chapter. It is clear that the two coincide when M and N are open in their ambient Euclidean spaces.

Assume first that f is A-smooth. We will consider charts $\tilde{\chi} : \tilde{U} \to \Omega$ of $\mathbb{R}^{\tilde{m}}$ adapted to M (see Chapter 1, Proposition 1.37 from Section 1.2), and similarly $\tilde{\chi}' : \tilde{U}' \to \Omega'$ adapted to N; they induce charts $\chi := \tilde{\chi}|_{U \cap M}$ for M, and similarly χ' for N. To prove that f is smooth around a point $p \in M$ it suffices to show that for any $\tilde{\chi}$ around p and $\tilde{\chi}'$ around $f(p)$, as above, $f^{\chi'}_{\chi}$ is smooth. Since f is A-smooth, we may assume that $f|_U = \tilde{f}|_U$ with $\tilde{f} : \tilde{U} \to \tilde{U}'$ is smooth; in turn, $\tilde{f}$ induces $F := \tilde{f}^{\tilde{\chi}'}_{\tilde{\chi}} : \Omega \to \Omega'$, whose restriction to $\Omega \cap (\mathbb{R}^m \times \{0\} \subset \mathbb{R}^{\tilde{m}}$ is precisely $f^{\chi'}_{\chi}$ (and takes values in $\Omega' \cap (\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{\tilde{n}})$. Hence we are in the situation of having a smooth map $F : \Omega \to \Omega'$ between opens in $\mathbb{R}^{\tilde{m}}$ and $\mathbb{R}^{\tilde{n}}$, taking $\Omega_0 = \Omega \cap (\mathbb{R}^m \times \{0\}$ to $\Omega_0' = \Omega' \cap (\mathbb{R}^n \times \{0\}$ and we want to show that $F|_{\Omega_0} : \Omega_0 \to \Omega_0'$ is smooth as a map between opens in $\mathbb{R}^m$ and $\mathbb{R}^n$; that should be clear.

Assume now that f is smooth. Tho check that it is A-smooth, we fix $p \in M$ and consider charts $\tilde{\chi}$ and $\tilde{\chi}'$ and proceed as above and with the same notation. This time but we do not have $\tilde{f}$, but having it is equivalent to having F extending $F_0 = f^{\chi'}_{\chi}$. Then we are in a similar general situation as above, when we have a smooth map $F_0 : \Omega_0 \to \Omega_0'$ between opens in $\mathbb{R}^m$ and $\mathbb{R}^n$ and we want to extend it to a smooth map F between opens in $\mathbb{R}^{\tilde{m}}$ and $\mathbb{R}^{\tilde{n}}$, defined in a neighborhood of $\chi(p)$. Using the decomposition $\mathbb{R}^{\tilde{m}} = \mathbb{R}^m \times \mathbb{R}^{\tilde{m}-m}$ and similarly for $\mathbb{R}^{\tilde{n}}$, we just set $F(x, y) = (F_0(x), 0)$. 😀

Example 2.32. An extreme is when $M = I \subset \mathbb{R}$ is an open interval (which, according to our conventions, is always endowed with the standard smooth structure- see Example 2.7) and N is an arbitrary manifold. Then smooth maps

$$\gamma : I \to N$$

are called **(smooth) curves in** N. On I one can use the atlas consisting of the identity chart only, hence the smoothness of γ is checked using charts (U, χ) for M; the resulting representation of γ in the chart χ,

$$\gamma^{\chi} := \chi \circ \gamma : I_{\chi} \subset \mathbb{R}^m$$

(with domain $I_{\chi} = \gamma^{-1}(U) \subset I$) will then have to be smooth in the usual sense.

Example 2.33. In particular, when $N = \mathbb{R}^n$ (again with the standard smooth structure) , one can use the small atlas $\mathscr{A}_{\mathbb{R}^n}$ (Example 2.12). We see that, for a map

$$f : M \to \mathbb{R}^n,$$

the smoothness of f is checked by using charts (U, χ) for M and looking at the representation of f w.r.t. χ

$$f_\chi = f \circ \chi^{-1} : \chi(U) \to \mathbb{R}^n.$$

Exercise 2.34. Show that for any chart (U, χ) of a manifold M, $\chi : U \to \mathbb{R}^m$ is a smooth map (where U is endowed with a smooth structure as in Example 2.18.

2.2.2 Observables (or: $\mathscr{C}^\infty(M)$)

Of particular interest is the collection of real-valued smooth functions on M:

$$\mathscr{C}^\infty(M) := \{f : M \to \mathbb{R} : f - \text{smooth}\},$$

denoted $\mathscr{C}^\infty(M, \mathscr{A})$ in the previous sections (see (2.1.1). Its importance is perhaps not so predictable at this point; so let us mention that, in some sense, $\mathscr{C}^\infty(M)$ together with its algebraic structure (being able to take sums and products) encodes entirely the manifold M. On step in that direction is the claim made in (2.1.2), which we are able to prove now. Top that end, we need a lemma which is very important in various other places. Conceptually, this is what makes smoothness more flexible then analytic, holomorphic, or polynomial functions.

Lemma 2.35 (the fundamental property of smooth functions). *For any manifold M, for any $p \in M$ and any open neighborhood U of p, there exists $f \in \mathscr{C}^\infty(M)$ that is supported in U and such that $f(p) \neq 0$. Actually, one may arrange f so that $f = 1$ in a (small enough) neighborhood of p.*

Proof. Given our discussion from $\mathbb{R}^m$ (namely the proof of Theorem 1.27), there is very little that we have to do: we may assume that U is the domain of a coordinate chart $\chi : U \xrightarrow{\sim} \mathbb{R}^m$ (why?) sending p to $0 \in \mathbb{R}^m$, then choose a smooth function $f : \mathbb{R}^m \to [0,1]$ supported in the ball $B(0,1)$ and such that $f(0) \neq 0$ (as in the proof of Theorem 1.27), then move it to U via χ, i.e. consider $f \circ \chi : U \to [0,1]$; given the support property of f, it follows that if we extend f to M by declaring it to be zero outside U, we get a smooth functions $\tilde{f} : M \to [0,1]$ satisfying the desired property.

For the last part, we would need a function f on $\mathbb{R}^m$ as above, with the extra-property that $f = 1$ in a neighborhood of the origin. For that, one has to slightly improve the choice of g in the proof of Theorem 1.27: choose $g : \mathbb{R} \to \mathbb{R}$ is any smooth function that is 1 when $|t| < \frac{1}{3}$ and is 0 when $t \geq \frac{1}{2}$ and set $f(x) = g(||x||^2)$. 😀

Corollary 2.36. *Let M be a set. Assume that we have two topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ on M, a smooth structure $\mathscr{A}_1$ on the space $(M, \mathscr{T}_1)$ and, similarly, a smooth structure $\mathscr{A}_2$ on the space $(M, \mathscr{T}_2)$. Assume that the two smooth structures give rise to the same notions of smoothness of $\mathbb{R}$-valued functions, i.e., for arbitrary functions $f : M \to \mathbb{R}$, one has:*

$$f \text{ is smooth w.r.t. } \mathscr{A}_1 \iff f \text{ is smooth w.r.t. } \mathscr{A}_2.$$

Then $\mathscr{T}_1 = \mathscr{T}_2$ and $\mathscr{A}_1 = \mathscr{A}_2$.

Proof. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two maximal smooth atlases in M such that $\mathscr{C}^\infty(M, \mathscr{A}_1) = \mathscr{C}^\infty(M, \mathscr{A}_2)$. First we show that the topologies $\mathscr{T}_i$ underlying $\mathscr{A}_i$, $i \in \{1,2\}$, must coincide. Let $U \in \mathscr{T}_1$. The previous lemma shows that, for any $p \in U$ we find $f_p \in \mathbb{C}^\infty(M, \mathscr{A}_1)$ such that

$$V_p := \{q \in M : f_p(q) \neq 0\} \subset U.$$

and q belongs to V_p. Since $\mathbb{C}^\infty(M, \mathscr{A}_1) = \mathbb{C}^\infty(M, \mathscr{A}_2)$, f_p is smooth also w.r.t. $\mathscr{A}_2$ and, therefore, $V_p \in \mathscr{T}_2$. Since $U = \cup_{p \in U} V_p$, it follows that $U \in \mathscr{T}_2$ as well. Hence $\mathscr{T}_1 \subset \mathscr{T}_2$ and, similarly, the other inclusion holds as well. Hence $\mathscr{T}_1 = \mathscr{T}_2$.

We now show that $\mathscr{A}_1 \subset \mathscr{A}_2$. Let $\chi \in \mathscr{A}_1$, $\chi : U \to \Omega \subset \mathbb{R}^m$, with $U \in \mathscr{T}_1 = \mathscr{T}_2$. Then each component χ_k of χ is smooth w.r.t. χ; however, we cannot say that it belongs to $\mathscr{C}^\infty(M, \mathscr{A}_1)$ because it is defined only on U. Therefore, for any point $p \in U$ we apply the previous lemma to find $\eta_p \in \mathscr{C}^\infty(M, \mathscr{A}_1)$ such that $\eta = 1$ in a neighborhood $V_p \subset U$ of p, and is 0 outside the closure in M of a larger neighborhood $\widetilde{V}_p \subset U$. It follows then that each $\eta_p \cdot \chi_k$, extended to the entire M by declaring to be 0 outside U, belongs to $\mathscr{C}^\infty(M, \mathscr{A}_1)$, hence also to $\mathscr{C}^\infty(M, \mathscr{A}_2)$. In particular, $\chi_k|_{V_p} = (\eta_p \cdot \chi_k)|_{V_p}$ is smooth w.r.t. $\mathscr{A}_2$. Since each $p \in U$ admits a neighborhood $V_p \subset U$ on which χ_k is smooth (w.r.t. $\mathscr{A}_2$ now), χ_k is smooth as a function on U, and so is χ (again, w.r.t. $\mathscr{A}_2$). Therefore, for any $\chi' \in \mathscr{A}_2$, $\chi_{\chi'} = (\chi \circ \chi')^{-1}$ will be smooth; the fact that $\mathscr{A}_2$ is maximal implies now that, indeed, $\chi \in \mathscr{A}_2$. This proves the inclusion $\mathscr{A}_1 \subset \mathscr{A}_2$, end then it suffices to invoke the maximality of $\mathscr{A}_2$.

And here is another interesting consequence:

Exercise 2.37. Show that, for any manifold M, $\mathscr{C}^\infty(M)$ is point separating, i.e. for any $p, q \in M$ distinct, there exists $f \in \mathscr{C}^\infty(M)$ such that $f(p) = 0$, $f(q) = 1$.

The importance of Lemma 2.35 is best understood from the perspective offered by the general discussion on subspaces of $\mathscr{C}(M)$, as discussed in the reminder on Topology, subsection 1.1.8. To apply Theorem 1.16 to $\mathscr{A} = \mathscr{C}^\infty(M)$: one has to check some simple algebraic properties which were taken care of in Exercise **??** above, and then there was the last, and more subtle, condition in that theorem; and that is precisely what the first part of the previous lemma is taking care of! Therefore we deduce:

Theorem 2.38. *On any manifold M, for any open cover $\mathscr{U}$ of M, there exists a smooth partition of unity on M subordinated to $\mathscr{U}$.*

Diving a bit more into the details of subsection 1.1.8 one see that, from the various properties discussed there, the one that is most subtle is "paracompactness". But Theorem 1.18 immediately implies that any manifold is automatically paracompact. One the other hand, when it comes to properties of subspaces $\mathscr{A} \subset C^\infty(M)$ discussed in subsection 1.1.8, the one that is more subtle is "normality"; from this perspective, the previous lemma checks the criteria for normality provided by Theorem 1.19. Therefore one also obtains:

Corollary 2.39. *On any manifold M, for any two disjoint closed subset $A, B \subset M$, there exists a smooth function $f : M \to [0,1]$ with the property that $f|_A = 0$, $f|_B = 1$.*

Remark 2.40 (For the curious students: a smooth Gelfand-Naimark ...). The "Gelfand-Naimark message" from Topology is that, for reasonable topological space X, the topological information on X can be completely recovered from the $\mathscr{C}(X)$ and its algebraic structure (the sums and products that make it into an algebra). As recalled in Remark 1.10, the way one "reconstructs" the space X from the algebra $\mathscr{C}(X)$ is by associating to any algebra A a topological space $X(A)$, called the spectrum of A, and defined as the set of all characters $\chi : A \to \mathbb{R}$ (see Remark 1.10).

Since the notion of character makes sense for any algebra, we can apply it to

$$A := \mathscr{C}^\infty(M),$$

the algebra of smooth functions on a manifold M. As before, any point $p \in M$ gives rise to a character

$$\chi_p : A \to \mathbb{R}, \quad \chi_p(f) := f(p);$$

and this gives rise to a map

$$\mathrm{GN} : M \to X(A), \quad p \mapsto \chi_p.$$

Note that the previous exercise says precisely that this map is injective! What about surjectivity? I.e., is is true that any character

$$\chi : \mathscr{C}^\infty(M) \to \mathbb{R}$$

is of type χ_p for some $p \in M$?

Theorem 2.41. *If M is a compact manifold then the map GN is 1-1.*

Proof. We are left with proving surjectivity. Hence we start with a character χ and we look for $p \in M$ such that $\chi = \chi_p$. First remark that, for the last equality to hold (given p), it suffices to require something apparently weaker condition

$$\text{if } f \in \mathscr{C}^\infty(M) \text{ is killed by } \chi \text{ (i.e. } \chi(f) = 0) \Longrightarrow f(p) = 0.$$

Indeed, for an arbitrary $f \in \mathscr{C}^\infty(M)$, since $f - \chi(f) \cdot 1$ is anyway killed by χ, the previous implication would imply $\chi(f) = f(p)$ for all f, i.e. $\chi = \chi_p$.

Therefore, it suffices to show that there exists $p \in M$ for which the previous implication holds. We proceed by contradiction. If no such p exists then we find, for each p, a function f_p with

$$\chi(f_p) = 0, \quad f_p(p) \neq 0.$$

We may actually assume that $f_p(p) > 0$ (otherwise replace f_p by its square). Since f_p is smooth (hence also continuous), for each p we find a neighborhood U_p of p s.t. $f_p > 0$ on the entire U_p. Then $\{U_p : p \in M\}$ forms an open cover of M and, using the compactness of M, we cab extract a finite subcover- corresponding to a finite number of points $p_1, \ldots, p_k$. Then the sum f of the corresponding functions f_{p_i} will have the property that

$$f \in \mathscr{C}^\infty(M), \quad \chi(f) = 0, \quad f > 0 \text{ everywhere on } M.$$

But then we can write the constant function $1 = f \cdot \frac{1}{f}$, with both f and $\frac{1}{f}$, so that

$$\chi(1) = \chi(f) \cdot \chi(\frac{1}{f}) = 0,$$

which provides us with a contradiction we were looking for.

Exercise 2.42. Adapt the previous proof to show that, if M is not necessarily compact then, for any character χ on $\mathscr{C}^\infty(M)$, there exists $p \in M$ such that $\chi(f) = f(p)$ for all $f \in \mathscr{C}^\infty(M)$ that are compactly supported (i.e. vanish outside some compact).

Exercise 2.43. Given two manifolds M and N, any smooth $F : M \to N$ induces

$$F^* : \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(M),$$

which is a morphism of algebras (i.e. is linear, multiplicative and sends the unit to the unit). Prove that conversely, any morphism from the algebra $\mathscr{C}^\infty(N)$ to $\mathscr{C}^\infty(M)$ arises in this way.

Then deduce that, for any two manifolds M and N,

$$M \text{ and } N \text{ are diffeomorphic} \quad \Longleftrightarrow \quad \mathscr{C}^\infty(M) \text{ and } \mathscr{C}^\infty(N) \text{ are isomorphic as algebras}$$

2.2.3 Special maps: Diffeomorphisms, immersions, submersions

Like in the case of $\mathbb{R}^m$, there are certain types of smooth maps that deserve separate names. The first one describes the correct notion of "isomorphisms" in Differential Geometry (analogous to linear isomorphisms in Linear Algebra, isomorphisms of groups in Group Theory, homeomorphisms in Topology).

Definition 2.44. A **diffeomorphism** between two manifolds M and N is a map $f : M \to N$ with the property that f is bijective and both f and f^{-1} are smooth.

Two manifolds M and N are said to be **diffeomorphic** if such a diffeomorphism exists.

A diffeomorphism allows one to pass from whatever differential geometric object/property on M to N and backwards; for that reason, two manifolds that are diffeomorphic are usually thought of as being "(basically) the same as manifolds".

Exercise 2.45. As a continuation of Exercise 2.45, show that for any chart (U, χ) of a manifold M, χ is a diffeomorphism between the domain U and its image $\Omega \subset \mathbb{R}^m$.

Exercise 2.46. In general, $\mathbb{R}^m$ has many smooth structures, though many of them (but not all!) are actually diffeomorphic. This exercise takes care of the simpler parts of these assertions. For a homeomorphism $\chi : \mathbb{R}^m \to \mathbb{R}^m$, consider

$$\mathscr{A}_\chi = \{\chi\},$$

(the atlas on $\mathbb{R}^m$ consisting of one single chart, namely χ itself).

Show that, for a general homeomorphism χ, $\mathscr{A}_\chi$ defines a smooth structure different than the standard one, but $\mathbb{R}^m$ endowed with the resulting smooth structure is diffeomorphic to the standard one.

Remark 2.47 (For the interested student: exotic smooth structures). As the previous exercise shows, the interesting question for $\mathbb{R}^m$, and as a matter of fact for any topological space M, is:

how many non-diffeomorphic smooth structures does a topological space M admit?

Even for $M = \mathbb{R}^m$ this is a highly non-trivial question, despite the fact that the answer is deceivingly simple: each of the spaces $\mathbb{R}^m$ with $m \neq 4$ admits only one such smooth structure, while $\mathbb{R}^4$ admits an infinite number of them (called "exotic" smooth structures on $\mathbb{R}^4$)!

Exercise 2.48. Show that if M is a smooth manifold with the property that its smooth structure can be induced by an atlas consisting of only one chart, then M is diffeomorphic to an open subset $\Omega \subset \mathbb{R}^m$ (endowed with the standard smooth structure- cf. Example 2.18).

With the notion of diffeomorphism at hand, and the fact that opens inside manifolds are automatically manifolds (as discussed in Example 2.18), one can now make sense of a smooth map $f : M \to N$ being a **local diffeomorphism** around a point $p \in M$: there are open neighborhoods U of p in M, and V of $f(p)$ in N, such that f restricts to a diffeomorphism $f|_U : U \to V$.

Exercise 2.49. First, convince yourself that the map

$$\exp : \mathbb{R} \to S^1, \quad t \mapsto e^{it} = (\cos t, \sin t)$$

is a local homeomorphism around any point. Put now a smooth structure on S^1 such that exp is a local diffeomorphism. Can you find more than one? Compare it with the smooth structure you found when doing Exercise 2.19.

The following shows how to reduce checking that a map is a diffeomorphism to the easier task of checking that it is a local diffeomorphism.

Lemma 2.50. *A smooth function $f : M \to N$ is a diffeomorphism if and only if it is a local diffeomorphism around any point, as well as a bijection.*

Proof. The direct implication is obvious. For the reverse one, the bijectivity assumption allows us to talk about the inverse $g : N \to M$ of f. The question is whether f and g are smooth. But smoothness is a local condition, that can be checked around points. Since f is a local diffeomorphism, it follows that both f as well as g are smooth around any point in their domain. Hence they are both smooth and, therefore, f is a diffeomorphism.

Next, we export the notion of immersion and submersion form Euclidean spaces to the general setting; the definition below is not the best one, but has the advantage that it can be given right away, before discussing tangent spaces (however, we will return to it later on- see Proposition **??**).

Definition 2.51. Let $f : M \to N$ be a smooth map between two manifolds. We say that f is an **immersion/-submersion at** p if its local representations $f_\chi^{\chi'}$ is an immersion/submersion at $\chi(p)$ (in the usual sense from Analysis) for any chart χ of M around p and χ' of N around $f(p)$.

We say that f is an **immersion/submersion** if it is one at all $p \in M$.

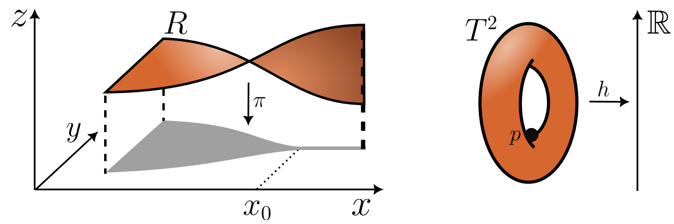


Fig. 2.5 Intuitively, an **immersion** at p is a map that locally around p does not crush the domain together, or in other words: wiggling p infinitesimally always corresponds to (first-order) movement of $f(p)$. Consider the map $\pi : R \to \mathbb{R}^3$ from the rotating ribbon R on the left that projects each point down to its shadow. It is an immersion at all points with $x < x_0$, but fails to be one as soon as the ribbon becomes exactly vertical for all $x \geq x_0$. A **submersion** at p is a map such that small wiggling away from p corresponds to movement of $f(p)$ in *all* directions in the codomain. Take the map $h : T^2 \to \mathbb{R}$ on the right that assigns to every point of the torus its height. At the marked point, it fails to be a submersion as any wiggling about p is horizontal, not changing the height to first order. Can you identify all the points where h is a submersion?

Exercise 2.52. Show that, in the previous definition, it is enough to check the required condition for (single!) one chart χ of M around p and one chart χ' of M' around $f(p)$.

Remark 2.53. As indicated above, once we will discuss that notion of tangent spaces, we will have another characterisation of the immersion and submersion conditions which does not make use of charts. That will provide another proof of, and actually extra-insight into, the previous exercise.

From the submersion and immersion theorems on Euclidean spaces, i.e. Theorem 1.32 and Theorem 1.33 from the previous chapter, we immediately deduce:

Theorem 2.54 (the submersion theorem). *If $f : M \to N$ is a smooth map between two manifolds which is a submersion at a point $p \in M$, then there exist charts χ of M around p and χ' of N around $f(p)$ such that $f_{\chi}^{\chi'} = \chi' \circ f \circ \chi^{-1}$ is given by*

$$f_{\chi}^{\chi'}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n).$$

Theorem 2.55 (the immersion theorem). *If $f : M \to N$ is a smooth map between two manifolds which is an immersion at a point $p \in M$, then there exist charts χ of M around p and χ' of N around $f(p)$ such that $f_{\chi}^{\chi'} = \chi' \circ f \circ \chi^{-1}$ is given by*

$$f_{\chi}^{\chi'}(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0).$$

Moreover, the charts $\chi : U \to \Omega \subset \mathbb{R}^m$ and $\chi' : U' \to \Omega' \subset \mathbb{R}^n$ can be chosen so that

$$f(U) = \{q \in U' : \chi'_{m+1}(q) = \ldots = \chi'_n(q) = 0\}.$$

Corollary 2.56. *Consider a smooth function $f : M \to N$, $p \in M$. Then the following are equivalent:*

(1) f is a local diffeomorphism around p.
(2) f is both a submersion as well as an immersion at p.
(3) f is a submersion at p and $\dim(M) = \dim(N)$.
(4) f is an immersion at p and $\dim(M) = \dim(N)$.

Proof. Using coordinate charts one may assume $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. The equivalence between (2), (3) and (4) is immediate from linear algebra: for a linear map $A : V \to W$ between two finite dimensional vector spaces, looking at the conditions: A is injective, A is surjective, $\dim(V) = \dim(W)$, any two implies the third. Since these actually imply that A is an isomorphism, making use also of the inverse function theorem, we obtain the equivalence with (1) as well. 😀

Exercise 2.57. Consider S^1 endowed with the smooth structure previously discussed (e.g. in Exercise 2.19) and $\mathbb{R}$ endowed with it canonical smooth structure. Prove that there are no submersions or immersions $f : S^1 \to \mathbb{R}$.

Exercise 2.58. Let M and N be two smooth manifolds, consider their product (with the product smooth structure as in Exercise 2.21). Show that the two projection maps $\mathrm{pr}_M : M \times N \to M$, $\mathrm{pr}_N : M \times N \to N$ are submersions.

Exercise 2.59. Let M be a manifold and consider the product $M \times M$ (with the product smooth structure as in Exercise 2.21). Show that the diagonal inclusion $\Delta : M \longrightarrow M \times M$, $\Delta(x) = (x,x)$, is an immersion.

2.3 Immersions and submanifolds

2.3.1 Embedded submanifolds; the regular value theorem

The characterization of embedded submanifolds of Euclidean spaces in terms of adapted charts (Proposition 1.37 in subsection 1.2.6) allows us to proceed more generally and talk about embedded submanifolds M of an arbitrary manifold N. Indeed, the notion of adapted chart that appears in Proposition 1.37 has an obvious generalization to this context:

Definition 2.60. Given an n-dimensional manifold N and a subset $M \subset N$, we say that M is an embedded m-dimensional submanifold of N if for any $p \in M$, there exists a chart

$$\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^n$$

for N around p with the property that

$$\widetilde{U} \cap M = \{p \in \widetilde{U} : \widetilde{\chi}_{m+1}(p) = \ldots = \widetilde{\chi}_n(p) = 0\} \tag{2.3.1}$$

or, equivalently,

$$\widetilde{\chi}(\widetilde{U} \cap M) = \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\}).$$

A chart $(\widetilde{U}, \widetilde{\chi})$ of N satisfying this equality is called **chart of N adapted to M**.

For any such chart one can talk about **the restriction of the adapted chart to** M and denoted

$$\widetilde{\chi}|_M : U \to \Omega.$$

Its domain is $U := \widetilde{U} \cap M$, its codomain is

$$\Omega := \widetilde{\Omega} \cap (\mathbb{R}^m \times \{0\}),$$

interpreted as an (open) subset of $\mathbb{R}^m$. With these we obtain an **induced smooth structure** on M.

Exercise 2.61. Check that, indeed, the collection of all charts of type $\widetilde{\chi}|_M$ obtained from charts $\widetilde{\chi}$ of N that are adapted to M, define a smooth structure on M.

Exercise 2.62. Assume that $M \subset N$ is an embedded submanifold and let P be another manifold. Show that:

1. a function $F : P \to M$ is smooth if and only if it is smooth as an N-valued function.

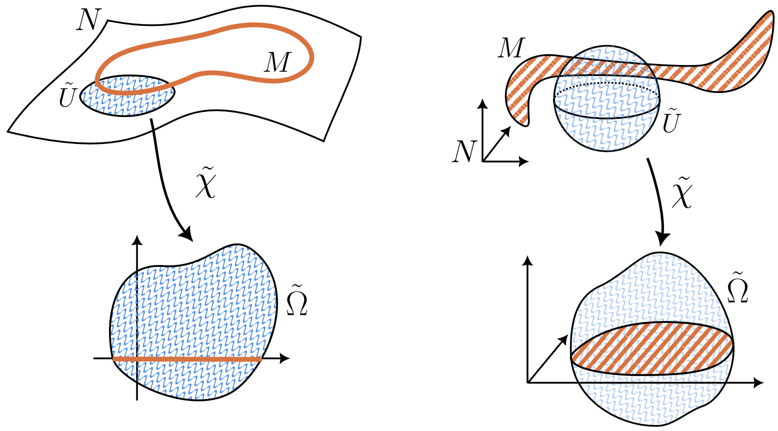


Fig. 2.6 Two examples of charts $\tilde{\chi} : \tilde{U} \to \tilde{\Omega}$ of an ambient manifold N adapted to an embedded submanifold M. The dimensions of the manifolds M, N are $(1,2)$ on the left hand side, and $(2,3)$ on the right hand side. Can you draw a picture for dimensions $(1,3)$?

2. if a function $F : M \to P$ admits a smooth extension to N, then F is smooth.
3. if M is closed in N, then the previous statement holds with "if and only if".

(Hint for the last point: use partitions of unity for N and don't forget that $N \setminus M$ is already open in N).

The notion of embedded submanifold allows us to introduce the smooth version of the notion of topological embedding. Recall that a map $F : M \to N$ between two topological spaces is a topological embedding if it is injective and, as a map from M to $F(M)$ (where the second space is now endowed with the topology induced from N), is a homeomorphism. The difference between the topological and the smooth case is that, while subspaces of a topological space inherit a natural induced topology, for smooth structures we need to restrict to embedded submanifolds.

Definition 2.63. A smooth map $F : M \to N$ between two manifolds M and N is called a **smooth embedding** if:

1. $F(M)$ is an embedded submanifold of N.
2. as a map from M to $F(M)$, F is a diffeomorphism.

Here is an useful criterion for smooth embeddings.

Theorem 2.64. *A map $F : M \to N$ between two manifolds is a smooth embedding if and only if it is both an immersion as well as a topological embedding.*

Proof. The direct implication should be clear, so we concentrate on the converse. We first show that $f(M)$ is an embedded submanifold. We check the required condition at an arbitrary point $q = F(p) \in N$, with $p \in M$. For that use the immersion theorem (Theorem 2.55) around p; we consider the resulting charts $\chi : U \to \Omega \subset \mathbb{R}^m$ for M around p and $\widetilde{\chi} : \widetilde{U} \to \widetilde{\Omega} \subset \mathbb{R}^n$ for N around q; in particular,

$$F(U) = \{q \in \widetilde{U} : \widetilde{\chi}_{m+1}(q) = \ldots = \widetilde{\chi}_n(q) = 0\}.$$

Since F is a topological embedding, $F(U)$ will be open in $F(M)$, i.e. of type

$$F(U) = F(M) \cap W$$

for some open neighborhood W of $q \in N$. It should be clear now that $\widehat{U} := \widetilde{U} \cap W$ and $\widehat{\chi} := \widetilde{\chi}|_{\widehat{U}}$ defines a chart for N adapted to M.

We still have to show that, as a map $F : M \to F(M)$, F is a diffeomorphism (where $F(M)$ is with the smooth structure induced from N). It should be clear that this map continues to be an immersion. Since M and $F(M)$ have the same dimension, it follows from Corollary 2.56 that $F : M \to F(M)$ is a diffeomorphism.

And here is a very useful consequence:

Corollary 2.65. *Let $F : M \to N$ be a smooth map between two manifolds M and N, with the domain M being compact. Then F is a smooth embedding if and only if it is an injective immersion.*

Proof. Use one of the main properties of compact spaces: injective maps from compacts to Hausdorff spaces are automatically topological embeddings!

The regular value theorem presented as Theorem 2.94 can now be generalized from Euclidean spaces to more general manifolds. The setting is as follows: we are looking at a smooth map

$$F : M \to N$$

and we are interested in its fiber above a point $q \in N$:

$$F^{-1}(q), \quad (\text{with } q \in N).$$

The condition that we need here is that q is a **regular value of** F in the sense that F is a submersion at all points $p \in F^{-1}(q)$.

Theorem 2.66 (the regular value theorem). *If $q \in N$ is a regular value of a smooth map*

$$F : M \to N,$$

then the fiber above q, $F^{-1}(q)$, is an embedded submanifold of M of dimension

$$dim(F^{-1}(q)) = dim(M) - dim(N).$$

Proof. In principle, this is just another face of the submersion theorem. Let $d = m - n$, where m and n are the dimension of M and N, respectively. We check the submanifold condition around an arbitrary point $p \in M_0 := f^{-1}(q)$. For that we apply the submersion theorem to f near p to find charts $\chi : U \to \Omega$ and $\chi' : U' \to \Omega'$ of M around p and of N around $F(p)$, respectively, such that $F(U) \subset U'$ and

$$F_{\chi}^{\chi'}(x) = (x_1, \ldots, x_n) \quad \text{for all } x \in \Omega.$$

After changing χ and χ' by a translation, we may assume that $\chi(p) = 0$ and $\chi'(q) = 0$. We claim that, up to a reindexing of the coordinates, χ is a chart of M adapted to M_0. Indeed, we have

$$\chi(U \cap M_0) = \{\chi(p') : p' \in U, F(p') = q\} = \{x \in \Omega : F_{\chi}^{\chi'}(x) = 0\},$$

or, using the form of $F_{\chi}^{\chi'}$,

$$\chi(U \cap M_0) = \{x = (x_1, \ldots, x_m) \in \Omega : x_1 = \ldots = x_n = 0\},$$

proving that M_0 is an $m - n$-dimensional embedded submanifold of M.

Finally, with smooth partitions of unity at hand one can use the same arguments as in the topological case and obtain the smooth version of Theorem 1.14 from Section 1.1 of Chapter 1:

Theorem 2.67. *Any (smooth) compact manifold can be smoothly embedded into some Euclidean space.*

Proof. We use the same map as in the proof of Theorem 1.14 from Section 1.1 of Chapter 1, just that we now make use of smooth partitions of unity (which exist by Theorem 2.38). It suffices to show that the resulting topological embedding

$$i = (\eta_1, \ldots, \eta_k, \tilde{\chi}_1, \ldots, \tilde{\chi}_k) : M \to \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{k \text{ times}} \times \underbrace{\mathbb{R}^d \times \ldots \times \mathbb{R}^d}_{k \text{ times}} = \mathbb{R}^{k(d+1)}.$$

is also an immersion. We check this at an arbitrary point $p \in M$. Since $\sum_i \eta_i = 1$, we find an i such that $\eta_i(p) \neq 0$. We may assume that $i = 1$. In particular, p must be in U_1, and then we can choose the chart χ_1 around p and look at the representation i_{χ_1} with respect to this chart; we will check that it is an immersion at $x := \chi_1(p)$. Hence assume that the differential of i_{χ_1} at x kills some vector $v \in \mathbb{R}^d$ (and we want to prove that $v = 0$). Then the differential of all the components of i_{χ_1} (taken at x) must kill v. But looking at those components, we remark that:

- the first $\mathbb{R}$-component is $\eta := \eta_1 \circ \chi_1^{-1} : U_1 \to \mathbb{R}$.
- the first $\mathbb{R}^d$ component is the linear map $f : U_1 \to \mathbb{R}^d, f(u) = \eta(u) \cdot u$.

Hence we must have in particular:

$$(d\eta)_x(v) = 0, \quad (df)_x(v) = 0.$$

From the formula of f we see that $(df)_x(v) = (d\eta)_x(v) \cdot x + \eta(x) \cdot v$; hence, using that previous equations we find that $\eta(x) \cdot v = 0$. But $\eta(x) = \eta_1(p)$ was assumed to be non-zero, hence $v = 0$ as desired. 😀

Exercise 2.68. Show that the map from Exercise 2.105 is an embedding of $\mathbb{P}^2$ in $\mathbb{R}^4$.

Recall that, while $S^1 \times S^1$ naturally embeds in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$, the interpretation of $S^1 \times S^1$ as a torus provides an even better embedding: inside $\mathbb{R}^3$. In the following you are asked to show that this can be propagated to higher dimensional tori. Here we use products of manifolds as discussed in Exercise 2.21

Exercise 2.69. Show that the n-dimensional torus,

$$T^n := \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$$

(sitting canonically inside $\mathbb{R}^{2n}$) can actually be embedded inside $\mathbb{R}^{n+1}$.

Exercise 2.70. With the same notations as in Exercise 2.110, show that each fiber of the Hopf map $h : S^3 \to S^2$ is an embedded submanifold of S^3 which is diffeomorphic to a circle.

Exercise 2.71. Consider the height function $f : T \to \mathbb{R}$ on the torus as indicated in Fig 2.7. At which points does f fail to be a submersion?

Exercise 2.72. Let M be a smooth m-dimensional manifold and let

$$f : M \longrightarrow \mathbb{R}$$

be a smooth function. For $\lambda \in \mathbb{R}$ define

$$M_\lambda = f^{-1}(\lambda), \quad M_{\leq \lambda} := \{p \in M : f(p) \leq \lambda\}.$$

Assume that λ is a regular value of f (so that, by the regular value theorem, M_λ is an $(m-1)$-dimensional manifold. Prove that $M_{\leq \lambda}$ is an m-dimensional manifold with boundary (see subsection **??**), with

$$\partial M_{\leq \lambda} = M_\lambda.$$

How many manifolds do you obtain in this way for the height function from the previous exercise?

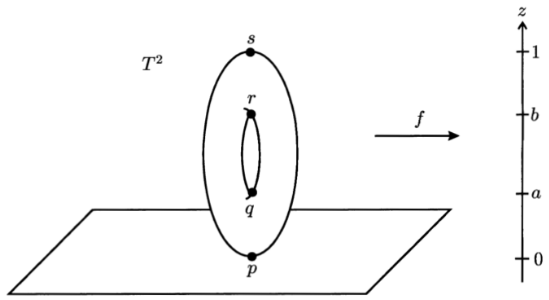


Fig. 2.7

Exercise 2.73 (part of the 2019/2020 exam). Consider

$$f : S^3 \to \mathbb{R}, \quad f(x,y,z,t) = x^2 + y^2 - z^2 - t^2.$$

(a) show that the zero-set $M_0 := f^{-1}(0)$ is an embedded submanifold of S^3.
(b) show that M_0 is diffeomorphic to the 2-torus.
(c) find all the points at which f is a submersion.

Exercise 2.74 (part of an exam from 2018; and you may want to skip (b) for now). Consider

$$M_4 := \{(x,y,z) \in \mathbb{R}^3 : x^4 + y^4 + z^4 = 1\} \subset \mathbb{R}^3,$$

and the map from M_4 to the 2-sphere S^2 given by

$$f : M_4 \to S^2, \quad f(x,y,z) = (x^2, y^2, z^2)$$

(a) Show that M_4 is a submanifold of $\mathbb{R}^3$ and f is a smooth map.
(b) Compute the tangent space of M_4 at the point $p = (\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, 0)$; more precisely, show that it is spanned by

$$\left(\frac{\partial}{\partial x}\right)_p - \left(\frac{\partial}{\partial y}\right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial z}\right)_p \in T_pM_4.$$

Similarly at the point $q = (\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$.
(c) Show that f is not an immersion at p, but it is a local diffeomorphism around q.
(d) Show that M_4 is not diffeomorphic to

$$M_3 := \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 1\} \subset \mathbb{R}^3.$$

but it is diffeomorphic to S^2.

2.3.2 General (immersed) submanifolds

In Topology (or, if you do not like that generality, just take $X = \mathbb{R}^n$) the leading principle for talking about "a subspace A of a space X" was to make sure that the inclusion $i : A \to X$ was continuous, and we chose "the best possible topology" on A doing that- and that gave rise to the induced (subspace) topology on any subset $A \subset X$. Looking for the analogous notion of "subspace" in the smooth context (i.e. a notion of "submanifold"), the first remark is that "inclusions" in the smooth context are expected to be immersions. Furthermore, starting with a subset M of a manifold N, there are several possible ways to proceed:

(1) try to make the space M (endowed with the topology induced from N) into a manifold such that the inclusion $i : M \hookrightarrow N$ is an immersion.
(2) forget about the induced topology on M and try to make the set M into a manifold such that the inclusion $i : M \hookrightarrow N$ is an immersion.
(yes, M will have a topology underlying the smooth structure, but it does not have to be the induced one).
(3) in the previous point choose "the best possible" manifold structure on M.

In (1), insisting that M has the endowed topology means that $i : M \hookrightarrow N$ is also an embedding (next to being an immersion) means that we are in the setting of Theorem 2.64. In other words, the previous section implemented precisely (1); gave rise to the notion of embedded submanifold.
The second possibility (i.e. item (2) above) gives rise to the notion of immersed submanifold.

Definition 2.75. Given a manifold N, an **immersed submanifold** of N is a subset $M \subset N$ together with a structure of smooth manifold on M, such that the inclusion $i : M \to N$ is an immersion.

We emphasize: an immersed submanifold is not just the subset $M \subset N$, but also the auxiliary data of a smooth structure on M (and, as the next exercise shows, given M, there may be several different smooth structures on M that make it into an immersed submanifold). Moreover, the required smooth structure on M induces, in particular, a topology on M; but we insist: this topology does not have to coincide with the one induced from N!

Exercise 2.76. Consider the figure eight in the plane $M = \mathbb{R}^2$. Show that it is not an embedded submanifold of $\mathbb{R}^2$, but it has at least two different smooth structures that make it into an immersed submanifold of $\mathbb{R}^2$.

Immersed submanifolds may look a bit strange/pathological, but they do arise naturally and one does have to deal with them. In general, any injective immersion

$$f : M \to N$$

gives rise to such an immersed submanifold: $f(M)$ together with the smooth structure obtained by transporting the smooth structure from M via the bijection $f : M \to f(M)$ (i.e. the charts of $f(M)$ are those of type $\chi \circ f^{-1}$ with χ a chart of M). And often immersed submanifolds do arise naturally in this way.

Example 2.77. For instance, the two immersed submanifold structures on the figure eight from the preceding exercise arise from two different injective immersions $f_1, f_2 : \mathbb{R} \to \mathbb{R}^2$ (with the same image: the figure eight), as indicated in the picture. Make pictures!

So: what happens for an embedded submanifold $M \subset N$? Endowed with "the smooth structure induced from N" (notion that only makes sense when M is embedded submanifold!) it clearly becomes an immersed submanifold. Can it be still given another smooth structure (as in the last example)? The answer is: no! Indeed, one has the following result which is a simplified version of Proposition 2.80 below.

Proposition 2.78. *If M is an embedded submanifold of the manifold N, then the set M admits precisely one smooth structure such that the inclusion into M becomes an immersion.*

To see the difference between "immersed" and "embedded" one just has to stare at the meaning of "immersions". If M (... together with a smooth structure on it) is an immersed submanifold of N, the immersion theorem tells us that there exist charts (U,χ) of N around arbitrary points $p \in M$ such that $U_M := M \cap \{q \in U : \chi_{n+1}(q) = \ldots = \chi_m(q) = 0\}$ is open in M and $\chi_M = \chi|_{U_M}$ is a chart for M. However, when saying that "U_M is open in M", we are making use of the topology on M induced by the smooth structure we considered on M, and not with respect to the topology induced from N (look again at the previous exercise!). This is the difference with the notion of "embedded submanifold".

Moving now to item (3), the best possible scenario would be when the subset $M \subset N$ has the following property: M admits a unique smooth structure that makes it into an immersed submanifold of N. When this happens we say that M has **the unique smooth structure property**. In this case there is no ambiguity what smooth structure we put on M, M becomes an immersed submanifold, and we are looking at those immersed submanifolds that are encoded just in the subset $M \subset N$ and no extra-data prescribed beforehand. There are two remarks here:

- but, again, even in this case, the underlying topological structure need not be the induced topology from N.
- on the other hand, embedded submanifolds do have this property (this is implied by the next result).

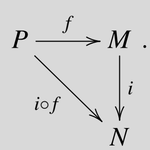
Of course, there are other examples of submanifolds with unique smooth structure besides the embedded ones: e.g. the "dutch figure eight" (exercise); for one more see Example 2.81 below.

However, it turns out that there is a smaller and more interesting class of submanifolds with unique smooth structures (and still includes the embedded submanifolds, as well as most of the other interesting examples):

Definition 2.79. An immersed submanifold M of a manifold N is called **an initial submanifold** if the following condition holds: *for any other manifold P and any map $f : P \to M$, f is smooth if and only if it is smooth as a map with values in N*. In other words,

$$f \text{ is smooth} \Longleftrightarrow i \circ f \text{ is smoooth},$$

where $i : M \hookrightarrow N$ is the inclusion; here one may want to think on the diagram:



What happens is that, while this condition is easier to check than the uniques smooth structure property, one has:

Proposition 2.80. *For a subset M of a manifold N one has the following:*

embedded submanifold $\Longrightarrow$ initial submanifold $\Longrightarrow$ the unique smooth structure property.

Proof. Consider the inclusion $i : M \hookrightarrow N$. For the first implication the main point is to show that if M is embedded and $f : P \to M$ has the property that $i \circ f : P \to N$ is smooth, then f is smooth. To check that f is smooth, we we use arbitrary charts χ of P and charts χ' of N adapted to M; we also use $\chi'_M = \chi'|_M$. Note that

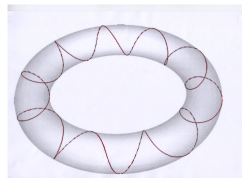
$$(i \circ f)_{\chi}^{\chi'} = f_{\chi}^{\chi'_M} : \Omega \to \mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n,$$

for some open $\Omega \subset \mathbb{R}^p$. Hence it suffices to remark that if a function $g : \Omega \to \mathbb{R}^m$ is smooth as a map with values in the larger space $\mathbb{R}^n$, then it is smooth.

For the second part assume that M is initial. In particular, it comes with a smooth structure defined by some maximal atlas $\mathscr{A}$, making the inclusion into N an immersion. To prove the uniqueness property, we assume that we have a second smooth structure with the same property, with associated maximal atlas denoted $\mathscr{A}'$. Applying the initial condition to $P = (M, \mathscr{A}')$ with f being the inclusion into N, we find that the identity map $\mathrm{id} : (M, \mathscr{A}') \to (M, \mathscr{A})$ is smooth. Applying the definition of smoothness, we see that any chart in $\mathscr{A}'$ must be compatible with any chart in $\mathscr{A}$. Therefore $\mathscr{A}' = \mathscr{A}$.

Example 2.81. When N is the 2-torus and one draws a curve in N winding around, there are two interesting cases:

- the curve closes up after a few turnings. That give a subset $M_0 \subset N$.
- the curve keeps on winding around infinitely, densely inside the torus. That gives another $M_1 \subset N$.



The picture shows M_0. Draw a picture for M_1! Note that the two types of curves are not even that far apart: any one that closes up, if perturbed a little, will become of the second type. However, M_0 and M_1 are quite different:

- M_0 is embedded in the torus.
- M_1 is only immersed.

For the last point, there is a map which should be clear on the picture $f : \mathbb{R} \to N$ with image precisely M_1 and which is an immersion. Therefore making M_1 into an immersed submanifold. However, we are in the better case: M_1 is an initial submanifold (... exercise!).

Exercise 2.82. In the previous example show that M_0 is diffeomorphic to S^1 while M_1 is diffeomorphic to $\mathbb{R}$.

Using arguments similar to those from Proposition 2.80 you can try the following:

Exercise 2.83. Let M be a subset of the manifold N. Then:

(i) For any topology on M, the resulting space M can be given at most one smooth structure that makes it into an immersed submanifold of N.

(ii) If we endow M with the induced topology, then the space M can be given a smooth structure that makes it into an immersed submanifold of N if and only if M is an embedded submanifold of N; moreover, in this case the smooth structure on M is unique.

2.4 Submersions and quotients

2.4.1 Quotient smooth structures: uniqueness

Another interesting corollary of the submersion theorem is that, for submersions $f : M \to N$, smooth maps on N can be detected by looking up on M (see below for the precise statement). That is interesting because, in practice, whenever we have a set N that we would like to make it into a a manifold in a "satisfactory way", very often N is closely related to an actual manifold M, through a surjective map $f : M \to N$. In such situations, the "satisfactory way" often means: such that f is a submersion.

Corollary 2.84. *If $\pi : M \to N$ is a surjective submersion then, for functions $f : N \to \mathbb{R}$, one has:*

$$f \in C^\infty(N) \Longleftrightarrow f \circ \pi \in C^\infty(M).$$

In particular, if $\pi : M \to N$ is a surjective map from a manifold M to a set N, there is at most one way to make N into a manifold such that $\pi : M \to N$ is a submersion.

Proof. The direct implication is clear, hence we concentrate on the reverse one. We will make use of the fact that smoothness of a map is a local property: for our f, it suffices to show that for any point $y \in N$, there exists a neighborhood V_y of y such that $f|_{V_y}$ is smooth. Fixing $y \in N$, we choose the coordinate charts around x and y that follow from the submersion theorem, and we set V_y to be the domain of the chart around y. Moving by those charts to opens inside Euclidean spaces, we may assume that π is the projection onto the first n coordinates-situation in which the fact that $f \circ \pi$-smooth implies that f is smooth is clear. The (pedantic) details are left as an exercise.

For the last part, one just invokes Corollary 2.36.

Exercise 2.85. Show that having a submersion $\pi : M \to N$ also forces the topology on N to coincide with the so-called **quotient topology** induced by π on N, defined by:

$$V - \text{open in } N \iff \pi^{-1}(V) - \text{open in } M$$

(which arises when looking for topologies on N making π continuous, and choosing the largest one among those; this was discussed also in the introductory course in Topology).

2.4.2 The case of group actions

Next, we look at actions of groups on manifolds. You have probably seen the notion of group actions (and quotients) in the Group Theory and Introduction to Topology courses, hence it is not a surprise that they will show up in differential geometry as soon as we start looking at the first examples (next chapter). So, let us discuss group actions in the realm of differential geometry, where sets or topological spaces are replaced by manifolds.

Let M be a manifold. We denote by $\text{Diff}(M)$ the set of all diffeomorphisms from M to M. Together with composition of maps, this is a group. Let Γ be another group, whose operation is denoted multiplicatively. An **action** of Γ on a manifold M is a group homomorphism

$$\phi : \Gamma \to \text{Diff}(M), \ \gamma \mapsto \phi_\gamma.$$

Hence, for each $\gamma \in \Gamma$, one has a diffeomorphism ϕ_γ of M ("the action of γ on M"), so that

$$\phi_{\gamma\gamma'} = \phi_\gamma \circ \phi_{\gamma'} \ \ \forall \, \gamma, \gamma' \in M.$$

Sometimes $\phi_\gamma(x)$ is also denoted $\gamma(x)$, or simply $\gamma \cdot x$. In other words, one encodes/think of an action as a map

$$\Gamma \times M \to M, \ (\gamma, x) \to \gamma \cdot x.$$

Exercise 2.86. Show that the fact that each ϕ_γ is a diffeomorphism is equivalent to the fact that the map $(\gamma, x) \to \gamma \cdot x$ is smooth, where $\Gamma \times M \to M$ is endowed with the smooth structure defined by charts of type $\{\gamma\} \times U \to \mathbb{R}^m$, $(\gamma, x) \mapsto \chi(x)$, one for each $\gamma \in \Gamma$ and each chart (U, χ) of M.

Given an action of Γ on M, the **Γ-orbit through** x is defined as $(\Gamma \cdot x) := \{\gamma \cdot x : \gamma \in \Gamma\}$, and the collection of all orbits forms the so-called **orbit space**

$$M/\Gamma := \{(\Gamma \cdot x) : x \in M\}.$$

Hence a single point of M/Γ is an entire orbit. Such points arise from points in M via the **canonical projection**

$$\pi_{\text{can}} : M \to M/\Gamma, \quad x \mapsto (\Gamma \cdot x),$$

just that two different points of M may give the same point in the orbit space; more precisely, one has:

$$(\Gamma \cdot x) = (\Gamma \cdot y) \iff \exists \, \gamma \in \Gamma \text{ s.t. } y = \gamma \cdot x. \tag{2.4.1}$$

In what follows, an action of a group Γ on M is said to be **free** if, for $\gamma \in \Gamma$ and $x \in M$, the equality $\gamma \cdot x = x$ can occur only when $\gamma = e$ is the identity element of Γ.

Theorem 2.87. *If Γ is a finite group acting freely on a manifold M, then the set M/Γ can be made into a smooth manifold in precisely one way such that the canonical projection $\pi_{can} : M \to M/\Gamma$ is a local diffeomorphism.*

Remark 2.88. M/Γ always carries a "natural topology", without any assumption on the group or on the action. It arises as "the best" topology on M/Γ that turns π_{can} into a continuous map, and it is precisely the quotient topology induced by π_{can} mentioned in Exercise 2.85.

In the smooth context however, M/Γ may fail to carry a similar "best smooth structure" (meaning, of course, that we require π to be a submersion). Corollary 2.84 shows that, if it exists, then it is unique, but the problem is with the existence. This problem arises already in simple situations such as $\Gamma = \mathbb{Z}_2$ (... finite!) acting on $\mathbb{R}$ by $\widehat{k} \cdot r = (-1)^k r$; we leave this as an exercise! 😰

Remark 2.89. Some of the discussion here may remind you of the introductory course on topology, where equivalence relations are discussed. If that is the case, you will notice that one can look at the entire discussion a bit differently: the very last condition in (2.4.1) defines an equivalence relation R_Γ between the points of M, the Γ-orbits are precisely R_Γ-equivalence classes and the orbit space M/Γ is precisely the quotient w.r.t. this equivalence relation. On the other hand, finding conditions on more general equivalence relations R on manifolds M to ensure that the quotient M/R can be made into a manifold into a satisfactory fashion (like the last theorem does for group actions) is more difficult.

Proof. It is interesting to point out the following consequences to the condition that Γ is finite s that the action is free, and that Γ is finite, respectively:

1. if the orbits through $x, y \in M$ do not coincide, then there exist neighborhoods U of x and V of y such that $U \cap \gamma \cdot V = \emptyset$ for all $\gamma \in \Gamma$.
2. if the action is also free then, around any $x \in M$ there exists an open neighborhood W which is Γ-small in the sense that $W \cap \gamma \cdot W = \emptyset$ for all $\gamma \in \Gamma$ distinct from the unit element e.

The first item is proven/used in Topology, when proving the quotients modulo finite groups are Hausdorff; briefly, it goes as follow: the orbits must be disjoint, hence $x \neq \gamma \cdot y$ for any γ; using that M is Hausdorff one finds neighborhoods U_γ of x and V_γ of y such that $U_\gamma \cap \gamma \cdot V_\gamma = \emptyset$; finally, one sets $U = \cap_\gamma U_\gamma$, and similarly for V, which, since Γ is finite, is a finite intersection of opens hence open. This proves the existence of the desired U and V. Notice that, while the condition on x and y is precisely that they induce two distinct points in M/Γ, the resulting U and V give rise to two disjoint subsets $\pi(U)$, $\pi(V)$. Since when endowing M/Γ with the quotient topology, $\pi_{\text{can}} : M \to M/\Gamma$ becomes an open map- i.e. sends opens to opens (why?), $\pi(U)$ and $\pi(V)$ will be disjoint opens and, therefore, M/Γ is Hausdorff.

For the second item we proceed similarly, but using that $x \neq \gamma \cdot x$ for all $\gamma \neq e$ (freeness). For each such γ we find neighborhoods W_γ and W'_γ of x such that $W_\gamma \cap \gamma \cdot W'_\gamma = \emptyset$ and consider

$$W = \left(\cap_{\gamma \neq e} W_\gamma\right) \cap \left(\cap_{\gamma \neq e} W'_\gamma\right).$$

It contains x, it is open (as a finite intersection of opens) and is Γ-small. Notice that being Γ-small is equivalent to the fact that $\pi|_W : W \to \pi(W)$ being a bijection and then, since π is continuous and open, to $\pi|_W$ being a local homeomorphism.

The second item can be used to exhibit the desired smooth structure on M/Γ. To that end, we consider charts (W, χ) on M that are Γ-small in the sense that W is. Item 2. implies that, around any point, one does find a Γ-small chart. Combining χ with the inverse of $\pi|_W$, one obtains a homeomorphism

$$\chi_0 : \pi(W) \to \chi(W) \subset \mathbb{R}^m.$$

We claim that these fit into a smooth atlas for M/Γ. So, let us fix two Γ-small charts on M, and prove that the induced charts (W, χ) and (W', χ') on M/Γ are smoothly compatible. We have to investigate the smoothness of

$$\chi_0' \circ \chi_0^{-1} = \chi' \circ (\pi|_{W'})^{-1} \circ (\pi|_W) \circ \chi,$$

on the domains where it is defined. Since χ and χ' are already smooth, we have to investigate the smoothness of

$$\tau_{W,W'} := (\pi|_{W'})^{-1} \circ (\pi|_W) : D \to D'$$

where $D \subset W$ and $D' \subset W'$ are the preimages of the overlap $\pi(W) \cap \pi(W')$ by $\pi|_W$ and $\pi|_{W'}$, respectively:

$$D = \{x \in W : \pi(x) \in \pi(W')\} \subset W, \quad D' = \{x \in W' : \pi(x) \in \pi(W)\} \subset W'.$$

We now check the smoothness of $\tau = \tau_{W,W'}$ around an arbitrary point $x_0 \in D$. Let $y_0 = \tau(x_0)$. Since x_0 and y_0 are mapped by π to the same point in M/Γ, we find γ_0 such that $y_0 = \gamma_0 \cdot x_0$. One then finds a neighborhood $D_0 \subset D$ such that $D_0' := \gamma_0 \cdot D_0$ sits inside D' and the, on D_0, τ must be given by the multiplication by γ_0 (hence smooth!).

In conclusion, we have a smooth structure on M/Γ which, by construction, makes the projection into a local diffeomorphism. Moreover, M/Γ is smooth. We leave it to the reader to prove that M/Γ is also second countable.

Remark 2.90. The proof above only used the properties mentioned in items 1. and 2. (well, 2. does imply that the action is free, but 1. and 2. may hold also when Γ is infinite):

1. if $(\Gamma \cdot x) \neq (\Gamma \cdot y)$ then there exist neighborhoods U of x and V of y such that $U \cap \gamma \cdot V = \emptyset$ for all $\gamma \in \Gamma$.
2. around any $x \in M$ there exists an open neighborhood W such that $W \cap \gamma \cdot W = \emptyset$ for all $\gamma \in \Gamma \setminus \{e\}$.

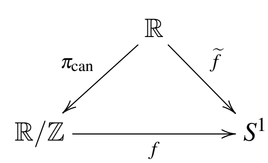
Some authors call such actions "properly discontinuous", but we find that a very unfortunate choice. We prefer to call them **free and proper** actions. Our choice also indicates the fact that it is not one single property, but a combination of two. More precisely, there is also the notion of **proper action**, defined without assuming any freeness: the condition is that, for any $K \subset M$ compact, the set of elements $\gamma \in \Gamma$ with the property that $\gamma \cdot K \cap K \neq \emptyset$ is finite. In particular, actions of finite groups are automatically proper. We leave it as an exercise (in Topology) to show that, indeed, an action satisfies 1. and 2. if and only if it is both free as well as proper.

Corollary 2.91. *The conclusion of the theorem holds more generally, for free and proper actions.*

Example 2.92. Giving an action of the group $(\mathbb{Z}, +)$ on a manifold M is equivalent to giving a diffeomorphism $\phi = \phi_1 : M \to M$. For instance, the translation $\mathbb{R} \to \mathbb{R}$, $r \mapsto r+1$ encodes the action

$$\mathbb{R} \times \mathbb{Z} \to \mathbb{R}, \quad (r,n) \mapsto \phi_n(r) = r+n.$$

The map $\widetilde{f} : \mathbb{R} \to S^1$, $t \mapsto (cos(2\pi t), sin(2\pi t))$ induces a bijection $f : \mathbb{R}/\mathbb{Z} \to S^1$ making the diagram below commutative



This action is both free as well as proper, hence $\mathbb{R}/\mathbb{Z}$ carries a smooth structure making π_{can} into a submersion. Actually, such a smooth structure can be obtained by transporting the smooth structure from S^1 via $\widetilde{f}$. By the uniqueness of such smooth structures, it follows that f is a diffeomorphism.

Example 2.93. Giving an action of the group $(\mathbb{Z}, +)$ on a manifold M is equivalent to giving an **involution** of M, i.e., a diffeomorphism $\phi : M \to M$ with the property that $\phi \circ \phi = \text{Id}_M$. Interesting examples are provided by reflection maps $\tau(x) = -x$, defined on various subspaces of the Euclidean space. E.g.:

- on $\mathbb{R}$, where it encodes a non-free action of $\mathbb{Z}_2$ (with quotient $\mathbb{R}_{\geq 0}$);
- on the circle $S^1 \subset \mathbb{R}^2$, or even on $\mathbb{R}^2 \setminus \{0\}$, to provide free actions of $\mathbb{Z}_2$ (what are the quotient ... manifolds???);
- on the higher dimensional spheres ... but that takes us to the next chapter.

2.4.3 Quotient smooth structures: general crieteria

2.5 Examples

2.5.1 Submanifolds of $\mathbb{R}^n$ via the regular value theorem

The regular value theorem is an immediate consequence of the discussion from subsection 1.2.6 (more precisely, of Theorem 1.38), which is the main tool to check whether a given subset of an Euclidean space, $M \subset \mathbb{R}^L$, is an embedded submanifold- and, therefore, a manifold on its own. Of course, in specific examples, there is a lot more to say about those manifolds hence, when interested in such specific example, this theorem is just the first step. A more general version will be presented a bit later, in Theorem 2.66.

Theorem 2.94 (the regular value theorem in $\mathbb{R}^n$). *Assume that $M \subset \mathbb{R}^n$ can be written as the zero-set of a smooth map $f : \Omega \to \mathbb{R}^k$ defined on an open subset $\Omega \subset \mathbb{R}^n$ and assume that f is a submersion at each point $p \in M$. Then M is an $m = n-k$ dimensional embedded submanifold of $\mathbb{R}^n$.*

Here are some

Exercise 2.95. Show that the sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$

is an embedded submanifold of $\mathbb{R}^3$; and similarly for the higher dimensional spheres.

Exercise 2.96. Show that

$$M = \{(x,y,z) \in \mathbb{R}^3 : x^3+y^3+z^3-3xyz = 1\}$$

is a 2-dimensional embedded submanifold of $\mathbb{R}^3$.

Exercise 2.97. Denote by $\mathscr{M}_{m\times m}(\mathbb{R})$ the set of $m \times m$ matrices with real coefficients, and let $O(m)$ be the subset consisting of orthonormal matrices

$$O(m) = \{A \in \mathscr{M}_{m\times m}(\mathbb{R}) : A \cdot A^T = I_m.\}$$

Identifying $\mathscr{M}_{m\times m}(\mathbb{R})$ with $\mathbb{R}^{m^2}$, show that $O(m)$ is a submanifold of dimension $\frac{m(m-1)}{2}$.

2.5.2 The spheres S^m

The first example in our list are the m-dimensional spheres

$$S^m := \{(x_0,\ldots,x_m) \in \mathbb{R}^{m+1} : (x_0)^2+(x_1)^2+\ldots+(x_m)^2 = 1\}.$$

Of course, this example fits into the general discussion of embedded submanifolds of Euclidean spaces mentioned already in Example 2.17 and is a consequence of the regular value theorem that we just discussed. However, it is one of the many examples which are instructive to consider separately, even as abstract manifolds, and notice their rather special properties.

First of all, as with any subspace of a Euclidean space, we endow it with the Euclidean topology: opens are intersections of S^m with opens in $\mathbb{R}^{m+1}$. In this way, S^m is a Hausdorff, second countable space (... even compact).

Intuitively it should be clear that, locally, S^m looks like (opens inside) $\mathbb{R}^m$- and that is something that we use everyday (we do live on some sort of sphere, remember?). For instance, drawing small disks on the sphere and projecting them on planes through the origin, one can easily build charts. For instance, the upper hemisphere

$$U_0^+ = \{(x_0,\ldots,x_m) \in S^m : x_0 > 0\}$$

(not even so small!) and the projection into the horizontal plane $\{0\} \times \mathbb{R}^m \subset \mathbb{R}^{m+1}$ gives rise to

$$\chi_0^+ : U_0^+ \to \mathbb{R}^m, \quad \chi(x_0, \ldots, x_m) = (x_1, \ldots, x_m).$$

Considering the similar charts with $x_0 < 0$ or using the other coordinates, and putting them all together, we get a smooth atlas

$$(U_0^+, \chi_0^+),\ (U_0^-, \chi_0^-),\ \ldots,\ (U_m^+, \chi_m^+),\ (U_m^-, \chi_m^-) \tag{2.5.1}$$

defining a "natural" smooth structure on S^m, called **the standard smooth structure on** S^m.

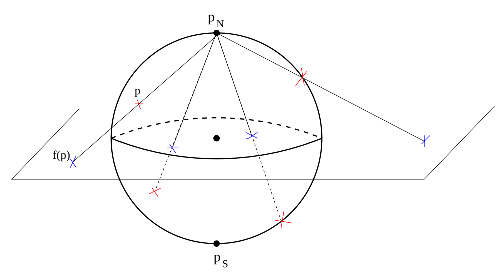
One can actually use the intuition to build similar charts and atlases; however, if you really follow your intuition, you will obtain the same smooth structure (and therefore the name "natural"). Here is an example of another smooth atlas on the sphere (... describing the same smooth structure). It is probably the most elegant one; at least it uses the least amount of charts: two. These are the so-called stereographic projections w.r.t. the north and the south poles,

$$p_N = (0, \ldots, 0, 1), \quad p_S = (0, \ldots, 0, -1) \in S^m,$$

respectively. The one w.r.t. to p_N is the map

$$\chi_N : S^m \setminus \{p_N\} \to \mathbb{R}^m$$

which associates to a point $p \in S^n$ the intersection of the line $p_N p$ with the horizontal hyperplane (see Figure 2.5.2).



The stereographic projection (sending the red points to the blue ones)

Fig. 2.8

Computing the intersections, we find the precise formula:

$$\chi_N : S^m \setminus \{p_N\} \to \mathbb{R}^n, \quad \chi_N(x_0, x_1, \ldots, x_m) = \left(\frac{x_0}{1 - x_m}, \ldots, \frac{x_{m-1}}{1 - x_m} \right)$$

(while for χ_S we find a similar formula, but with +s instead of the −s). Reversing the process (i.e. computing its inverse), we find

$$\chi_N^{-1} : \mathbb{R}^m \to S^m \setminus \{p_N\}, \quad \chi_N^{-1}(u_1, \ldots, u_m) = \left(\frac{2u_1}{|u|^2 + 1}, \ldots, \frac{2u_m}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1} \right)$$

and we deduce that χ_N is a homeomorphism. And similarly for χ_S. Computing the change of coordinates between the two charts we find

$$\chi_S \circ \chi_N^{-1} : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}, \quad \chi_S \circ \chi_N^{-1}(u) = \frac{u}{|u|^2}.$$

This is clearly smooth, composed with itself is the identity, therefore it is a diffeomorphism (... therefore also solving Exercise 1.58 from Chapter 1). We deduce that the two charts

$$(S^m \setminus \{p_N\}, \chi_N), \quad (S^m \setminus \{p_S\}, \chi_S) \tag{2.5.2}$$

define a smooth m-dimensional atlas on S^m.

Exercise 2.98. Show that the stereographic projections give rise to the standard smooth structure on S^m. Or, more precisely, show that the smooth atlas (2.5.2) induces the same smooth structure as (2.5.1).

Exercise 2.99. Consider the height function

$$f : S^2 \to \mathbb{R}, \quad f(x,y,z) = z.$$

At which points in the sphere does f fail to be a submersion?

Remark 2.100 (For the curious students: exotic spheres). As for the Euclidean spaces, it becomes very interesting (and exciting!) when we ask about the existence of other smooth structures on S^m (endowed with the Euclidian topology). Of course, for the reasons we explained, we only look at non-diffeomorphic smooth structures. Again, the answers are deceivingly simple (and the proofs highly non-trivial):

- except for the exceptional case $m = 4$ (see below), for $m \leq 6$ the standard smooth structure on S^m is the only one we can find (up to diffeomorphisms).
- S^7 admits precisely 28 (!?!) non-diffeomorphic smooth structures.
- S^8 admits precisely 2.
- ...
- and S^{11} admits 992, while S^{12} only one!
- and S^{31} more than 16 million, while S^{61} only one (and actually, next to the case S^m with $m \leq 6$, S^{61} is the only odd-dimensional sphere that admits only one smooth structure).
- ...

Moreover, some of the exotic (i.e. non-diffeomorphic to the standard) spheres can be described rather simply. For instance, fixing $\varepsilon > 0$ small enough, inside the small sphere of radius ε, $S^9_\varepsilon \subset \mathbb{C}^5$,

$$W_k := \{(z_1, z_2, z_3, z_4, z_5) \in S^9_\varepsilon : z_1^{6k-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0\} \subset \mathbb{C}^5 \cong \mathbb{R}^{10}$$

(one for each integer $k \geq 1$) are all homeomorphic to S^7, they are all endowed with smooth structures induced from the standard (!) one on $\mathbb{R}^{10}$, but the first 28 of them are each two non-diffeomorphic (and after that they start repeating). In this way one obtains a rather explicit description of all the 28 smooth structures on S^7.

While the number of smooth structures on S^m is well understood for $m \neq 4$, the case of S^4 remains a mystery: is there just one smooth structure? Is there a finite number of them (and how many?)? Or there is actually an infinite number of them? The smooth Poincare conjecture says that there is only one; however, nowadays it is believed that the conjecture is false.

2.5.3 The projective spaces $\mathbb{P}^m$

Probably the simplest example of a smooth manifold that does not sit *naturally* inside a Euclidean space (therefore for which the abstract notion of manifold is even more appropriate) is the m-dimensional projective space $\mathbb{P}^m$. Recall that it consists of all lines through the origin in $\mathbb{R}^{m+1}$:

$$\mathbb{P}^m = \{l \subset \mathbb{R}^{m+1} : l - \text{one dimensional vector subspace}\}.$$

Each point $x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}$ gives rise to the line l_x through the origin and x, hence $l_x := \mathbb{R} \cdot x \subset \mathbb{R}$, so that we can write

$$\mathbb{P}^m = \left\{l_x : x \in \mathbb{R}^{m+1} \setminus \{0\}\right\}, \quad \text{where } (l_x = l_y) \Longleftrightarrow (y = \lambda \cdot x \text{ for some } \lambda \in \mathbb{R}^*)$$

$\mathbb{P}^m$ is best understood by relating it back to $\mathbb{R}^{m+1} \setminus \{0\}$, via the map

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{P}^m, \quad x \mapsto l_x.$$

For instance, this gives rise to a natural topology on $\mathbb{P}^m$: the quotient topology w.r.t. π (already mentioned before, see e.g. Exercise 2.85)- that is, the largest one that makes π continuous, explicitly described as follows:

$$U \subset \mathbb{P}^m \text{ is open} \quad \Longleftrightarrow \quad \pi^{-1}(U) \text{ is open in } \mathbb{R}^{m+1}.$$

All these define $\mathbb{P}^n$ as a set, and as a topological space. For the smooth structure, it is handy to change the notation for the lines l_x to

$$[x_0 : x_1 : \ldots : x_m] := l_x = \mathbb{R} \cdot x, \quad \text{for } x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}.$$

The use of the symbol : in the notation should suggest "division" and is motivated by

$$[x_0 : x_1 : \ldots : x_m] = [y_0 : y_1 : \ldots : y_m] \Longleftrightarrow y_k = \lambda \cdot x_k \text{ for some } \lambda \in \mathbb{R}^* \text{ and all } k\text{s}. \tag{2.5.3}$$

The coordinate notation is also more appropriate when one is searching for coordinate charts. The natural smooth structure on $\mathbb{P}^m$ is obtained starting from a simple observation: the last equality above allows us in principle to make the first coordinate equal to 1:

$$[x_0 : x_1 : \ldots : x_m] = \left[1 : \frac{x_1}{x_0} : \ldots : \frac{x_m}{x_0}\right],$$

i.e. to use just coordinates from $\mathbb{R}^m$; with one little problem- when $x_0 = 0$ (this "little problem" is what forces us to use more than one chart). We arrive at a very natural chart for $\mathbb{P}^m$: with domain

$$U_0 = \{[x_0 : x_1 : \ldots : x_m] \in \mathbb{P}^m : x_0 \neq 0\}$$

and defined as

$$\chi^0 : U_0 \to \mathbb{R}^m, \ \chi^0([x_0 : x_1 : \ldots : x_m]) = \left(\frac{x_1}{x_0}, \ldots, \frac{x_m}{x_0}\right).$$

And similarly when trying to make the other coordinates equal to 1:

$$\chi^i : U_i \to \mathbb{R}^m, \ \chi^0([x_0 : x_1 : \ldots : x_m]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_m}{x_i}\right),$$

defined on U_i defined by $x_i \neq 0$.

Exercise 2.101. Show that, indeed,

$$(U_0, \chi^0), (U_1, \chi^1), \ldots, (U_m, \chi^m),$$

is, indeed, a smooth atlas (hence it gives rise to a smooth structure on $\mathbb{P}^m$).

Remark 2.102 (gluing the antipodal points of the sphere). By rescaling elements $x \in \mathbb{R}^{m+1} \setminus \{0\}$ to force them to be of norm one or, more geometrically, by intersecting the lines through the origin with the unit sphere, one can replace $\mathbb{R}^{m+1} \setminus \{0\}$ by S^m in the discussion above. Hence

$$\mathbb{P}^m = \{l_x : x \in S^m\} \quad \text{where, for } x, y \in S^m\text{: } \ (l_x = l_y) \Longleftrightarrow (y = x \text{ or } y = -x) \tag{2.5.4}$$

and it is not very difficult to show that the quotient topology induced by π coincides with the one induced by its restriction

$$\pi|_{S^m} : S^m \to \mathbb{P}^m, \quad x \mapsto l_x.$$

These allow us to think of $\mathbb{P}^m$ as being obtained from S^m by gluing any point $x \in S^m$ to its antipodal $-x$. More elegantly, one may say that we deal with an action of $\mathbb{Z}_2$ on S^m, and

$$\mathbb{P}^m = S^m / \mathbb{Z}_2,$$

at least as topological spaces. Notice that one of the gains of using S^m is that it follows right away that $\mathbb{P}^m$ is compact (as the quotient of a compact space). On the other hand, one could also invoke Theorem 2.87 to make $\mathbb{P}^m$ into a smooth manifold. Using the uniqueness part of the theorem, the following should be a rather easy exercise:

Exercise 2.103. Show that the smooth structure induced by the charts (U_k, χ_k) coincides with the one obtained by applying Theorem 2.87 to the action of $\mathbb{Z}_2$ on S^n given by the reflection $x \mapsto -x$.

The following exercise show that, when $m = 1$, $\mathbb{P}^1$ is not really new: it is diffeomorphic to the circle $S^1 \subset \mathbb{R}^2$.

Exercise 2.104. Consider the function

$$f : \mathbb{P}^1 \to S^1, \quad f([x:y]) = \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{2xy}{x^2 + y^2} \right)$$

and do the following:

1. show that f is well-defined and smooth.
2. show hat f is actually a diffeomorphism.
3. after identifying $\mathbb{P}^1$ with S^1 using f, what does the map $H : S^1 \to \mathbb{P}^1$ become?
4. explain f with a picture.

Exercise 2.105. Let $f : \mathbb{P}^2 \to \mathbb{R}^4$ be the function given by

$$f([x:y:z]) = (xy,\ yz,\ zx,\ y^2 - z^2) \quad \text{for } (x,y,z) \in S^2.$$

Please do the following:

1. check that f is well-defined and write the general formula for f (not only for (x,y,z) in the sphere).
2. compute the representation of f with respect to the charts χ^i of $\mathbb{P}^2$ (and the identity chart for $\mathbb{R}^4$).
3. show that f is an immersion.
4. we also compose f with the projection $\mathbb{R}^4 \to \mathbb{R}^3$ on the first three coordinates,

$$g := \mathrm{pr} \circ f : \mathbb{P}^2 \to \mathbb{R}^3;$$

   Show that its image is the following explicit subspace of $\mathbb{R}^3$:

$$R := \left\{ (X,Y,Z) \in \mathbb{R}^3 : |X|, |Y|, |Z| \le \frac{1}{2},\ (XY)^2 + (YZ)^2 + (ZX)^2 = XYZ \right\} \dots$$

5. ... but this is not an embedded submanifold of $\mathbb{R}^3$.
6. show that, however, $g : \mathbb{P}^2 \to \mathbb{R}^3$ is an immersion everywhere except for six points $p_1, \dots, p_6$ ...
7. ... but even after removing those six points, $g(\mathbb{P}^2 \setminus \{p_1, \dots, p_6\})$ is not an embedded submanifold of $\mathbb{R}^3$.
8. show that, on the other hand,

$$R_0 = \{(X,Y,Z) \in R : XYZ \neq 0\}$$

   is an open dense subset of R which is an embedded submanifold of $\mathbb{R}^3$, and there is open dense subset $\mathbb{P}^2_0 \subset \mathbb{P}^2$ such that $g|_{\mathbb{P}^2_0} : \mathbb{P}^2_0 \to R_0$ is a diffeomorphism.

The image R of g is know as "the Roman surface", or the "Steiner surface" (discovered by Steiner in Rome in 1844, according to Wikipedia). One can actually show that $\mathbb{P}^2$ cannot be embedded in $\mathbb{R}^3$. The g and R above can be seen as an attempt to find an immersion of $\mathbb{P}^2$ in $\mathbb{R}^3$ (non-injective, of course). You may be surprised to hear that such immersions actually exist. Finding an explicit one is quite a bit more difficult but also very interesting, and gives rise to Boy's surface in $\mathbb{R}^3$ ... but I let you google this one ...

Exercise 2.106. Show that the smooth structure on $\mathbb{P}^m$ discussed here makes the canonical map

$$H : S^m \to \mathbb{P}^m, \quad H(x_0, \dots, x_m) = [x_0 : \dots : x_m]$$

into a submersion.

Exercise 2.107. As a continuation of the previous exercise: show that it is the unique smooth structure on $\mathbb{P}^m$ for which H is a submersion.

Remark 2.108 (For the interested students: minimal number of charts). Note that this smooth structure on $\mathbb{P}^m$ is also a first example in which, at least intuitively, we need many more than just two charts to form an atlas. The question, valid for any manifold M, asking what is the minimal number of charts one needs to obtain an atlas of M, is a very interesting one. In general one can show that one can always find an atlas consisting of no more than $m+1$ charts where m is the dimension of M (not very deep, but not trivial either!). Therefore, denoting by $N_0(M)$ the minimal number of charts that we can find, one always has $N_0(M) \leq \dim(M)+1$. The atlas described above confirms this inequality for $\mathbb{P}^m$. However, in concrete examples, we can always do better. E.g. we have seen that $N_0(S^m) = 2$ (well, why can't it be 1?). The same holds for all compact orientable surfaces. Thinking a bit (but not too long) about $\mathbb{P}^m$ one may expect that $N_0(\mathbb{P}^m) = m+1$; however, that is not the case. The precise computation was carried out by M. Hopkins, except for the cases $n = 31$ and $n = 47$. For those cases we know that $N_0(\mathbb{P}^{31})$ is either 3 or 4, while $N_0(\mathbb{P}^{46})$ is either 5 or 6. For all the other cases, writing $m = 2^k a - 1$ with a odd, one has

$$N_0(\mathbb{P}^m) = \begin{cases} \max\{2,a\} & \text{if } k \in \{1,2,3\} \\ \text{the least integer} \geq \frac{m+1}{2(k+1)} & \text{otherwise} \end{cases} \qquad \square$$

2.5.4 *The complex projective spaces* $\mathbb{CP}^m$

Returning to the basics, note that there is a complex analogue of $\mathbb{P}^m$; for that reason $\mathbb{P}^m$ is sometimes denoted $\mathbb{RP}^m$ and called **the real projective space**. The m-dimensional **complex projective space** $\mathbb{CP}^m$ is defined completely analogously but using complex lines $l_z \subset \mathbb{C}^{m+1}$, i.e. 1-dimensional complex subspaces of $\mathbb{C}^{m+1}$ (and 1-dimensional is in the complex sense). Again, one can write

$$\mathbb{CP}^m = \left\{[z_0 : z_1 : \ldots : z_m] : (z_0, z_1, \ldots, z_m) \in \mathbb{C}^{m+1} \setminus \{0\}\right\}$$

where $[z_0 : z_1 : \ldots : z_m]$ is just a notation for the line through the origin and the point $z = (z_0, \ldots, z_m) \in \mathbb{C}^{m+1}$. Hence

$$[z_0 : z_1 : \ldots : z_m] = [\lambda \cdot z_0 : \lambda \cdot z_1 : \ldots : \lambda \cdot z_m] \quad \text{for } \lambda \in \mathbb{C}^*.$$

Analogously to the real case, one can realize

$$\mathbb{CP}^m = \left(\mathbb{C}^{m+1} \setminus \{0\}\right)/\mathbb{C}^*.$$

Using $\mathbb{R}^{2m} = \mathbb{C}^m$ we can also represent the $(2m+1)$-dimensional sphere as:

$$S^{2m+1} = \{(z_0, \ldots, z_m) \in \mathbb{C}^{m+1} : |z_0|^2 + \ldots + |z_m|^2 = 1\}$$

and there is an obvious map

$$H : S^{2m+1} \to \mathbb{CP}^m, \quad H(z_0, \ldots, z_m) = [z_0 : \ldots : z_m].$$

Intersecting the (complex) lines with this sphere we can realize each line as as $[z]$ with $z \in S^{2m+1}$; two like this, $[z_1]$ and $[z_2]$, are equivalent if and only if $z_2 = \lambda \cdot z_1$, where this time $\lambda \in S^1$; i.e. the group $\mathbb{Z}_2$ from the real case is replaced by the group S^1 of complex numbers of norm 1 (endowed with the usual multiplication). We obtain

$$\mathbb{CP}^m = S^{2m+1}/S^1.$$

As for the smooth structure one proceeds completely analogously, keeping in mind the identification $\mathbb{C}^m = \mathbb{R}^{2m}$: we get a (smooth) atlas made of $m+1$ charts.

$$\mathscr{A} = \{(U_0, \chi^0), \ldots, (U_m, \chi^m)\}$$

given by

$$U_i = \{[z_0 : z_1 : \ldots : z_m] \in \mathbb{CP}^m : z_i \neq 0\},$$
$$\chi^i : U_i \to \mathbb{C}^m = \mathbb{R}^{2m},$$
$$\chi^i([z_0 : z_1 : \ldots : z_m]) = \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_m}{z_i}\right).$$

Remark 2.109 (For the interested students: minimal number of charts for $\mathbb{CP}^m$). Note that the change of coordinates is actually given by holomorphic maps- therefore $\mathbb{CP}^m$ is also a complex manifold (see Section 2.1.4). Note that m is the complex dimension of $\mathbb{CP}^m$; as a manifold, it is $2m$-dimensional. As a curiosity: for the minimal number of charts needed to cover $\mathbb{CP}^m$, the answer is much simpler in the complex case:

$$N_0(\mathbb{CP}^m) = \begin{cases} m+1 & \text{if } m \text{ is even} \\ \frac{m+1}{2} & \text{if } m \text{ is odd} \end{cases}. \quad \square$$

Similar to Exercise 2.104, the following exercise show that, when $m = 1$, $\mathbb{CP}^1$ is not really new: it is diffeomorphic to the 2-sphere. This will be discussed together with the map

$$H : S^3 \to \mathbb{CP}^1, \quad H(z_0, z_1) = [z_0 : z_1].$$

The notation H (and h for the map in the next exercise) are related to the name of Hopf (... fibration).

Exercise 2.110. Consider the map

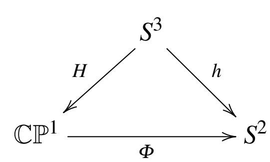
$$h : S^3 \to S^2, \quad h(z_0, z_1) := (|z_0|^2 - |z_1|^2, 2i \cdot \overline{z_0} \cdot z_1)$$

or, using real coordinates, h sends $(x,y,z,t) \equiv (x + i \cdot y, z + i \cdot t)$ to

$$h(x,y,z,t) = (x^2 + y^2 - z^2 - t^2, 2(yz - xt), 2(xz + yt)).$$

Show that:

1. h is well defined and it is a smooth submersion.
2. H is a smooth submersion.
3. There exists and is unique a map $\Phi : \mathbb{CP}^1 \to S^2$ such that $h = \Phi \circ H$ i.e. a commutative diagram:



.

4. Φ is smooth.
5. Φ is actually a diffeomorphism.

Exercise 2.111. Returning to arbitrary dimensions and the canonical map $H : S^{2m+1} \to \mathbb{CP}^m$ then, as in Exercise 2.106, show that the smooth structure on $\mathbb{CP}^m$ discussed here is uniquely determined by the condition that the projection H becomes a submersion. Try to further generalize, and eventually deduce something about smooth structures on quotients of smooth manifolds.

2.5.5 The torus T^2

We now concentrate on 2-dimensional manifolds (surfaces). Some of them (e.g. the 2-sphere and the 2-torus) sit rather canonically inside Euclidean spaces, but others (e.g. the projective plane or the Klein bottle) do not. The point is that each one of them is an interesting manifold on its own (completely independent on how it may sit inside an Euclidean space), each one of them can be "embedded" in an Euclidean space (providing "concrete models"), but while some of those embeddings are very natural (e.g. the concrete realization of the sphere in $\mathbb{R}^2$), other are not (e.g. any model of the Klein bottle). Therefore, it is interesting to free our mind and to think of them as abstract manifolds.

We start with "the torus"- which is the standard name for subspaces of $\mathbb{R}^3$ which look like the surface of a doughnut; we use it generically for any manifold that is diffeomorphic to such a doughnut surface (or just homeomorphic, if you are doing Topology ...). There are various ways to produce explicit models and, for each one of them, the mathematics to make it precise.

Knutsel model: All the surfaces can be obtained from a planar figure, usually a square, after gluing some of its sides. For the torus, one simply glues each pair of opposite sides of a square, as shown in Figure 2.9. The outcome is a real-life torus.

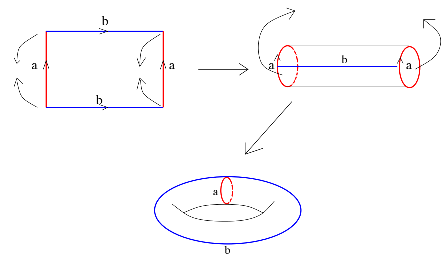


Fig. 2.9

The mathematical tool to make this precise is the notion of equivalence relation: given a set/space/manifold M, an equivalence relation R on M can be thought of as encoding the "gluing information" (so that we will be gluing precisely the equivalent points of M), and the the quotient space M/R can be thought of as the result of the gluing. Recall that

$$M/R = \{R(x) : x \in M\},$$

where $R(x)$ is the R-orbit through $x \in M$ (also called the equivalence class of x), $R(x) = \{y \in M : (x,y) \in R\}$. Hence an element/point of M/R is an entire equivalence class, and the intuition that "we glue precisely the equivalent points" takes the precise form:

$$R(x) = R(y) \Longleftrightarrow (x,y) \in R.$$

The fact that "the points of M/R are obtained from those of M" is made precise by/encoded in the surjective map

$$\pi_{\mathrm{can}} : M \to M/R$$

that sends x to $R(x)$, called canonical projection. This projection is very important since it allows one to understand/study the quotient M/R by relating it to M For instance, if M is a topological space, then it is natural to look for topologies on M/R that make π_{can} continuous and, among those, pick up "the most interesting one". That is what is known in Topology as the **quotient topology** on M/R, defined by:

$$V - \text{open in } M/R \iff \pi_{\mathrm{can}}^{-1}(V) - \text{open in } M.$$

When M is a manifold, the question of whether M/R admits a "canonical smooth structure" is more subtle; we will return to it later on.

Back to the torus, the relevant space is $M = [0,1] \times [0,1]$, two points $x, y \in M$ are (declared) equivalent if they get glued to each other, i.e. if

R1: $x = y$, or
R2: $x = (t,0)$ and $y = (t,1)$ with $t \in [0,1]$, or the other way around, or
R3: $x = (0,s)$ and $t = (1,s)$ with $s \in [0,1]$, or the other way around.

This defines an equivalence relation R_{torus} and the conclusion is that $[0,1] \times [0,1]/R_{\mathrm{torus}}$ is a precise (though quite abstract) version of the torus, defined now as a topological space.

One way to exhibit a smooths structure on the abstract model without making use of the concrete ones (to be discussed below) is by noticing that the equivalence relation R_{torus} comes from a group actions on the entire plane $\mathbb{R}^2$. The group is $(\mathbb{Z}^2, +)$ acting on $\mathbb{R}^2$ by

$$(n,m) \cdot (x,y) = (x+n, y+m).$$

Notice now that the corresponding equivalence relation $R_{\mathbb{Z}^2}$ on $\mathbb{R}^2$ has the following properties:

- each point in $\mathbb{R}^2$ is equivalent with at least one point in the square $[0,1] \times [0,1]$;
- the restriction from $\mathbb{R}^2$ to the square is precisely the equivalence relation R_{torus} from the knutsel model.

It follow that

$$[0,1] \times [0,1]/R_{\text{torus}} \equiv R^2/\mathbb{Z}^2.$$

Finally, $R^2/\mathbb{Z}^2$ has a smooth structure (unique, by Corollary 2.84) with the property that the projection $\pi : \mathbb{R}^2 \to R^2/\mathbb{Z}^2$ is a submersion. This can be seen directly, or one could now invoke Corollary 2.91.

Remark 2.112 ("knutsel models" for S^2 and $\mathbb{P}^2$). For the surfaces that arise in the last two sections, the sphere S^2 and the projective plane $\mathbb{P}^2$, one can describe similar "knutsel models". For $S2$, the procedure is pretty clear and is described in Figure 2.10.

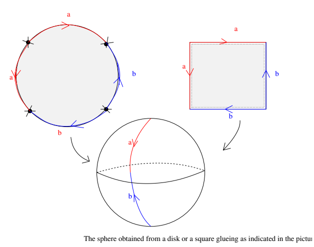


Fig. 2.10

For $\mathbb{P}^2$, since it cannot be realised inside $\mathbb{R}^3$, it is a bit harder to use the intuition; actually, one may even claim that a "knutsel model" helps in picturing how the projective plane actually looks like.

The model is shown in Figure 2.11 but it still requires a little explanation: the disk in the picture is the unit disk $D^2 \subset \mathbb{R}^2$. The reason that it appears is that it can be used to "parametrise" $\mathbb{P}^2$. More precisely, any $y \in D^2$ gives rise to a line $l(y) \subset \mathbb{R}^3$ as follows: the perpendicular on the horizontal plane $x_2 = 0$ going through y intersects the northern hemisphere $S^2_+ = \{x \in S^2 : x_2 \geq 0\}$ into a point $\widetilde{y}$, and then $l(y)$ is the line through the origin and $\widetilde{y}$. All lines arise in this way, just that some different ys may give the same line. Actually, one remarks right away that

$$l(y) = l(y') \Longleftrightarrow y = y' \text{ or } y \in \partial D^2 (= S1) \text{ and } y = -y'.$$

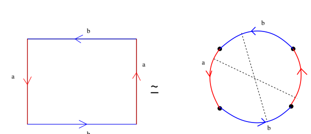


Fig. 2.11

Therefore, $\mathbb{P}^2$ can be thought of as obtained from D^2 by gluing any point in the boundary with its opposite. This is precisely what the disk in the picture describes, while the passing to a square model should be clear. ✓

Beweging model: For the next model of the torus one has to move a bit as follows: place yourself perpendicularly on the XOY plane with your chest at the origin O, take a circle of some radius r (... smaller than the length $\widetilde{r}$ of your hand), hold it by its center (you find a way to do that ...) and then rotate yourself a full 360 degrees. The locus spanned by the circle will be a torus. This description translates mathematically into a parametrised version of the torus. To that end, notice that the points in the resulting tous are determined by the angles a and b as shown in the picture.

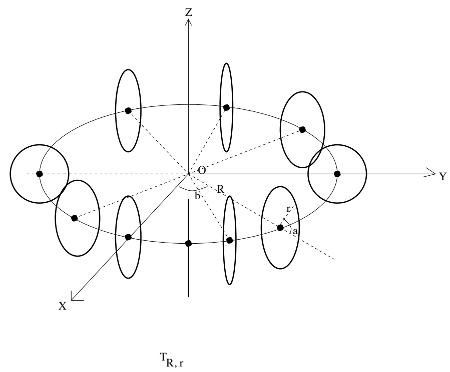


Fig. 2.12

Computing the resulting coordinates, one obtains the parametric description:

$$T^2 = T^2_{\widetilde{r},r} = \{((\widetilde{r}+r\cdot\cos a)cos b,\ (\widetilde{r}+r\cdot\cos a)sin b,\ r\cdot\sin a):\ a,b\in[0,2\pi]\}\subset\mathbb{R}^3. \tag{2.5.5}$$

Denoting the coordinates by x,y,z and eliminating a and b using $\sin^2+\cos^2=1$ one finds the implicit description

$$T^2 = \{(x,y,z)\in\mathbb{R}^3 : \left(\sqrt{x^2+y^2}-\widetilde{r}\right)^2+z^2=r^2\}. \tag{2.5.6}$$

Exercise 2.113. Use the regular value theorem in $\mathbb{R}^3$ (see Theorem 2.94) to conclude that $T^2_{\widetilde{r},r}$ is a manifold.

This concrete model of the torus is related to quotient one as follows. First of all, it is related to the square via

$$\widetilde{f}:[0,1]\times[0,1]\to T^2,\quad (t,s)\mapsto((\widetilde{r}+r\cdot\cos 2\pi t)cos 2\pi s,\ (\widetilde{r}+r\cdot\cos 2\pi t)sin 2\pi s,\ r\cdot\sin 2\pi t).$$

This map has the property that $f(x)=f(y)$ holds precisely in one of the cases (R0)-(R2) described above, hence it induces a bijection between $[0,1]\times[0,1]/R$ and $T^2_{\widetilde{r},r}$ (and one can also check that this is actually a homeomorphism).

The geography model: But probably the shortest description of the torus is simply $S^1\times S^1$. The fact that such a product, a priori sitting inside $\mathbb{R}^2\times\mathbb{R}^2=\mathbb{R}^4$, can be realised inside $\mathbb{R}^3$, is interesting on its own. On the other hand, the fact that the standard models inside $\mathbb{R}^3$ can be interpreted as such a product amounts to realising that the torus carries a longitude $\times$ latitude grid. This is indicated in Figure 2.13, where the red/blue circles stand for "longitude/latitude lines"; looking at a general point in the torus one sees that, through it, there passes precisely one blue and one red circle. Fixing some reference point on the torus (which will play the role of "null Island point") one obtains a 0-longitude (red) circle and a 0-latitude (blue) circle, which will play the role of "coordinate axes": any point on the torus will have a "longitudinal coordinate" obtained by intersecting the red circle through it with the 0-latitude (blue) circle and, similarly, a "latitudinal coordinate". We see that the coordinates are now elements of the circle and, therefore, one gets a pair (z_1,z_2) with $z_1,z_2\in S^1\times S^1$ as coordinates.

In the explicit description (2.5.5), the coordinates of a point in the torus will be simply $z_1=e^{i\cdot a}$ and $z_2=e^{i\cdot b}$. To emphasise that everything works out smoothly we formulate the next exercise, but we should also mention that the most elegant way to do it can be carried out only later on, after we discuss tangent spaces (Exercise 2.114 to come).

Exercise 2.114. Prove that

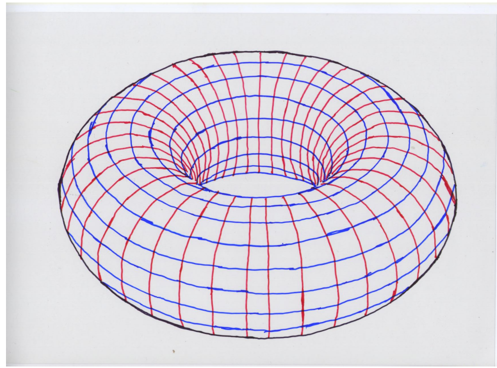


Fig. 2.13

$$f_0 : S^1 \times S^1 \to T^2, \quad (e^{i\cdot a}, e^{i\cdot b}) \mapsto ((\widetilde{r} + r\cdot \cos a)cos b,\ (\widetilde{r} + r\cdot \cos a)sin b,\ r\cdot \sin a)$$

is a diffeomorphism, where $T^2 = T^2_{\widetilde{r},r}$.

All together, one obtains a sequence of diffeomorphisms

$$\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1 \cong T^2_{\widetilde{r},r}, \quad (t,s) + \mathbb{Z}^2 \mapsto \left(e^{2i\pi t}, e^{2i\pi s}\right) \mapsto (\widetilde{r} + r\cdot \cos 2\pi t) sin 2\pi s,\ r\cdot \sin 2\pi t).$$

Notice however that that the description of the torus as $\mathbb{R}^2/\mathbb{Z}^2$ is closest to the one provided by the knutsel model. Indeed, the square $[0,1]\times[0,1] \subset \mathbb{R}^2$ interacts with equivalence relation R_Γ ($\Gamma = \mathbb{Z}_2$) on $\mathbb{R}^2$ as follows:

- each point in $\mathbb{R}^2$ is equivalent with at least one point in the square $[0,1]\times[0,1]$;
- the restriction of the equivalence relation from $\mathbb{R}^2$ to the square is precisely the equivalence relation R_{torus} from the knutsel model.

The full torus: The torus also comes with a "full-version". Playing the game from Fig 2.12, one would now need to grab and rotate a full 2-dimensional disk of radius r,

$$D^2_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le \widetilde{r}^2\}.$$

The outcome will be a subspace

$$T^2_{\widetilde{r},r,\text{solid}} \subset \mathbb{R}^3.$$

Extending the computation we did for the torus to the full torus, one obtains a parametrization

$$T^2_{\widetilde{r},r,\text{solid}} = \left\{(\widetilde{r} + ru)\cos a, (\widetilde{r} + ru)\sin a, v) : a \in [0,2\pi], (u,v) \in D^2\right\},$$

where D^2 is the unit disk; this gives rise to the following analogue of (2.5.6):

$$T^2_{\widetilde{r},r,\text{solid}} = \{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2} - \widetilde{r}\right)^2 + z^2 \le r^2\}.$$

as well as to a diffeomorphism

$$f : S^1 \times D^2 \to T^2_{\widetilde{r},r,\text{solid}}, \quad (e^{ia}, (u,v)) \mapsto (\widetilde{r} + ru)\cos a, (\widetilde{r} + ru)\sin a, rv). \tag{2.5.7}$$

Exercise 2.115. Show that T^2_{solid} is a manifold with boundary, and that it is diffeomorphic to $S^1 \times D^2$.

Of course, the diffeomorphism from the exercise, when restricted to the boundary, will become the diffeomorphism between T^2 and $S^1 \times S^1$ discussed above.

And here is a very nice property of the solid torus, that one sometimes states simply as: the 3-sphere can be obtained by gluing together two solid tori along their boundary. The following exercise explains this process and its interaction with the Hopf map from Exercise 2.110.

Exercise 2.116. Consider again the Hopf map in the explicit form

$$h : S^3 \to S^2, \quad h(x,y,z,t) = (x^2+y^2-z^2-t^2, 2(yz-xt), 2(xz+yt)).$$

Consider also the decomposition of S^2 into the open upper and lower hemispheres,

$$S^2_+ = \{(u,v,w) \in S^2 : u \geq 0\}, \quad S^2_- = \{(u,v,w) \in S^2 : u \leq 0\}$$

with the common intersection the circle identified with

$$S^1 = S^2_+ \cap S^2_+ = \{(u,v,w) \in S^2 : u = 0\}.$$

Show that:

1. each of the hemispheres are manifolds with boundary diffeomorphic to the unit disk D^2.
2. the pre-image of the common circle S^1 via h is homeomorphic to the torus.
3. the pre-image of the upper/lower hemisphere via h is homeomorphic to the solid torus.

Therefore, the pre-image via h of the decomposition $S^2 = S^2_+ \cup S^2_-$ becomes a decomposition of S^3 into two copies of the solid torus.

Note that we do not quite have yet the theoretical foundation to make this exercise into a ”smooth one”; but, with the right concepts, all the homeomorphisms will become smooth (diffeomorphisms).

2.5.6 The Moebius band

The most popular manifold with boundary is the Moebius band. According to wikipedia, it “is a surface that can be formed by attaching the ends of a strip of paper together with a half-twist. As a mathematical object, it was discovered by Johann Benedict Listing and August Ferdinand Moebius in 1858, but it had already appeared in Roman mosaics from the third century CE.” Here we look at it following the same lines of the discussion of the torus.

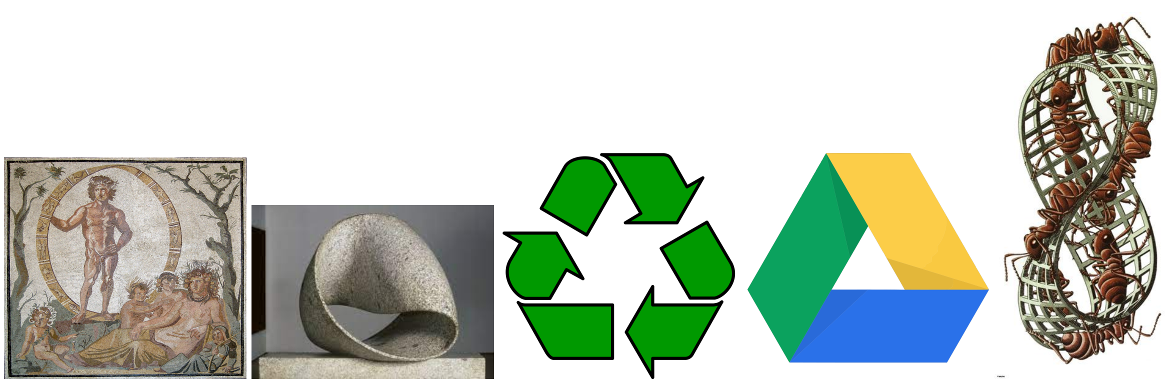


Fig. 2.14

Knutsel model: Most probably, the way that the Moebius was first shown to you was by using a long strip of paper and gluing two of its edges not in the most obvious way (that would give you a cylinder), but first twisting the strip creating a ribbon. Of course, "long strip" is so as to have a nice model made of paper (without rupturing the paper); in principle, this is again about the unit square $[0,1] \times [0,1]$, where we now glue two of its opposite edges while changing their orientation, as shown in Figure 2.15.

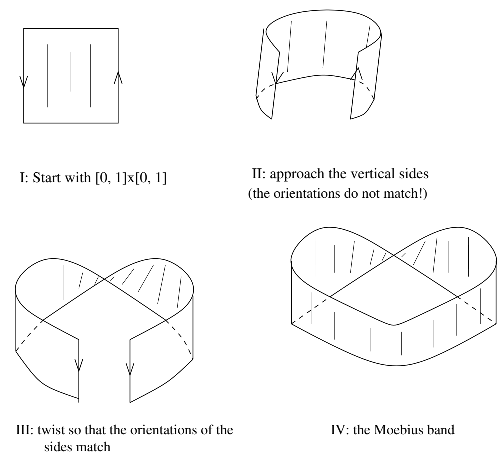


Fig. 2.15

More precisely (from a mathematical viewpoint), we deal with the quotient space $[0,1] \times [0,1]/R_{\text{Moe}}$ where R_{Moe} is the equivalence relation described as follows: two points $x = (t,s), x' = (t',s') \in [0,1] \times [0,1]$ are declared equivalent if:

R1: $(t,s) = (t',s')$, or
R2: $t = 0$, $t' = 1$ and $s' = 1 - s$, or
R3: $t = 1$, $t' = 0$ and $s' = 1 - s$.

This makes sense of the Moebius band as a topological space, living on its own (independent on how we choose to embed it inside an Euclidean space).

Beweging model: To produce a model inside $\mathbb{R}^3$ that can be described in formulas, we proceed like we did for the torus. We place ourselves in the same position (perpendicular on the XOY plane, etc), but now holding a stick by its middle, ands while rotating ourselves a full 360 degrees, we rotate the stick by 180 degrees (turning it upside down). Of course, we assume the rotations to be uniform (constant spead). See Figure 2.16. To write down explicit formulas then, again, we denote by $\widetilde{r}$ the length of the arm, and by r half of the length of the stick. The starting position is the segment A_0B_0 perpendicular on XOY with middle point $P_0 = (\widetilde{r},0,0)$. At any moment, the segment stays in the plane through the origin and its middle point, which is perpendicular on the XOY plane.

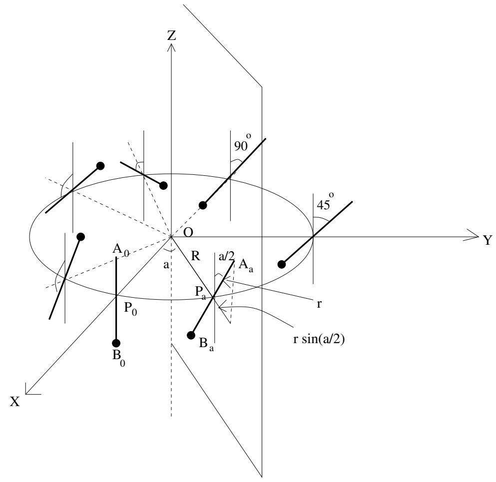
We parametrise the movement by the angle a shown in the picture. At any moment a, the segment stays in the plane perpendicular on the XOY plane that goes through the origin and the middle point

$$P_a = (\widetilde{r}\cos(a), \widetilde{r}\sin(a), 0).$$

At this moment, the precise position of the segment, denoted A_aB_a, is determined by the angle that it makes with the perpendicular on the plane XOY through P_a; call it b. This angle depends on a. Due to the assumptions (namely that while a goes from 0 to 2π, b only goes from 0 to π, and that the rotations are uniform), we have $b = a/2$ (see 2.16). We deduce

$$A_a = \left(\left(\widetilde{r} + r\sin\frac{a}{2}\right)\cos(a), \left(\widetilde{r} + r\sin\frac{a}{2}\right)\sin(a), r\cos\frac{a}{2}\right).$$

and a similar formula for B_a (obtained by replacing r by $-r$). One obtains the following explicit model for the Moebius band:



Explicit realization of the Moebius band in |R³ : M R, r

Fig. 2.16

$$M_{\widetilde{r},r} = \left\{ \left(\left(\widetilde{r} + tr\sin\frac{a}{2}\right)\cos(a), \left(\widetilde{r} + tr\sin\frac{a}{2}\right)\sin(a), tr\cos\frac{a}{2}\right) : a \in [0,2\pi], t \in [-1,1] \right\} \quad (2.5.8)$$
$$= \{((\widetilde{r} + tr\sin b)\cos 2b, (\widetilde{r} + tr\sin b)\sin 2b, tr\cos b) : b \in [0,\pi], t \in [-1,1]\}$$

Exercise 2.117. Show that $M_{\widetilde{r},r} \subset \mathbb{R}^3$ is, indeed, a manifold with boundary, and a different choice of r and $\widetilde{r}$ produces diffeomorphic manifolds.

To fix one concrete model, one usually takes $\widetilde{r} = 2$ and $r = 1$.

Exercise 2.118. Consider $f : [0,1] \times [0,1] \to \mathbb{R}^3$ given by

$$f(t,s) = ((2 + (2s-1)\sin(\pi t))\cos(2\pi t), (2 + (2s-1)\sin(\pi t))\sin(2\pi t), (2s-1)\cos(\pi t)).$$

Check that $f(t,s) = f(t',s')$ holds only in one of the situations R1-R3 above. Deduce that the abstract Moebius band $[0,1] \times [0,1]/R_{\text{Moe}}$ is homeomorphic to the $M_{2,1} \subset \mathbb{R}^3$.

The model inside the tautological bundle over $\mathbb{P}^1$: The explicit model $M_{\widetilde{r},r}$ clearly lives inside the solid torus $T_{\widetilde{r},r}$ and, via the identification of the solid torus with $S^1 \times D^2$, see (2.5.7), it is clear that $M_{\widetilde{r},r}$ will be identified to the following subspace

$$\left\{ \left(e^{ia}, te^{\frac{ia}{2}}\right) : e^{ia} \in S^1, t \in [-1,1] \right\} \subset S^1 \times D^2.$$

Now, the key remark is that the appearance of e^{ia} and $e^{\frac{ia}{2}}$ together is at the very heart of the diffeomorphism between $\mathbb{P}^1$ and S^1 mentioned in Exercise 2.104; indeed, that diffeomorphism takes the line through $e^{\frac{ia}{2}}$ precisely to the point $e^{ia} \in S^1$. Therefore, pairs $\left(e^{ia}, te^{\frac{ia}{2}}\right)$ as above may be interpreted as a pairs (l_z, tz) with $z \in S^1$ and $t \in [-1,1]$. We discover the so-called tautological line bundle over $\mathbb{P}^1$,

$$E = \{(l,v) \in \mathbb{P}^1 \times \mathbb{R}^2 : v \in l\} \subset \mathbb{P}^1 \times \mathbb{R}^2.$$

Inside it there are various subspaces obtained by posing conditions on $|v|$, such as

$$E_{\leq 1} = \{(l,v) \in E : |v| \leq 1\}, \quad E_{<1} = \{(l,v) \in E : |v| < 1\}, \text{ etc},$$

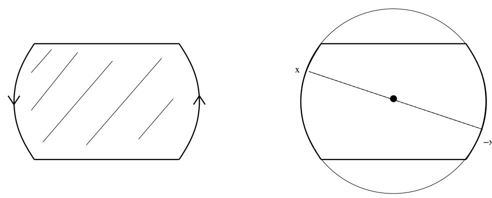
and the conclusion is that we have a diffeomorphism

$$E_{\leq 1} \to M_{\widetilde{r},r}, \quad (l,z) \mapsto ((\widetilde{r} + tr\sin b)\cos 2b, (\widetilde{r} + tr\sin b)\sin 2b, tr\cos b),$$

where $b \in [0,\pi]$ and $t \in [-1,1]$ are chosen so that l is the line through the origin and e^{ib}, and $z = te^{ib}$. Of course, $E_{<1}$ corresponds to the open Moebius band, while the smaller $E_{\leq \varepsilon} \to E_1$ correspond to thiner Moebius bands.
The model inside $\mathbb{P}^2$: Another copy of the Moebius band can be detected inside the projective plane $\mathbb{P}^2$. To visualise it, we use the model for the projective plane that was mentioned in Remark 2.112: as the space obtained from D^2 by gluing the antipodal points on its boundary. Consider inside D^2 the "band"

$$B = \{(x,y) \in D^2 : -\frac{1}{2} \leq y \leq \frac{1}{2}\}. \subset D^2.$$

The gluing process that produces $\mathbb{P}^2$ affects B in the following way: it glues the "opposite curved sides" of B as in the picture (Figure 2.17), and gives us the Moebius band.



The Moebius band inside the projective plane

Fig. 2.17

Paying attention to what happens to $D^2 - \mathrm{int}(B)$ in the gluing process, you see that it contributes with a copy of the closed disk.

Exercise 2.119. Fill in the details and deduce that $\mathbb{P}^2$ can be obtained by starting from a Moebius band and a disk, and gluing them along the boundary circle.

As a quotient modulo a group: To continue the analogy with the discussion from the torus, let us point out that the Moebius band can also be realised as a quotient modulo a group action. Again, the group is $\mathbb{Z}^2$ acting on the space $\mathbb{R}^2$, but with the action:

2.5.7 The Klein bottle

A good friend of the torus is the so called Klein bottle, call it K.

Knutsel model: As for the torus, it can be obtained from a square by gluing each pair of opposite edges, as shown in Figure 2.18, i.e., changing the orientation in one of the pairs.
Beweging model: We leave it to the imagination of the reader to find motions that produce the Klein bottle. One ends up with an explicit model of K as a subspace of $\mathbb{R}^4$:

$$K = \{((2+cos(a))cos(b), (2+cos(a))sin(b), sin(a)cos(b/2), sin(a)sin(b/2)) : a, b \in [0, 2\pi]\}. \tag{2.5.9}$$

Exercise 2.120. Explain why this subspace of $\mathbb{R}^4$ can be interpreted as the result of the gluing from Fig. 2.18.

Crafting the torus: Next, we point out that the Klein bottle is to the torus what the projective space is to the sphere: a quotient which is obtained by gluing "antipodal points". To see this, we go back to the square used to obtain the Klein bottle. Since the gluing is so problematic, let us mirror the square as in the picture; to get the Klein bottle we would first have to fold the longer square back (a gluing process itself) and perform the original gluing. However, in the longer square, the gluing that of the opposite side (which was problematic at the beginning) disappears: we

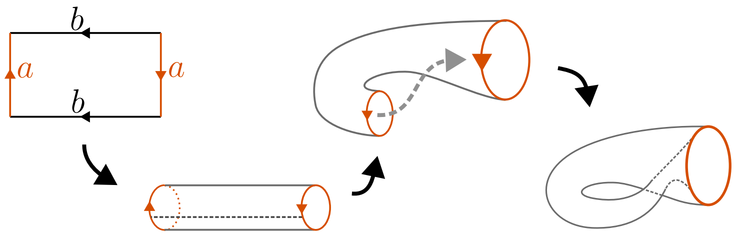


Fig. 2.18 Construction of the Klein bottle by gluing: Take a rectangle and glue the sides b together to get a cylinder. Then twist one end around so that the remaining a sides can be glued together in opposite orientation from what we would do to get a torus. Note that when visualizing this in $\mathbb{R}^3$, we are forced to create a self intersection when pulling the a sides close, but this does not affect the abstract topological gluing procedure.

can perform it and we get precisely the torus! However, to get the Klein bottle we see we have to keep on going and finish gluing the rest. Inspecting the picture, it will eventually be clear that what we still have to do is to glue points in the torus which are "antipodal" (reflections of each other with respect to the origin).

This can be further interpreted as a quotient of the torus modulo a $\mathbb{Z}_2$-action, similar to $\mathbb{P}^2 = S^2/\mathbb{Z}_2$:

$$K = T/\mathbb{Z}_2.$$

Realising the torus as $T = S^1 \times S^1$, the action of $\mathbb{Z}_2$ is simply given by $(z_1, z_2) \mapsto (-z_1, -z_2)$. While part of this discussion is purely topological and "hand waving", the last description we achieved is precise and makes sense in the realm of smooth manifolds. Actually, one can now apply Theorem 2.87 right away and make K into a smooth manifold.

Gluing two Moebius bands: In general, given two manifolds with boundary (subsection 2.1.4) N_1 and N_2, if they have the same boundary, they can be glued along their boundary producing a new manifold,

$$N_1 \cup_\partial N_2,$$

(now a manifold without boundary!). Making this precise is not completely trivial, but the statement should be clear intuitively. For instance, if N_1 is the unit disk, and $N_2 \subset \mathbb{R}^2$ consists of vectors of norm ≥ 1, then $N_1 \cup_\partial N_2$ is precisely $\mathbb{R}^2$.

This construction can be applied in particular to $N_1 = N_2 = M$ the Moebius band. Using the knutsel model for M, Figure 2.19 indicates that the outcome is precisely the Klein bottle.

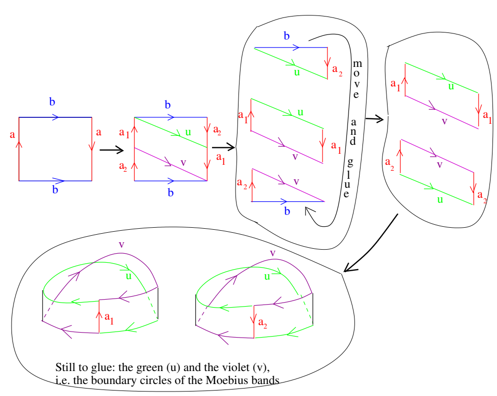


Fig. 2.19

Exercise 2.121. Explain how the picture describes the Klein bottle as the result of gluing two Moebius bands.

2.5.8 ... and the rest

Of course, the spaces that we have mentioned here: the torus, the Klein bottle, the 2-sphere S^2 and the projective plane $\mathbb{P}^2$, they are all examples of surfaces or, in our language, of 2-dimensional manifolds. One may remember from other courses (Meetkunde een Topologie?) that, at least topologically, surfaces are classified into:

- orientable ones: the sphere S^2, the torus T and then, for each g, the torus with g-wholes T_g (g-times connected sum of T with itself).
- non-orientable ones: the projective plane $\mathbb{P}^2$ and, similar to T_g, the spaces P_h obtained as the connected sum of h copies of $\mathbb{P}^2$.

The same classification result holds also in the smooth context. But, before that, one has to make sense of all these spaces as smooth manifolds. And that can be done using some basic very general operations with manifolds:

- Manifolds with boundary (subsection 2.1.4), N_1 and N_2, if they have the same boundary, they can be glued along their boundary producing a new manifold,
$$N_1 \cup_\partial N_2,$$
now a manifold without boundary!
- Given an m-dimensional manifold without boundary M, after removing a small open ball (in a coordinate chart) one obtains a manifold with boundary; let us denote it M° and call it "cut-M". It boundary is just S^{m-1}. Hence, if we have two m-dimensional manifolds M_1 and M_2 and we consider their cuts, we can just apply the gluing from the previous point. Hence one gets a new m-dimensional manifold,
$$M_1 \sharp M_2 := M_1^\circ \cup_{S^{m-1}} M_2^\circ,$$
called the connected sum of M_1 and M_2.

To do all of this in detail and properly (e.g. to see that the connected sum does not depend on how we remove the balls) requires a bit of work which goes beyond the scope of this course. But the intuition should be clear. And it can be applied right away: in this way all the surfaces

$$T_g = \underbrace{T\sharp \ldots \sharp T}_{g \text{ times}}$$

and

$$P_h = \underbrace{\mathbb{P}^2\sharp \ldots \sharp \mathbb{P}^2}_{h \text{ times}}$$

become smooth manifolds. Furthermore, various constructions that you may have seen already in Topology, makes sense in the context of smooth manifolds. For instance the fact that the projective space $\mathbb{P}^2$ can be described as obtained by gluing a disk D^2 to the Moebius band:

$$\mathbb{P}^2 = \text{Moebius} \cup_\partial D^2.$$

In turn, this property can be read a bit differently: the cut projective space (i.e. after removing a ball) is the Moebius band. Therefore, for any 2-dimensional manifold M, the operation of taking the connected sum with $\mathbb{P}^2$,

$$\mathbb{P}^2 \sharp M$$

means: remove a ball from M and, along the boundary circle, glue back a Moebius band.

Exercise 2.122. You should convince yourself now that the Klein bottle can also be described as

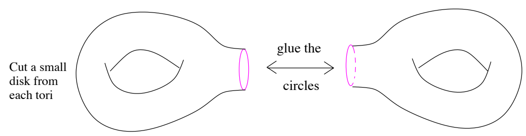


Fig. 2.20

$$K = \mathbb{P}^2 \sharp \mathbb{P}^2.$$

2.5.9 Grassmanians

Similar to considering lines through the origin (i.e. 1-dimensional vector subspaces) in $\mathbb{R}^{n+1}$ like we did for $\mathbb{P}^n$, Grassmannians are obtains when considering higher dimensional subspaces. More precisely, one defines

$$G_k(\mathbb{R}^{n+k}) := \{V \subset \mathbb{R}^{n+k} : V \text{ is a } k\text{-dimensional vector subspace of } \mathbb{R}^{n+k}\}.$$

To introduce the smooth structure, we fix an point $V \in G_k(\mathbb{R}^{n+k})$ and look at "nearby points". Using the standard inner product on $\mathbb{R}^{n+k}$, we consider the orthogonal $V^\perp \subset \mathbb{R}^{n+k}$, so that we have a direct sum decomposition

$$\mathbb{R}^{n+k} = V \oplus V^\perp.$$

The key idea is based on the intuition that, if V is thought of as an "OX" axis (and $V^\perp$ as OY), k-dimensional sub-spaces that are "close to V" look like graphs of function (from our OX to our OY). This takes the precise form of associating to any linear map $L : V \to V^\perp$ the k-dimensional subspace of $\mathbb{R}^{n+k}$:

$$V_L := \{v + L(v) : v \in V\} \subset \mathbb{R}^{n+k}.$$

Of course, V corresponds to $L = 0$. We now define

$$\mathrm{Op}(V) := \left\{V_L : L \in \mathrm{Lin}\left(V, V^\perp\right)\right\}.$$

2.5.10 More examples: classical groups and ... Lie groups

One very interesting (and special) class of manifolds are the so-called Lie groups: they are both groups, as well as manifolds, and the two structures are compatible:

Definition 2.123. A **Lie group** G is a group which is also a manifold, in a compatible fashion, i.e., such that the multiplication $m : G \times G \to G$, $m(g,h) = gh$, and the inversion $\iota : G \to G$, $\iota(g) = g^{-1}$, are smooth maps.

A **Lie group homomorphism** between two Lie groups G and H is any group homomorphism $f : G \to H$ which is also smooth. When f is also a diffeomorphism, we say that f is an isomorphism of Lie groups.

Example 2.124. The unit circle S^1, identified with the space of complex numbers of norm one,

$$S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

is a Lie group with respect to the usual multiplication of complex numbers. Actually, we see that

$$S^1 = SU(1).$$

Example 2.125. Similarly, the 3-sphere S^3 can be made into a (non-commutative, this time) Lie group. For that we replace $\mathbb{C}$ by the space of quaternions:

$$\mathbb{H} = \{x + iy + jz + kt : x, y, z, t \in \mathbb{R}\}$$

where we recall that the product in $\mathbb{H}$ is uniquely determined by the fact that it is $\mathbb{R}$-bilinear and $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. Recall also that for

$$u = x + iy + jz + kt \in \mathbb{H}$$

one defines

$$u^* = x - iy - jz - kt \in \mathbb{H}, \ \ |u| = \sqrt{uu^*} = \sqrt{x^2 + y^2 + z^2 + t^2} \in \mathbb{R}.$$

Then, the basic property $|u \cdot v| = |u| \cdot |v|$ still holds and we see that, identifying S^3 with the space of quaternionic numbers of norm 1, S^3 becomes a Lie group.

Remark 2.126 (For the interested students: spheres, Lie groups, etc). One may wonder: which spheres can be made into Lie groups? Well, it turns out that S^0, S^1 and S^3 are the only ones!

On the other hand, it is interesting to understand what happens with the arguments we used for S^1 and S^3 (and which were very similar to each other) in higher dimensions. The main point there was the multiplication on $\mathbb{C}$ and $\mathbb{H}$ and the presence of a norm such that

$$|x \cdot y| = |x| \cdot |y| \tag{2.5.10}$$

(so that, for two elements in the sphere, i.e. of norm one, their product is again in the sphere). So, to handle S^n similarly, we would need a "normed division algebra" structure on $\mathbb{R}^{n+1}$, by which we mean a multiplication "$\cdot$" on $\mathbb{R}^{n+1}$ that is bilinear and a norm satisfying the previous condition. Again, it is only on $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^4$ that such a multiplication exists. If we do not insist on the associativity of the multiplication, there is one more possibility: $\mathbb{R}^8$ (the so called octonions). But nothing else! And this was known since the 19th century! But why did people care about such operations in the 19th century? Well ... it was number theory and the question of which numbers (integers) can be written as a sum of two, three, etc squares. For sum of two squares the central formula which shows that a product of two numbers that can be written as a sum of two squares can itself be written as a sum of two squares is:

$$(x^2 + y^2)(a^2 + b^2) = (xa - yb)^2 + (xb + ya)^2.$$

Or, in terms of the complex numbers $z_1 = x + iy, z_2 = a_ib$, the norm equation (2.5.10). The search for similar "magic formulas" for sum of three squares never worked, but it did for four:

$$(x^2 + y^2 + z^2 + t^2)(a^2 + b^2 + c^2 + d^2) =$$

$$(xa + yb + zc + td)^2 + (xb - ya - zd + tc)^2 +$$
$$+ (xc + yd - za - tb)^2 + (xd - yc + zb - ta)^2.$$

This is governed by the quaternions and its norm equation (2.5.10).

Example 2.127 (GL_n). Probably the most important example is the **general linear group** $GL_n(\mathbb{R})$, consisting of $n \times n$ invertible matrices with real entries:

$$GL_n(\mathbb{R}) = \{A \in \mathscr{M}_{n\times n}(\mathbb{R}) : \det(A) \neq 0\}$$

They sit inside the space of all $n \times n$ matrices (for some n natural number)- which is itself a Euclidean space (just that the variables are arranged in a table rather than in a row):

$$\mathscr{M}_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}.$$

Furthermore, since the determinant is continuous as a map

$$\det : \mathscr{M}_{n\times n}(\mathbb{R}) \to \mathbb{R},$$

it follows that $GL_n(\mathbb{R})$ is open inside $\mathscr{M}_n(\mathbb{R})$. Therefore $GL_n(\mathbb{R})$ inherits a natural smooth structure: the one induced by the atlas consisting of one single chart (with image open in $\mathbb{R}^{n^2}$):

$$GL_n(\mathbb{R}) \ni A = (A^i_j)_{ij} \quad \overset{\chi}{\longmapsto} \quad (A^1_1,\ldots,A^1_n,A^2_1,\ldots,A^2_n,\ldots,A^n_1,\ldots,A^n_n) \in \mathbb{R}^{n^2}.$$

Hence, indeed, $GL_n(\mathbb{R})$ is a Lie group of dimension n^2. A similar discussion applies to matrices with complex coefficients, giving rise to a similar Lie group $GL_n(\mathbb{C})$; identifying $\mathbb{C}$ with $\mathbb{R}^2$, one has:

$$\mathscr{M}_{\times n}(\mathbb{C}) \cong \mathbb{R}^{(2n)^2},$$

so that $GL_n(\mathbb{C})$ will be a Lie group of dimension $4n^2$.

Example 2.128 (subgroups of GL_n). Inside the GL_ns one can find several interesting Lie groups such as

- **the orthogonal group**: $O(n) = \{A \in GL_n(\mathbb{R}) : A\cdot A^T = I\}$ (where A^T denotes the transpose of A).
- **the special orthogonal group**: $SO(n) = \{A \in O(n) : \det(A) = 1\}$.
- **the special linear group**: $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}$.
- **the unitary group**: $U(n) = \{A \in GL_n(\mathbb{C}) : A\cdot A^* = I\}$ (where A^* is the conjugate transpose).
- **the special unitary group**: $SU(n) = \{A \in GL_n(\mathbb{C}) : A\cdot A^* = I, \det(A) = 1\}$.
- **the symplectic group**: $Sp_n(\mathbb{R}) := \{A \in GL_{2n}(\mathbb{R}) : A^T JA = J\}$, where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in GL_{2n}(\mathbb{R}). \tag{2.5.11}$$

Each one of these caries a natural smooth structure; actually, they are all embedded submanifolds of the corresponding (Euclidean) spaces of matrices (see below). Since the group operations are restrictions of the ones of the GL_ns, they will continue to be smooth- so that all these groups will be Lie groups.

For the claim that they are embedded submanifolds of $GL_n \subset \mathscr{M}_{n\times n}$, since they are all given by (algebraic) equations, there is a natural to proceed: use the regular value theorem (Theorem 2.66). As an illustration, we provide the details for the case of $O(n)$.

Example 2.129 (the regular value theorem approach in the case of $O(n)$). The equation defining $O(n)$ suggest we should be looking at

$$f : GL_n(\mathbb{R}) \to \mathscr{M}_{n\times n}(\mathbb{R}), \quad f(A) = A\cdot A^T.$$

However, it is important to note that the values of this function takes values in a smaller Euclidean space. Indeed, since in general, $(A\cdot B)^T = B^T\cdot A^T$, the matrices of type $A\cdot A^T$ are always symmetric. Therefore, denoting by $\mathscr{S}_n$ the space of symmetric $n\times n$ matrices (again a Euclidean space, of dimension $\frac{n(n+1)}{2}$), we will deal with f as a map

$$f : GL_n(\mathbb{R}) \to \mathscr{S}_n.$$

(If we didn't remark that f was taking values in $\mathscr{S}_n$, we would not have been able to apply the regular value theorem; however, the problem that we would have encountered would clearly indicate that we have to return and make use of $\mathscr{S}_n$ from the beginning; so, after all, there is no mystery here).

Now, f is a map from an open in a Euclidean space (and we could even use f defined on the entire $\mathscr{M}_{n\times n}(\mathbb{R})$), with values in another Euclidean space; and it is clearly smooth since it is given by polynomial expressions. So, at each point $A \in GL_n(\mathbb{R})$, the differential of f at A,

$$(df)_A : \mathscr{M}_{n\times n}(\mathbb{R}) \to \mathscr{S}_n$$

can be computed by the usual formula:

$$(df)_A(X) = \frac{d}{dt}\Big|_{t=0} f(A+tX) = \frac{d}{dt}\Big|_{t=0} \left(A\cdot A^T + t(A\cdot X^T + A^T\cdot X) + t^2 X\cdot X^T\right) = A\cdot X^T + X\cdot A^T.$$

Since $O(n) = f^{-1}(\{I\})$, we have to show that $(df)_A$ is surjective for each $A \in O(n)$. I.e., for $Y \in \mathscr{S}_n$, show that the equation $A\cdot X^T + X\cdot A^T = Y$ has a solution $X \in \mathscr{M}_{n\times n}(\mathbb{R})$. Well, it does, namely: $X = \frac{1}{2}YA$ (how did we find it?). Therefore $O(n)$ is a smooth submanifold of GL_n of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Exercise 2.130. Do the same for the other groups in the list. At least for one more.

On the other hand, instead of doing a case-by-case analysis, one can also invoke the following general result:

Theorem 2.131. *Any closed subgroup of $GL_n(\mathbb{R})$ or $GL_n(\mathbb{C})$ is automatically an embedded submanifold and, therefore, becomes a Lie group.*

Of course, this theorem is very useful and one should at least be aware of it. A proof will be given at the end of this section.

Example 2.132 (low dimensions). For n small, the classical subgroups of GL_n have can be further recognized as manifolds/groups that we have already looked at. Here are a couple of examples which we leave as exercises:

Exercise 2.133. Show that

$$f : S^1 \to SO(2),\ \ f(x,y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

is an isomorphism of Lie groups.

Exercise 2.134. Show that

$$F : S^3 \to SU(2),\ \ F(\alpha,\beta) = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

where we interpret S^3 as $\{(\alpha,\beta) \in \mathbb{C}^2 : |\alpha|^2 + |\beta|^2 = 1\}$, is an isomorphism of Lie groups.

By analogy with the last example, one may expect that S^3 is isomorphic also to $SO(3)$ (at least the dimensions match!). However, the relation between these two is more subtle:

Exercise 2.135. Recall that we view S^3 inside $\mathbb{H}$. We also identify $\mathbb{R}^3$ with the space of pure quaternions

$$\mathbb{R}^3 \xrightarrow{\sim} \{v \in \mathbb{H} : v + v^* = 0\},\ (a,b,c) \mapsto ai + bj + ck.$$

For each $u \in S^3$, show that $A_u(v) := u^*vu$ defines a linear map $A_u : \mathbb{R}^3 \to \mathbb{R}^3$ which, as a matrix, gives an element $A_u \in SO(3)$. Then show that the resulting map

$$\phi : S^3 \to SO(3),\ u \mapsto A_u$$

is smooth, is a group homomorphism, is a surjective local diffeomorphism, but each fiber has two elements (it is a 2-1 cover). Finally, deduce that $SO(3)$ is diffeomorphic to the real projective space $\mathbb{P}^3$.

Exercise 2.136. The aim of this exercise is to show how $GL_n(\mathbb{C})$ sits inside $GL_{2n}(\mathbb{R})$, and similarly for the other complex groups. Well, there is an obvious map

$$j : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R}), \quad A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

Show that:

1. j is an embedding, with image $\{X \in GL_{2n}(\mathbb{R}) : J \cdot X = X \cdot J\}$, where J is the matrix (2.5.11).
2. j takes $U(n)$ into $O(2n)$.
3. actually j takes $U(n)$ into $SO(2n)$.

(Hint for (3): prove that $\det(j(Z)) = |\det(Z)|^2$ for all $Z \in GL_n(\mathbb{C})$. For this: show that for B invertible, there exists a matrix $X \in GL_n(\mathbb{R})$ such that

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & X^{-1} \\ -I & X \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

Proof (of Theorem 2.131 ... for the interested students). We will be using the exponential map of matrices ($\exp(A) = \sum_n \frac{A^n}{n!}$), which can be seen as a smooth map

$$\exp : \mathscr{M}_{n\times n} \to GL_n \subset \mathscr{M}_{n\times n}$$

Since its differential at the zero-matrix is

$$(d\exp)_0 : \mathscr{M}_{n\times n} \to \mathscr{M}_{n\times n}, \quad X \mapsto \frac{d}{dt}\Big|_{t=0} \exp(tX) = X,$$

the inverse function theorem implies that exp restricts to a diffeomorphism between some open neighborhood of $0 \in \mathscr{M}_{n\times n}$ to an open neighborhood of $I \in \mathrm{GL}_n$. The resulting inverse, that could be called "logarithm", can then be interpreted as chart for GL_n.

For closed subgroups $G \subset GL_n$ there is a rather simple idea to exhibit a smooth structure on G: looking for matrices whose exponential lands in G one introduces

$$\mathfrak{g} := \{X \in \mathscr{M}_{n\times n} : \exp(tX) \in G \ \forall\, t \in \mathbb{R}\} \subset \mathscr{M}_{n\times n}(\mathbb{R}), \tag{2.5.12}$$

and the idea is to use $\exp|_{\mathfrak{g}}$ to produce a chart for G around $\exp(0) = I$ (the identity matrix); for charts around other points $A \in \mathrm{GL}_n$, one uses the group structure to move from I (and around) to A. Of course, exhibiting charts on G to make G into a manifold would not be necessary if we manage to prove (as stated) that G is an embedded submanifold. The point of the simple idea above is that it can actually be adjusted to provide not only charts for $\mathfrak{g}$ but also adapted charts- therefore proving the stronger statement that G is an embedded submanifold. Time to start the proof!

Therefore, assume that $G \subset GL_n$; for simplicity, assume we work over $\mathbb{R}$. We introduce $\mathfrak{g}$ given by (2.5.12). It is handy to realize what G being closed implies about $\mathfrak{g}$: to ensure that $\exp(tX) \in G$ holds *for all* $t \in \mathbb{R}$, it suffices that it holds for a sequence $t_i \to 0$ of nonzero real numbers. The proof actually reveals something slightly stronger:

Lemma 2.137. *Given a matrix X, if there exists a sequence of nonzero real numbers with $t_i \to 0$, and a sequence of matrices $X_i \to X$ such that $\exp(t_iX_i) \in G$ for all i, then $\exp(tX) \in G$ for all t (i.e., $X \in \mathfrak{g}$).*

Proof. Since $\exp(-Y) = \exp(Y)^{-1}$ for all Y, after eventually changing the signs of some t_is and t, we may assume that $t_i > 0$ for all i and that we are looking at $\exp(tX)$ with $t > 0$. Fix t. For each i, the positive number t/t_i sits in an interval $[k_i, k_i + 1]$ for some integer k_i. One finds that $t - t_i \leq k_i t_i \leq t$, i.e. we managed to write our t as a limit of integer multiples of the t_is- namely $t = \lim_{i\to\infty} k_i t_i$. In turn, this allows us to write $\exp(tX)$ as a limit of elements of G:

$$\exp(tX) = \lim_{i\to\infty} \exp(k_i t_i X_i) = \lim_{i\to\infty} \exp(t_i X_i)^{k_i}.$$

Since G is closed, we obtain $\exp(tX) \in G$.

With this lemma at hand one can now show that the sum of two matrices $X, Y \in \mathfrak{g}$ is again in $\mathfrak{g}$. Indeed, writing

$$e^{tX} e^{tY} = e^{f(t)}, \quad \text{with} \quad \lim_{t\to 0} \frac{f(t)}{t} = X + Y,$$

take t_n to be any sequence of real numbers converging to 0 and $X_n = \frac{f(t_n)}{t_n}$. It follows that $\mathfrak{g}$ is a vector subspace of $\mathscr{M}_{n\times n}$. In turn, this allows us to choose a complement $\mathfrak{g}'$ of $\mathfrak{g}$ in $\mathscr{M}_{n\times n}(\mathbb{R})$- which is needed to adapt the original idea we mentioned above to proving that G is an embedded submanifold. More precisely, we aim to show that one can find an open neighborhood V of the origin in $\mathfrak{g}$, and a similar one V' in $\mathfrak{g}'$, so that

$$\phi : V \times V' \to GL_n, \phi(X, X') = e^X \cdot e^{X'}$$

is a local diffeomorphism onto an open neighborhood of the identity matrix and so that

$$G \cap \phi(V \times V') = \{\phi(v,0) : v \in V\}. \tag{2.5.13}$$

This means that the submanifold condition is verified around the identity matrix and then, using left translations, it will hold at all points of G. Note also that the differential of ϕ at 0 (with ϕ viewed as a map defined on the entire $\mathfrak{g} \times \mathfrak{g}'$) is the identity map:

$$(d\phi)_0 : \mathfrak{g} \times \mathfrak{g}' \to T_I GL_n = \mathscr{M}_{n\times n}(\mathbb{R}), (X, X') \mapsto \frac{d}{dt}_{t=0} e^{tX} e^{tX'} = X + X'.$$

In particular, ϕ is indeed a local diffeomorphism around the origin. Hence the main condition that we have to take care of is (2.5.13), i.e. that if $X \in V$, $X' \in V'$ satisfy $e^X \cdot e^{X'} \in G$, then $X' = 0$. Hence it suffices to show that one can find V' so that

$$X' \in V', e^{X'} \in G \Longrightarrow X' = 0$$

(then just choose any V small enough such that ϕ is a diffeomorphism from $V \times V'$ into an open neighborhood of the identity in GL_n). For that we proceed by contradiction. If such V' would not exist, we would find a sequence $Y_n \to 0$ in $\mathfrak{g}'$ such that $e^{Y_n} \in G$. Since Y_n converges to 0, we find a sequence of integers $k_n \to \infty$ such that $X_n := k_n Y_n$ stay in a closed bounded region of $\mathfrak{g}'$ not containing the origin (e.g. on $\{X \in \mathfrak{g}' : 1 \leq ||X|| \leq 2\}$, for some norm on $\mathfrak{g}'$); then, after eventually passing to a subsequence, we may then assume that $X_n \to X \in \mathfrak{g}'$ for non-zero X. On the other hand, setting $t_n = \frac{1}{k_n}$, we see that $e^{t_n X_n} = e^{Y_n} \in G$ hence Lemma 2.137 would imply $X \in \mathfrak{g}$, providing the desired contradiction. 😀

For closed subgroups $G \subset GL_n$, the spaces $\mathfrak{g}$ used in the proof above (given by (2.5.12)) are not just a tool but, as we shall see above, are of fundamental importance: they are are the linear counterpart of Lie groups and, together with the relevant algebraic structure, they contain almost all the information on G. The relevant structure is, besides that of vector space, the one provided by the commutator of matrices:

$$[X,Y] = XY - YX.$$

Using an argument similar to the one we used to prove that $\mathfrak{g}$ is a vector subspace of $\mathscr{M}_{n\times n}$, you can now try to show the following:

Exercise 2.138. Given a closed subgroup $G \subset GL_n$, show that $\mathfrak{g}$ is closed under the commutator bracket of matrices:

$$X, Y \in \mathfrak{g} \Longrightarrow [X,Y] \in \mathfrak{g}.$$

(Hint: use an argument similar to the one we used to prove that $\mathfrak{g}$ is a vector subspace of $\mathscr{M}_{n\times n}$, making use of Lemma 2.137).