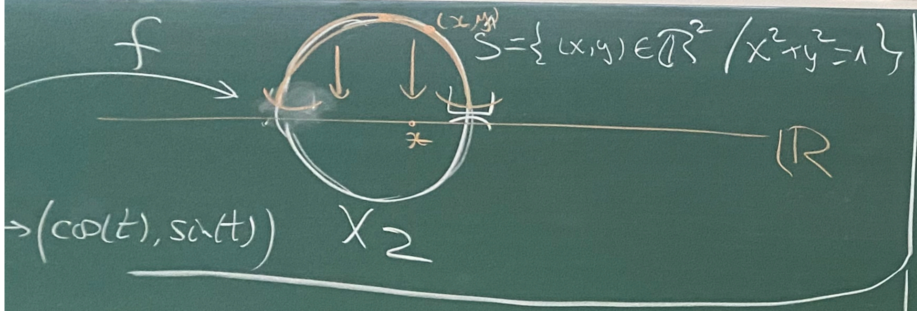
1 Course	objects:	morphisms (maps)	Some keywords:	Isomorphisms/equivalences	
Linear Algebra	vector spaces $(V, +, \cdot)$	linear maps (matrices)	addition of vectors, multiplication by scalars, vector subspaces,	linear isomorphisms $f: V_1 \longrightarrow V_2$ linear & bijective	vectors $v \in V$ $(\Rightarrow f^{-1} = \text{linear})$
Group Theory	groups $(G, \circ, inv)$	group (homo)morphisms	multiplication, inverse, associativity, identity, normal subgroup, order,	group isomorphism $f: G_1 \longrightarrow G_2$ homomorphism & bijective	elements $g \in G$ $(\Rightarrow f^{-1} = \text{morphism})$
Topology	topological spaces $(X, \mathcal{T})$	continuous functions	opens, topology, convergent sequences, continuity of map, subspaces, ...	homeomorphisms $f: X_1 \longrightarrow X_2$ continuous & bijective bij & f, f^{-1} - continuous	points $p \in X$ $\mathbb{R}, \mathbb{R}^2$ $\mathbb{R}^2, \mathbb{R}^3$ $(\not\Rightarrow f^{-1} = \text{continuous})$
Manifolds (Differential Geometry)	Manifolds $(M, \mathcal{A})$	smooth maps	smoothness, charts, speed of curves, tangent vectors, differential forms	diffeomorphisms	$\mathbb{R}^2, \mathbb{R}^3$ points $p \in M$.



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hic?

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Charts: (2)

- a chart on a set M : a ~~triple~~ pairs $(U, x, \cancel{\Omega})$ $x(U)$
  m-dimensional

  with $\begin{cases} U \subseteq M \text{ open} \\ x(U) \subseteq \mathbb{R}^m \text{ open} \\ x : U \longrightarrow \Omega \ \ \cancel{\text{bijection}} \text{ homeomorphism} \end{cases}$

  Call U: the domain of the chart and, for $p \in U$ $(x_1(p), \dots, x_m(p))$ the coordinates of p w.r.t (U, x)

- a topological chart on a topological Space M.
  m-dim. chart

- a smooth m-dim chart on M is the same, but requiring now $x =$ diffeomorphism

③ Smoothness of maps $f: M \to \mathbb{R}^k$;
when / for which Ms do we know what it means?

For general sets? NO. For:

① $M = \mathbb{R}^m$, $M = \Omega \subseteq \mathbb{R}^m$ open subsets:
(Def 1.20, page 15) done in 1st lectures in Analysis

② $M \subseteq \mathbb{R}^L$ subset:
$(\forall) p \in M$, $\exists\, \underline{U \subseteq M \text{ open}}$, $p \in U$
$\exists\, \underline{\tilde{U} \subseteq \mathbb{R}^L \text{ open}}$ s.t. $U = \underline{\tilde{U} \cap M}$
$\exists\, \tilde{f}: \tilde{U} \to \mathbb{R}^k$ $\underline{\text{smooth}}$ s.t. $\tilde{f}|_U = f|_U$.

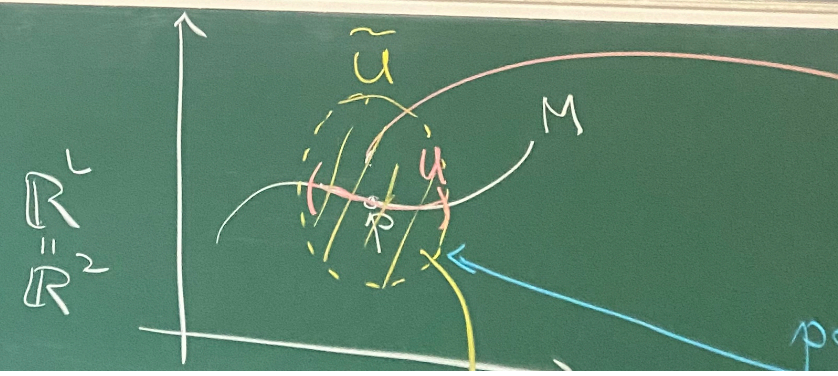
$\Rightarrow$ we can talk about diffeomorphisms between subsets of Euclidean spaces: f = bij, with f, f^{-1} = smooth

Ex: $f: S^1 \to \mathbb{R}$, $f(x,y) = x$: $U = S^1$, $\tilde{U} = \mathbb{R}^2$, $\tilde{f}(x,y) = x$

Ex: $f: S^1 \to \mathbb{R}$, $f(x,y) = \dfrac{x}{x^4 + y^4 - \frac{1}{4}}$, Given $p = (x_0, y_0)$ $\tilde{U} = \ldots$ $\tilde{f}(x,y) = \dfrac{x}{x^4+y^4-\frac{1}{4}}$ defined on $\tilde{U}$.
$x^4 + y^4 \geq \dfrac{(x^2+y^2)^2}{2} = \dfrac{1}{2}$

...jection
... - count
...hisms

...quivalent:



Theorem 1.38 (& 1.37) ④: Given $M \subseteq \mathbb{R}^L$, $p \in M$. The following conditions are equivalent:

(1) "the m-dimensional manifold condition at p"; i.e.,

$\exists$ smooth chart of M

$$x : \underset{\underset{M}{\cap \text{open}}}{U} \longrightarrow \underset{\underset{\mathbb{R}^m}{\cap \text{open}}}{\Omega} \quad \text{with } p \in U, \text{ diffeomorphism.}$$

(2) "m-dimensional parametrization at p", i.e.,

$\exists$ homeomorphism

$$par : \underset{\underset{\mathbb{R}^m}{\cap \text{open}}}{\Omega} \longrightarrow \underset{\underset{M \subseteq \mathbb{R}^L}{\cap}}{U} \quad \text{with } p \in U,$$

which as a map $par : \Omega \to \mathbb{R}^L$ is a smooth immersion

(3) "m-dimensional implicit eq at p", i.e.,

$\exists$ smooth submersion

$$eq : \underset{\underset{\mathbb{R}^L}{\cap \text{open}}}{\tilde{U}} \longrightarrow \mathbb{R}^{L-m} \quad \text{s.t. } M \cap \tilde{U} = \{ q \in \tilde{U} : eq(q) = 0 \}$$

$p \in \tilde{U}$

set: ... $\Rightarrow$ we can talk about diffeomorphisms between subsets of Euclidean spaces: f = bij with f, f^{-1} = smooth

$\subseteq M$ open, $p \in U$

$\subseteq \mathbb{R}^L$ open s.t. $U = \tilde{U} \cap M$

$\{ x : U \to \Omega$ ~~bijection~~ homeomorphism

Call U: the domain of the chart and, for $p \in U$: $(x_1(p), \dots, x_m(p))$ the coordinates of p w.r.t. (U, x)

2) a topological chart on a topological ...

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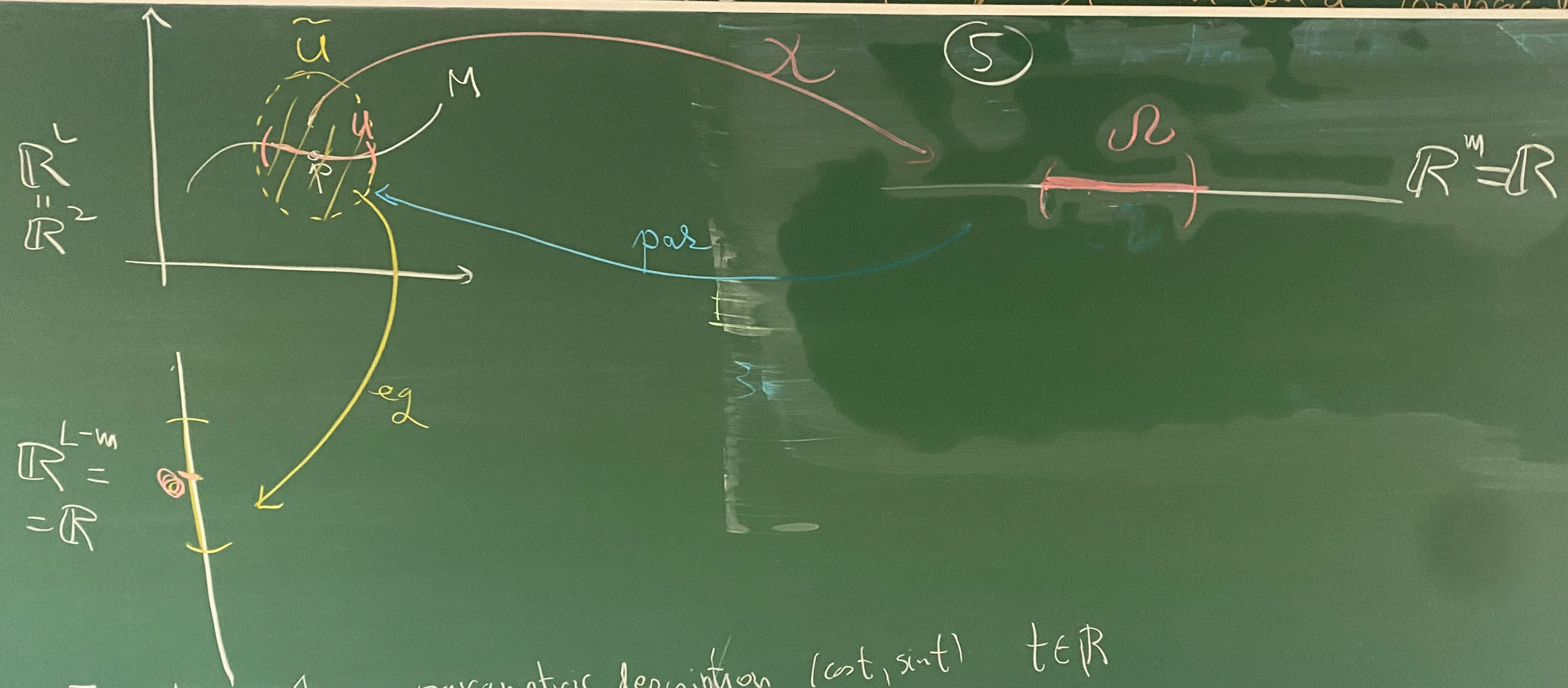

Ex: $M = S^1$ parametric description $(\cos t, \sin t)$ $t \in \mathbb{R}$

equation $x^2 + y^2 = 1$