1

Reminder:

Charts:

- an m-dimensional chart on a set M: a pair (U, X)

M=set

$$\begin{cases} U \subseteq M & \text{open} \\ \Omega = X(U) \subseteq \mathbb{R}^m & \text{open} \\ X : U \longrightarrow \Omega = X(U) & \cancel{\text{bijection}} \quad \cancel{\text{homeomorphism}} \quad \text{diffeomorphism} \\ \quad\ p \mapsto (x_1(p), \dots, x_m(p)) & \end{cases}$$

the coordinates of p w.r.t. (U, X)

M= space

- an m-dimensional topological chart on a topological space M: with additions

M=?

- given "a nice M": an m-dimensional smooth chart on M: with additions:

("diffeom" $f \equiv f$ bijection s.t. f, f^{-1} are "smooth")

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Thm 1.38 & 1.37: Given $M \subseteq \mathbb{R}^L$, $p \in M$, the follo[wing] are equivalent:

(1) manifold condition at p:

$\exists$ smooth classical chart $X : \underset{\substack{\cap\,\text{open} \\ M}}{U} \longrightarrow \underset{\substack{\cap\,\text{open} \\ \mathbb{R}^m}}{\Omega}$ with $p \in U$

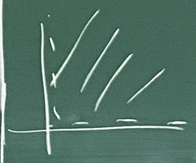
(2) parametrization condition at p:

Cases in which "diffeomorphism" (hence ②
also "smooth chart" makes sense:

① $M = \mathbb{R}^L$ (or open in $\mathbb{R}^L$): most usual notion of smoothness $\Longrightarrow$

$\Rightarrow$ can talk about classical smooth chart on $\mathbb{R}^L$

ex. $U = \{(x,y) \in \mathbb{R}^2 : x>0, y>0\}$



$x : U \longrightarrow (0,\infty)\times\left(0,\frac{\pi}{2}\right)$, $(x,y) \longmapsto (r,\theta)$ s.t. $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$

② $M \subseteq \mathbb{R}^L$ any subset: use the improvised Def. 1.34 $\Rightarrow$

$\Rightarrow$ can talk about classical smooth charts on M
(... if they exist)

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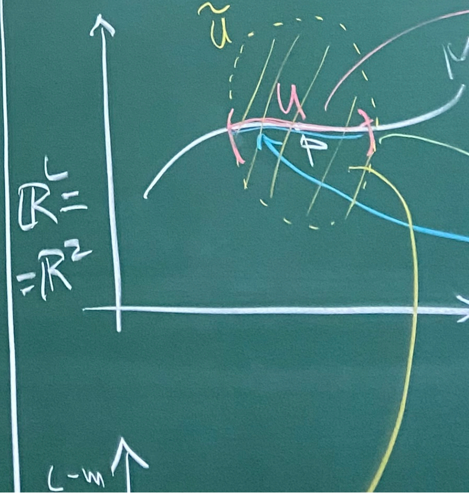
(4) adapted chart condition at p: ④

$\exists$ classical smooth chart of $\mathbb{R}^L$

$$\tilde{x} : \underset{\underset{\mathbb{R}^L}{\cap\!\!|\,\circ}}{\tilde{U}} \longrightarrow \underset{\underset{\mathbb{R}^L}{\cap\!\!|\,\circ}}{\tilde{\Omega}}, \quad p \in \tilde{U}$$

that is adapted to M, i.e.: denoting

$L=2$, $m=1$, $L-m=$



$M =$: space

- an m-dimensional topological chart on a topological space M : with additions

$M = ?$

- given "a nice M": an m-dimensional Smooth chart on M : with additions:

("diffeom" $f \equiv f$ bijection s.t. f, f^{-1} are "smooth")

Thm 1.38 & 1.37 ③: Given $M \subseteq \mathbb{R}^L$, $p \in M$, the followi[ng] are equivalent:

(1) manifold condition at p:

$\exists$ smooth chart (classical) $x : U \longrightarrow \Omega$ with $p \in U$, $U \underset{\text{open}}{\subseteq} M$, $\Omega \underset{\text{open}}{\subseteq} \mathbb{R}^m$

(2) parametrization condition at p:

$\exists$ homeomorphism $par : \Omega \longrightarrow U$ with $p \in U$, $\Omega \underset{o}{\subseteq} \mathbb{R}^m$, $U \underset{o}{\subseteq} M \subseteq \mathbb{R}^L$

Such that, as a map $par : \Omega \to \mathbb{R}^L$, it is a smooth immersion (NICE).

(3) equation condition at p:

NICE $\exists$ Smooth Submersion $eq : \tilde{U} \longrightarrow \mathbb{R}^{L-m}$, $\tilde{U} \underset{o}{\subseteq} \mathbb{R}^L$ such that

$$M \cap \tilde{U} = \{ q \in \tilde{U} \mid eq(q) = 0 \}$$

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Def
in $\mathbb{R}$
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the improvised Def. 1.34 $\Rightarrow$

$\Rightarrow$ can talk about classical smooth chart

(... if they exist)

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(4) adapted chart condition at p: (4)

$\exists$ classical smooth chart of $\mathbb{R}^L$

$$\begin{array}{ccc} \tilde{x} : \tilde{U} & \longrightarrow & \tilde{\Omega} \\ \cap \text{l} \circ & & \cap \text{l} \circ \\ \mathbb{R}^L & & \mathbb{R}^L \end{array}, \quad p \in \tilde{U}$$

that is adapted to M, i.e.: denoting

$$U = \tilde{U} \cap M$$

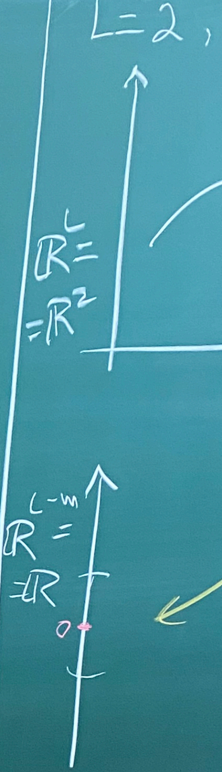
$$\Omega = \tilde{\Omega} \cap (\mathbb{R}^m \times \{0\})$$

one has

$$\tilde{x}(U) = \Omega$$

NICE smooth immersion

Def: An "m-dimensional embedded manifold in $\mathbb{R}^L$" is any $M \subseteq \mathbb{R}^L$ s.t., $\forall p \in M$, these (equivalent) conditions hold true.



$(x_1, \dots, x_k), x_{k+1}, \dots$

smooth charts on M

exist)

... locally, g "looks like"

in the appropr

$L = 2,\ m = 1,\ L - m = 1$

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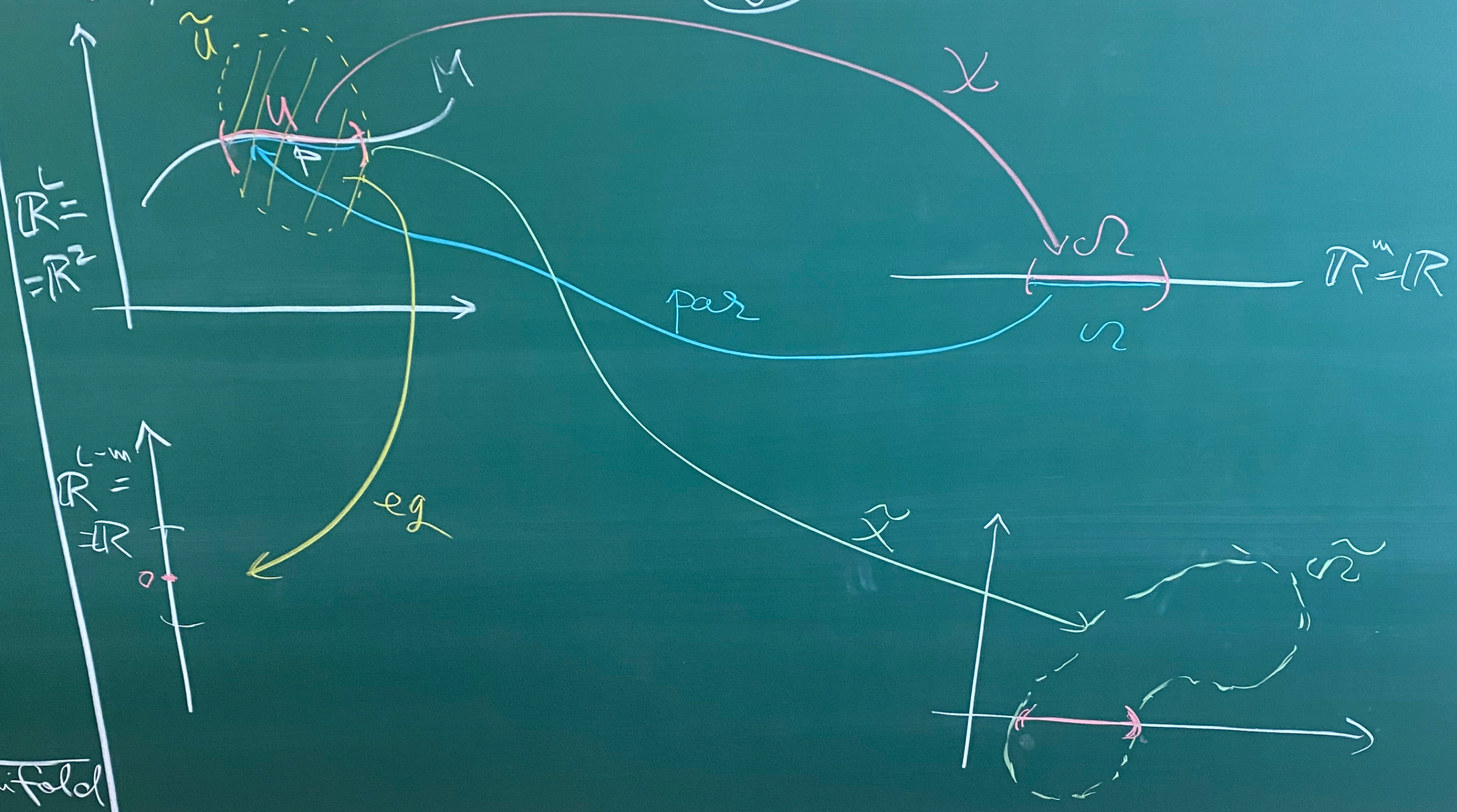
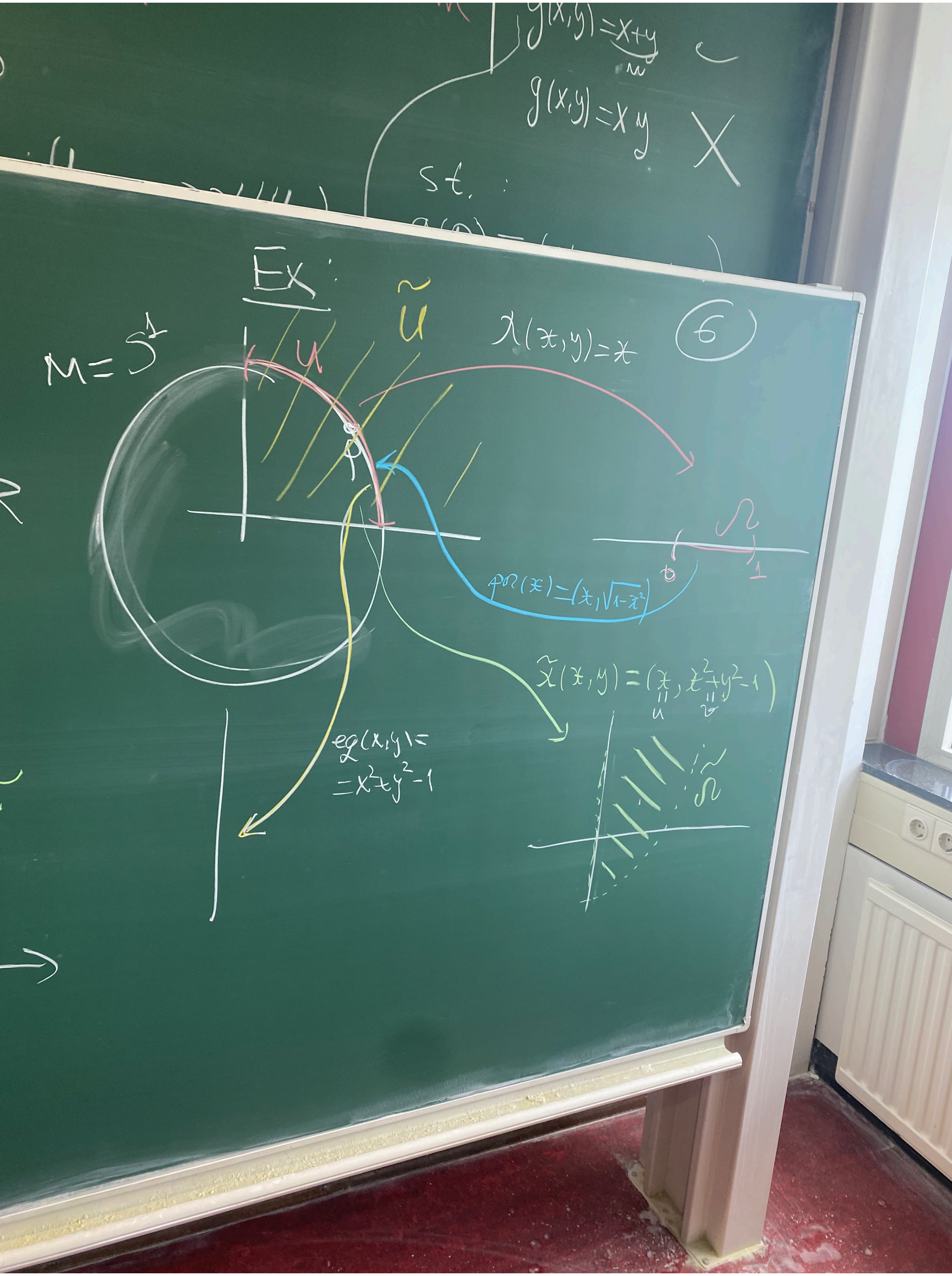


ung

manifold

$\forall, p \in M,$

true.


g(x,y) = x+y
g(x,y) = xy
st.
Ex:
M = S^1
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χ(x,y) = x
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p
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0
1
φΩ(x) = (x, √(1−x²))
χ̃(x,y) = (x, x²+y²−1)
u
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Ω̃
eg(x,y) =
= x²+y²−1

on $\mathbb{R}^L$

st.

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on M

Submersion: $g: \Omega \longrightarrow \mathbb{R}^k$, $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^L$

Submersion if it is a submersion at all $p \in \Omega$:

i.e.

- $(dg)_p : \mathbb{R}^L \longrightarrow \mathbb{R}^k$ surjective

$\Updownarrow$

- $L \geq k$ & $\left(\frac{\partial g_i}{\partial x_j}(p)\right)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq L}}$ has (maximal) rank k

Submersion theorem $\Downarrow$

- $\begin{cases} \exists\, U_p \subseteq \Omega \text{ open}, p \in U_p \\ \exists \text{ classical smooth chart } \chi : U_p \to \chi(U_p) \end{cases}$

$U_p \subseteq \Omega$, $\chi(U_p) \subseteq \mathbb{R}^L$

$\chi = (\underbrace{\chi_1, \dots, \chi_k}, \chi_{k+1}, \dots, \chi_L)$

Intuitively:

$g: \mathbb{R}^2 \to \mathbb{R}$

$g(x,y) = x$ ✓

$g(x,y) = x+y$ ✓

$g(x,y) = xy$ ✗

st.

$g(p) = (\chi_1(p), \dots, \chi_k(p))$

ex. $\chi(x,y) = (\underbrace{x+y}_{u}, \underbrace{x}_{v})$

$\mathcal{C}(M, \mathbb{R})$

Submersion: $g : \Omega \longrightarrow \cancel{\mathbb{R}^k}\ \mathbb{R}^L$

$\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^m$ ~~$\mathbb{R}^L$~~

~~submersion~~ immersion if it is ~~submersion~~ immersion at all $p \in \Omega$:

i.e.:

- $(dg)_p : \mathbb{R}^{\cancel{k}\, m} \longrightarrow \mathbb{R}^{\cancel{k}\, L}$ ~~surjective~~ injective

$\Updownarrow$

- $L \geq \cancel{k}\, m$ & $\left(\dfrac{\partial g_i}{\partial x_j}(p)\right)_{\substack{1 \leq i \leq \cancel{k} L \\ 1 \leq j \leq \cancel{k} m}}$ has (maximal) rank $\cancel{k}\, m$

Submersion theorem $\Updownarrow$

- ~~$\exists\, U_p \subseteq \Omega$ open, $p \in U_p$~~
  ~~$\exists$ classical smooth chart $\chi : U_p \longrightarrow \chi(U_p)$, $U_p \subseteq \Omega$, $\chi(U_p) \subseteq \mathbb{R}^L$, $\chi = (\chi_1, \ldots, \chi_k, \chi_{k+1}, \ldots, \chi_L)$~~

- ... locally, g "looks like" $(x_1, \ldots, x_m) \longrightarrow (x_1, \ldots, x_m, \underbrace{0, \ldots, 0}_{L-m})$ in the appropriate coordinates

Intuitively:

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$g(x,y) = x$ ✓

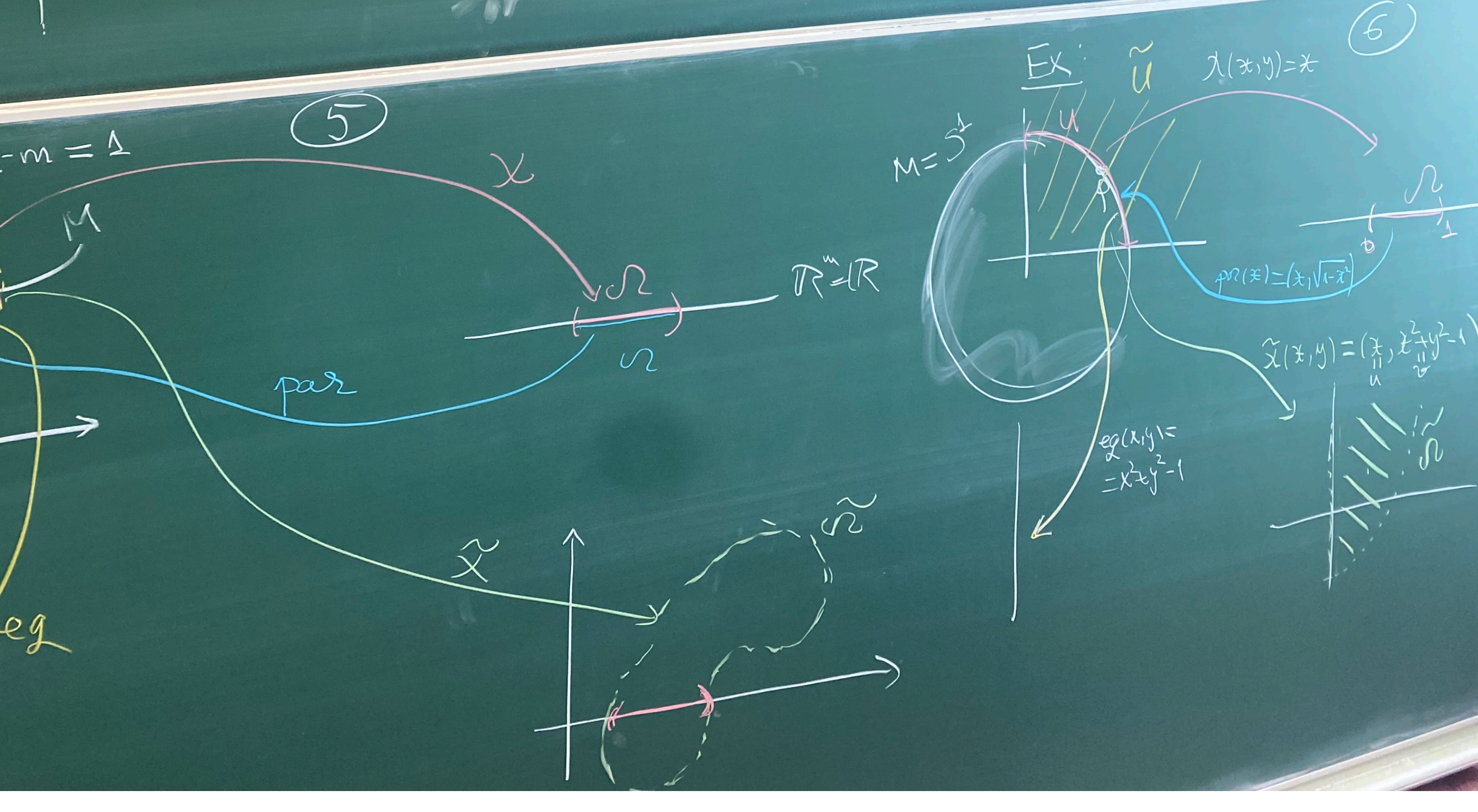
$g(x,y) = \underbrace{x+y}_{u}$ ✓

$g(x,y) = xy$ ✗

s.t. $g(p) = (\chi_1(p), \ldots, \chi_k(p))$

ex. $\chi(x,y) = (\underbrace{x+y}_{u}, \underbrace{x}_{v})$

$g(x,y) = u$



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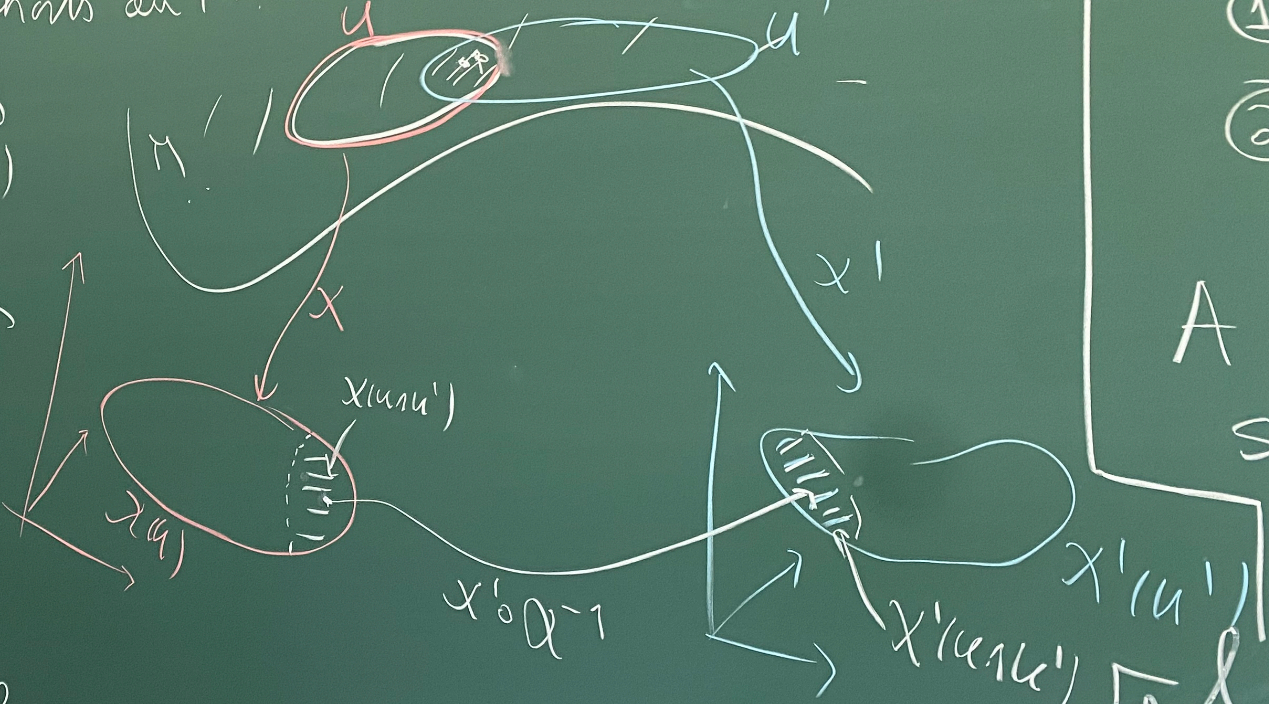
move to chapter II : now $\boxed{M = \text{top. space}}$

Use m-dimensional topological charts (U, x) of M.
$$x : \underset{\substack{\cap \text{open} \\ M}}{U} \longrightarrow x(U) = \underset{\substack{\cap \text{open} \\ \mathbb{R}^m}}{\Omega}$$

Assume (U, x), (U', x') two charts on M.

define the change of coordinates
$$c_x^{x'} = x' \circ x^{-1} : \underset{\substack{\cap \text{open} \\ \mathbb{R}^m}}{x(U \cap U')} \longrightarrow \underset{\substack{\cap \text{open} \\ \mathbb{R}^m}}{x'(U \cap U')}$$



Rk: $\underbrace{c = x' \circ x^{-1}} = (c_1, \dots, c_m) \Rightarrow c \circ x = x' \Rightarrow$

$$\Rightarrow \begin{cases} x_1'(p) = c_1(x_1(p), \dots, x_m(p)) \\ x_2'(p) = c_2(x_1(p), \dots, x_m(p)) \end{cases}$$

Def: Say (U, x) and (U', x') are smoothly compatible if $c_x^{x'}$ is a diffeomorphism (in the classical sense)

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Def: An m-dimensional smooth atlas on M is a collection $\mathcal{A}$ of m-dimensional topological charts of M s.t.:

① $(\forall)\ p \in M$, $\exists\ (U, x)$ chart from $\mathcal{A}$ s.t. $p \in U$

② each two charts from M are smoothly compatible.

A smooth structure on M is such a smooth atlas satisfying also the maximality condition:

all topological charts on M with the property that it is smoothly compatible with all charts from $\mathcal{A}$, must be in $\mathcal{A}$

$x'(u')$

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$\mathcal{A}_{\mathbb{R}^m}$:

$\mathcal{A}'_{\mathbb{R}^m}$:

$\mathcal{A}^{max}_{\mathbb{R}^m}$

Rk: A

on M

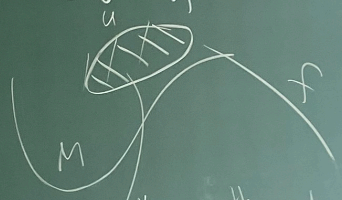
Comments: (10)

(A) How does such a collection $\mathcal{A}$ allow us to talk about a function $f: M \to \mathbb{R}$ being "smooth"?

Def: Say f is smooth w.r.t. $\mathcal{A}$ if

$(\forall (U, x) \in \mathcal{A}),\ f \circ x^{-1} : x(U) \to \mathbb{R}$

is smooth in the classical sense.



valid for any atlas (only ① & ② required)

(B) Examples of smooth atlases on $M = \mathbb{R}^m$:

$\mathcal{A}_{\mathbb{R}^m}$: consists of only $(\mathbb{R}^m, Id_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m)$

$\mathcal{A}'_{\mathbb{R}^m}$: consists of (Ω, id_Ω), one for each $\Omega \subseteq \mathbb{R}^m$ open

$\mathcal{A}^{max}_{\mathbb{R}^m}$: all classical smooth charts of $\mathbb{R}^m$

Rk: They all give rise to same notion of smoothness.

(C) Any smooth atlas $\mathcal{A}$ (① & ②) gives rise to a maximal one, $\mathcal{A}^{max}$

Hence any atlas induces a smooth str.