Reminder: Given M = (topological) space, $m \in \mathbb{N}$ (1)

- an (m-dimensional) topological chart (U, x) on M:

$$\begin{array}{ccc} x : U & \longrightarrow & x(U) \\ \cap_{\circ} & & \cap_{\circ} \\ M & & \mathbb{R}^m \end{array} \quad \text{homeomorphism}$$

- when $M \subseteq \mathbb{R}^L$ also makes sense: classical smooth chart: required classical diffeomorphism

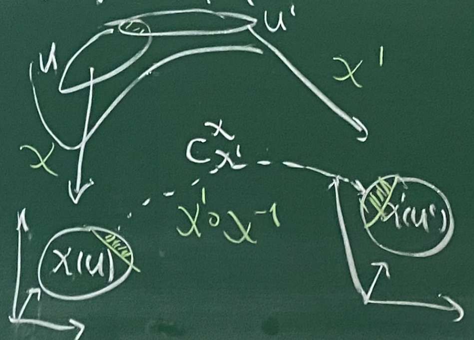
when M admits enough such $\Rightarrow$ talk about "m-dimensional embedded manifold in $\mathbb{R}^L$"

- when M is just a space: "smooth chart": makes no sense! (not defined)

- smooth compatibility of charts $(U, x), (U', x')$ on M:

$$c^{x'}_{x} := x' \circ x^{-1} : x(U \cap U') \to x'(U \cap U')$$

required to be a classical diffeomorphism



Q1: Given a smooth manifold $(M, \mathcal{A}_M)$

- a smooth (m-dimensional) atlas on M: collection $\mathcal{A}$ of topological charts s.t.
  1. $(\forall)\ p \in M$ is in the domain of some chart $(U,x) \in \mathcal{A}$
  2. each two $(U,x), (U',x')$ from $\mathcal{A}$ are smoothly compatible
- smooth structure on M: $\mathcal{A}$ as above but also satisfying:
  3. any topological chart on M that is smoothly compatible with all $(U,x) \in \mathcal{A}$ must be in $\mathcal{A}$.
- Smooth (m-dimensional) manifold: any top space M which is Hausdorff and 2^{nd} countable TOGETHER WITH a smooth structure $\mathcal{A}_M$.

  $(M, \mathcal{A}_M)$ ... or: "the manifold M"

  its members: "the smooth charts of the manifold M"
- Smooth functions between manifolds $(M, \mathcal{A}_M)$, $(N, \mathcal{A}_N)$: any $f: M \to N$ continuous s.t.

  $(\forall)\ (U,x) \in \mathcal{A}_M$, $(U',x') \in \mathcal{A}_N$ one has: $x' \circ f \circ x^{-1}: \underset{\mathbb{R}^m}{\underset{\cap}{\text{Domain}}} \to \underset{\mathbb{R}^n}{\underset{\cap}{\text{Codomain}}}$ required to be smooth in the classical sense
- diffeomorphisms: f s.t. f = bijective, f, f^{-1} = smooth

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Next: stare at the definitions ... (3)

(1) $\boxed{\text{Any (smooth, } m\text{-dimensional) atlas } \mathcal{A} \text{ on } M \text{ induces a smooth structure on } M}$

Namely: $\mathcal{A}^{max} := \left\{ \begin{array}{l} \text{those topological charts} \\ \text{of } M \; / \; \text{which are } \mathcal{A} \end{array} \right\}$ $<$ smooth atlas $\Rightarrow$ smooth structure; satisfies (3)

Ex: On $M = \mathbb{R}^m$: $\mathcal{A}_{\mathbb{R}^m} = \{ Id_{\mathbb{R}^m} \}$

smooth atlasses on the topological space $\mathbb{R}^n$

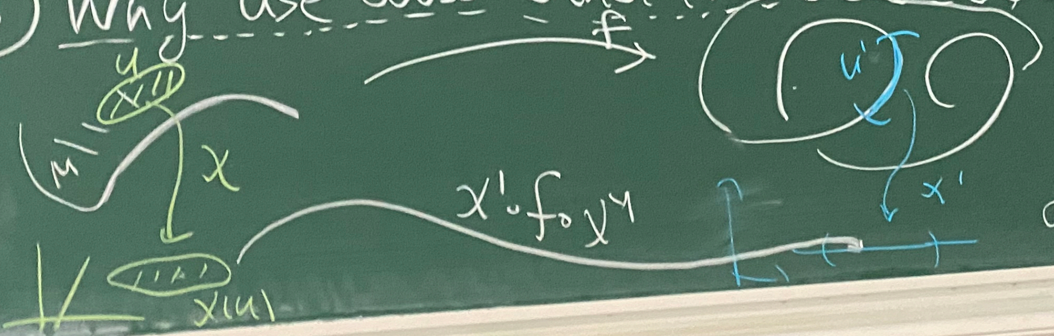
$\mathcal{A}_1 = \{ Id_U : U \subseteq \mathbb{R}^m \text{ open} \}$, $\mathcal{A}_2 = \{ Id_B : B \subseteq \mathbb{R}^n \text{ open ball} \}$

$\mathcal{A}^{max}_{\mathbb{R}^m} = \{ (U, X) \; / \; (U,X) = \underline{\text{classical smooth chart of } \mathbb{R}^m} \}$

All induce $\mathcal{A}^{max}_{\mathbb{R}^m}$ (smooth structure)

(2) (2a) Why use maximal atlases? Because they are canonical (always there, uniquely associated to the manifold)

(2b) Why use also other (non-maximal) atlases? Smaller is easier to handle and contains all the info.



...ological charts ② s.t.

... $\mathcal{A}$

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...e with all $(U,X)\in\mathcal{A}$

...d 2^{nd} countable

...ture $\mathcal{A}_M$.

... M"

...e smooth

...e manifold M"

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... in the classical sense

Next: stare at the definitions ... ③

① Any (smooth, m-dimensional) atlas $\mathcal{A}$ on M induces a smooth structure on M

Namely: $\mathcal{A}^{max} := \left\{ \begin{array}{l} \text{those topological charts} \\ \text{of } M \end{array} \Big/ \text{ which are } \right\}$ < smooth atlas / satisfies ③ $\Rightarrow$ smooth structure

Ex: On $M=\mathbb{R}^m$: $\mathcal{A}_{\mathbb{R}^m} = \{ Id_{\mathbb{R}^m} \}$

$\mathcal{A}_1 = \{ Id_U : U \subseteq \mathbb{R}^m \text{ open} \}$, $\mathcal{A}_2 = \{ Id_B : B \subseteq \mathbb{R}^m \text{ open ball} \}$

$\mathcal{A}^{max}_{\mathbb{R}^m} = \{ (U,X) / (U,X) = \text{classical smooth chart of } \mathbb{R}^m \}$

smooth atlasses on the topological space $\mathbb{R}^m$

All induce $\mathcal{A}^{max}_{\mathbb{R}^m}$ (smooth structure)

② (2a) Why use maximal atlases? Because they are canonical (always there, uniquely associated to the manifold)

(2b) Why use also other (non-maximal) atlases? easier

Ex: In context of Def of smooth maps, assuming that

A'_M = atlas on M inducing A_M } then f is smooth iff (*)

A'_N = —"— A_N } holds for (∀) $(U,X)\in A'_M$, $(U',X')\in A'_N$.

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③ for $M \subseteq \mathbb{R}^L$ a "m-dimensional embedded manifold in $\mathbb{R}^L$": ④

charts ② 5.1

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Next: stare at the definitions ... ③

① $\boxed{\text{Any (smooth, m-dimensional) atlas } \mathcal{A} \text{ on } M \text{ induces a smooth structure on } M}$

③ for $M \subseteq \mathbb{R}^L$ a "m-dimensional embedded manifold in $\mathbb{R}^L$": ④

$\mathcal{A}_M :=$ the collection of all classical smooth charts.

Convention: by default, such M will be endowed with THIS smooth structure (classical smooth structure on M)

In particular: for any smooth manifold M, can talk about:

- $\mathbb{R}$-valued smooth functions on M, $f : M \to \mathbb{R}$

  $\mathcal{C}^\infty(M) = \{ f : M \to \mathbb{R} \mid f = \text{smooth} \}$

- (smooth) curve in M: $\gamma : \mathbb{R} \to M$

  $\gamma : (a,b) \to M$.

④ For $M \subseteq \mathbb{R}^L$ as above, with the induced topology: are there other smooth (different than the classical one) on M? YES! HERE IS AN EXAMPLE:

Ex: $M = \mathbb{R}$ Choose $\chi_0 : \mathbb{R} \to \mathbb{R}$, $\chi_0(t) = \sqrt[3]{t}$

Set: $\mathcal{A}_{\chi_0} := \{ (\mathbb{R}, \chi_0) \}$ a smooth atlas on $\mathbb{R}$

$\Rightarrow$ a smooth structure $\mathcal{A}^{max}_{\chi_0}$ on $\mathbb{R}$

$(\mathbb{R}, \chi_0) \in \mathcal{A}^{max}_{\chi_0}$ ← not a (classical) smooth diffeomorphism

…ace, $m \in \mathbb{N}$ (1)

• a smooth (m-dimensional) atlas on M: collection $\mathcal{A}$ of topological charts (2) s.t.

…t is the (5)'

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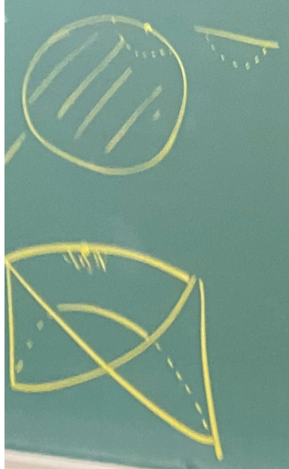


(6)

Remark: However, $(\mathbb{R}, \mathcal{A}_{x_0}^{max})$ und $(\mathbb{R}, \mathcal{A}_{\mathbb{R}}^{max})$ are diffeomorphic ($\exists$ diffeomorphism between them)

$$(\mathbb{R}, \mathcal{A}_{\mathbb{R}}^{max}) \xrightarrow[f(t)=t^3]{f = x_0^{-1}} (\mathbb{R}, \mathcal{A}_{x_0}^{max})$$

$$(\mathbb{R}, \mathcal{A}_{\mathbb{R}}^{max}) \xleftarrow[x_0(t)=\sqrt[3]{t}]{x_0} (\mathbb{R}, \mathcal{A}_{x_0}^{max})$$

are not diffeomorphic in the classical sense

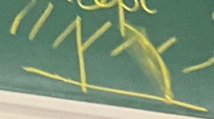
(5) A bit about 2nd countability of topological space

- automatic for any $M \subseteq \mathbb{R}^L$ with the induced topology
- used only later to "add up local objects to global ones" (partitions of unity)
  - to integrate over manifolds
  - to prove embedding theorems.

"boundary points"

(6) Variations

- C^k-smooth manifolds
- replace $\mathbb{R}^m$ by $\mathbb{C}^m$ and smooth by holomorphic $\Rightarrow$ the concept of complex manifold.
- replace $\mathbb{R}^m$ by $\mathbb{R}^m_+ = \{(x_0, x_1, \dots, x_m) \in \mathbb{R}^m / x_m \geq 0\}$



$\mathbb{R}^2_+ \Rightarrow$ manifold with bou…

Reminder: Given $M =$ (topological) space, $m \in \mathbb{N}$ ①

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⑤

Q1: Given a smooth manifold $(M, \mathcal{A}_M)$ what is the smallest atlas $\mathcal{A}$ s.t. $\mathcal{A}^{max} = \mathcal{A}_M$.

~~Q2: On a given space M, can one find more than one smooth structure? And how many different ones are there?~~


Reminder: Given M = (topological) space, m ∈ ℕ
• a smooth (m-dimensional) atlas on M: collection A of topological charts s.t.
Next: stare at the definitions ...
Q1: Given a smooth manifold (M, A_M) what is the smallest atlas A s.t. A^max = A_M.
Q2. On a given space M, can one find more than one smooth structure? And how many different ones are there?
Q2: Given space M, how many non-diffeomorphic smooth structures can one find on M?
Answer M = ℝ : 1
M = ℝ² : 1
M = ℝ³ : 1
M = ℝ⁴ : ∞
M = S¹ : 1
M = S²
M = S⁷ : 28 !!
Milnor
manifolds with bdry
Remark: However, (ℝ, A_{x₀}^max) and (ℝ, A_ℝ^max) are diffeomorphic (∃ diffeomorphism between them)
(ℝ, A_ℝ^max) —f = x₀⁻¹→ (ℝ, A_{x₀}^max)
f(t) = t³
(ℝ, A_ℝ^max) ←x₀— (ℝ, A_{x₀}^max)
x₀(t) = ∛t
are not diffeomorphic in the classical sense
5 A bit about 2nd countability of topological space
automatic for any M ⊆ ℝ^L with the induced topology
used only later to "add up local objects to global ones" (partitions of unity)
to integrate over manifolds
to prove embedding theorems.
6 Variations
• C^k-smooth manifolds
• replace ℝ^m by ℂ^m and smooth by holomorphic ⇒ the concept of complex manifold
• replace ℝ^m by ℝ^m_+ = {(x₀, x₁, ..., x_m) ∈ ℝ^m / x_m ≥ 0}
ℝ²_+ ⇒ manifold with boundary
"boundary points"
3 for M ⊆ ℝ^L a "m-dimensional embedded manifold in ℝ^L":
A_M := the collection of all classical smooth charts.
Convention: by default, such M will be endowed with THIS smooth structure (classical smooth structure on M)
In particular, for any smooth manifold M, can talk about
• ℝ-valued smooth functions on M, f: M → ℝ
C^∞(M) = {f: M → ℝ / f = smooth}
• (smooth) curve in M γ: ℝ → M
γ: (a, b) → M
4 For M ⊆ ℝ^L as above, with the induced topology: are there other smooth (different than the classical one) on M? YES! HERE IS AN EXAMPLE
Ex: M = ℝ Choose x₀: ℝ → ℝ, x₀(t) = ∛t
Set: A_{x₀} := {(ℝ, x₀)} a smooth atlas on ℝ
⇒ a smooth structure A_{x₀}^max on ℝ
(ℝ, x₀) not a (classical) smooth diffeomorphism
CASIO

$x: U \to x(U)$ homeomorphism
$\quad\ \ \cap \qquad \cap$
$\quad\ \ M \qquad \mathbb{R}^m$

- when $M \subseteq \mathbb{R}^L$ also makes sense: Smooth chart: classical C^k- required classical C^k-diffeomorphism
when M admits enough such $\Rightarrow$ talk about "m-dim…

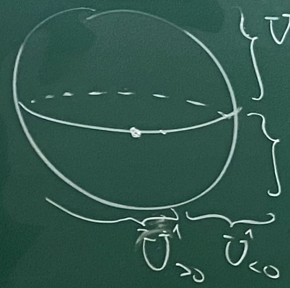
② each two $(U,x),(U',x')$ from $\mathcal{A}$ are smo…

- Smooth structure on M: $\mathcal{A}$ as above but also ③ any topological chart on M that is smoothly compa… must be in $\mathcal{A}$.
- Smooth manifold: any top space M which is H…

Ex: $M = S^2 \subseteq \mathbb{R}^3$ is a 2-dimensional embedded manifold in $\mathbb{R}^3$ $\Rightarrow$ M carries its classical Smooth structures

Other smooth structures? Other smooth atlases inducing the

Possible atlases:



$\{U^0_{>0}$, $U^0_{<0}$, $U^1_{>0}$, $U^1_{<0}$

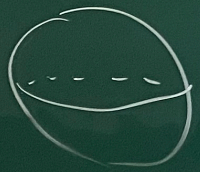
$x^0_{>0}: U^0_{>0} \to \mathbb{R}^2$
$(x_0,x_1,x_2) \longmapsto (x_1,x_2)$

$(U^0_{>0}, x^0_{>0}), (U^0_{<0}, x^0_{<0})$,
$(U^1_{>0}, x^1_{>0}), (U^1_{<0}, x^1_{<0})$
$(U^2_{>0}, x^2_{>0}), (U^2_{<0}, x^2_{<0})$

6 charts! make a smooth atlas.

induced smooth structure: the

Another nice smooth atlas: use stereogr. projection.



$U_N = S^2 \setminus \{p_N\}$
$U_S = S^2 \setminus \{p_S\}$
$\Rightarrow$ an atlas with only two charts!