-1-

Reminder: M = topological space, $m \in \mathbb{N}$ ("m-dimensional")

→ smooth structure on M: a maximal smooth atlas $\mathcal{A}$ on M

collection $\mathcal{A}$ of topological charts on M, $x: U \to x(U)$ homeomorphism, (U, x), $U \subset M$, $x(U) \subset \mathbb{R}^m$

satisfying:

(1) $(\forall)\, x \in M$ is in the domain U of some $(U, x) \in \mathcal{A}$.

(2) any two $(U,x), (U',x') \in \mathcal{A}$ are smoothly compatible

i.e. $c_x^{x'} = x' \circ x^{-1}$ is a classical diffeomorphism.

i.e. → (3) any topological chart of M that is smoothly compatible with all charts $\in \mathcal{A}$ must $\in \mathcal{A}$.

(U, x) in $\mathcal{A}_M$: smooth charts of the manifold M

→ manifold: $(M, \mathcal{A}_M)$

M ← topological space which is Hausdorff & 2^{nd} countable

$\mathcal{A}_M$ ← smooth structure

$\boxed{\text{Rk 1}}$ Any smooth atlas $\mathcal{A}$ induces a smooth structure $\mathcal{A}^{max}$

"the smooth structure induced by the smooth atlas $\mathcal{A}$"

Rk 2: Given manifold $(M, \mathcal{A}_M)$, any smooth atlas $\mathcal{A} \subseteq \mathcal{A}_M$ induces $\mathcal{A}_M$ ($\mathcal{A}^{max} = \mathcal{A}_M$!)
Exrcise

→ **Smooth maps between manifolds**: $f : (M, \mathcal{A}_M) \to (N, \mathcal{A}_N)$
means $f : M \to N$ continuous & satisfying:

$$(*) \quad \left.\begin{array}{l} (\forall)\ (U, x) \in \mathcal{A}_M \\ (\forall)\ (\tilde{U}, \tilde{x}) \in \mathcal{A}_N \end{array}\right\} \text{ one has that } \boxed{\tilde{x} \circ f \circ x^{-1}} \text{ is smooth in the classical sense}$$

how f looks in coordinates

Rk 3: If we have smaller atlases $\mathcal{A}'_M \subseteq \mathcal{A}_M$, $\mathcal{A}'_N \subseteq \mathcal{A}_N$: suffices to check
$(*)$ for $(U,x) \in \mathcal{A}'_M$, $(\tilde{U}, \tilde{x}) \in \mathcal{A}'_N$

Rk 4: Also good to know: "for embeded in Euclidean spaces" $M \subseteq \mathbb{R}^L$, $N \subseteq \mathbb{R}^{L'}$:

$$\underbrace{f : (M, \mathcal{A}_M^{max}) \to (N, \mathcal{A}_N^{max})}_{\text{smooth}} \iff f \text{ is smooth in the classical sense}$$

Example

Conditi

Th

fold

-3-

...mple: The classical smooth structure on M

... whenever $M \subseteq \mathbb{R}^L$ embedded manifold

namely: $\mathcal{A}_M^{max} :=$ the collection of all classical smooth charts on M

...tion!! ... Very useful criteria:

Thm (the RVT in $\mathbb{R}^L$): Assume that $M = \mathrm{Zero}(f) = \{p \in \mathbb{R}^L \,/\, f(p) = 0\}$

for some smooth $f : \mathbb{R}^L \longrightarrow \mathbb{R}^k$

$m := L - k$

Assume: f = submersion at each $p \in M$ (rank of $\mathrm{Jac}(f)$ is k at all such p)

Conclusion: M = m-dimensional embedded manifold in $\mathbb{R}^L$

...tions in coordinates ("page -2-")

Example: $M = S^2 \subseteq \mathbb{R}^3$

Various possible ways to make it into a smooth 2-dimensional manifold

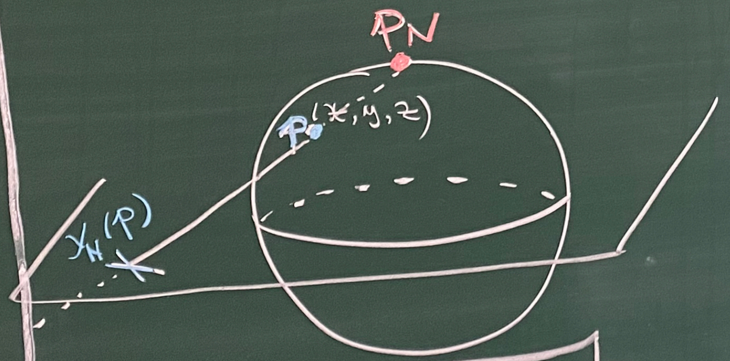
E.g.:

① Appeal to the RVT: $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$, $f(x,y,z) = x^2+y^2+z^2-1$

$(Df)_p = (2x \quad 2y \quad 2z)$ rank 1 $(\forall)$ $p = (x,y,z) \in S^2$ (since $0 \notin S^2$).

② use open half-spheres $\Rightarrow$ an smooth atlas made of 6 charts

③ use stereographic projection w.r.t. p_N & p_S $\Rightarrow$ an atlas made of 2 charts



(U_N, x_N) with

$U_N := S^2 \setminus \{p_N\}$, $x_N : U_N \to \mathbb{R}^2$

$x_N(p) = \overline{p\,p_N} \cap$ plane $\mathbb{R}^2 \times \{0\}$

$= \left(\underbrace{\frac{x}{1-z}}_{u}, \underbrace{\frac{y}{1-z}}_{v}\right)$ For $p = (x,y,z)$

Good to have at hand: The inverse of x_N: $x_N^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{\cdots}\right)$

Ex: all give rise to the same smooth structure on S^2!!

Some examples of how to represent functions in coordinates

(see "page -2-")

<u>Ex 1</u>: $f: S^2 \longrightarrow \mathbb{R}$, $f(p)$ = the 1st Euclidean coordinate of p

$(f(x,y,z)=x)$.

Represent it w.r.t. (U_N, x_N) for S^2 and $(\mathbb{R}, id_{\mathbb{R}})$ for $\mathbb{R}$

$$\begin{tikzcd} p & S^2 \arrow[rr, "f"] & & \mathbb{R} \\ (u,v) & U_N \arrow[u, phantom, "\cup"] \arrow[rr, bend left] \arrow[d, "x_N"'] & & \mathbb{R} \arrow[u, equal] \arrow[d, "id_{\mathbb{R}}"] \\ & \mathbb{R}^2 \arrow[rr, dashed] & & \mathbb{R} \end{tikzcd}$$

$\frac{2u}{u^2+v^2+1}$

$$id_{\mathbb{R}} \circ f \circ x_N^{-1}(u,v) = \frac{2u}{u^2+v^2+1}$$

<u>Ex 2</u>: $id: S^2 \longrightarrow S^2$ w.r.t. (U_N, x_N) on the first S^2, (U_S, x_S) on second

$$\begin{tikzcd} S^2 \arrow[r, "id"] & S^2 \\ U_N \arrow[u, phantom, "\cup"] \arrow[d, "x_N"'] & U_S \arrow[u, phantom, "\cup"] \arrow[d, "x_S"] \\ \mathbb{R}^2 \arrow[r, dashed] & \mathbb{R}^2 \end{tikzcd}$$

$$x_S \circ id \circ x_N^{-1}(u,v) = \frac{1}{u^2+v^2}\cdot(u,v) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)$$

... a maximal smooth atlas $\mathcal{A}$ on M

collection $\mathcal{A}$ of topological charts on M, $x: U \longrightarrow x(U)$ homeo-

Some abstract examples/constructions:

Induced smooth structures: $(M, \mathcal{A}_M)$, $U_0 \subseteq M$ open $\Big\}$ ⇝ form the induced smooth structure on U_0

$$\mathcal{A}_M|_{U_0} := \{(U,x) \in \mathcal{A}_M \,/\, U \subseteq U_0\}.$$

Products: $(M, \mathcal{A}_M)$, $(N, \mathcal{A}_N)$ given manifolds

Look at $M^m \times N^n = \{(x,y) \,/\, x \in M, y \in N\}$

For $(U,x) \in \mathcal{A}_M$, $(U',x') \in \mathcal{A}_N$ $\Big\}$ ⇝ can form $x \times x' : U \times U' \longrightarrow x(U) \times x'(U')$

$U \times U' \subseteq M \times N$ (chart of $M \times N$), $\quad x(U) \times x'(U') \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$

$\Longrightarrow$ we get an atlas on $M \times N \Rightarrow$ a smooth structure (product smooth str.)

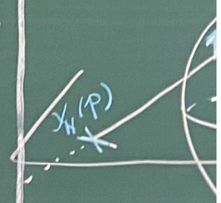
... can start forming $\mathbb{R} \times M$, $S^1 \times M$, $S^1 \times S^1 \times S^2$

Example

Various

E.g.:

①



②

③

Ex: all gi... the same s... structure

— 1 —

...ogical space, $m \in \mathbb{N}$ ("m-dimensional")

...n M: a maximal smooth atlas $\mathcal{A}$ on M

... of topological charts on M, $x: U \longrightarrow x(U)$ homeomorphism

(U, x), $U \subseteq M$, $x(U) \subseteq \mathbb{R}^m$

...) $(\forall)\, x \in M$ is in the domain U of some $(U,x) \in \mathcal{A}$.

...) any two $(U,x), (U',x') \in \mathcal{A}$ are smoothly compatible

i.e. $C^{x'}_{x} = x' \circ x^{-1}$ is a classical diffeomorphism.

3) any topological chart of M that is smoothly compatible with all charts $\in \mathcal{A}$ must $\in \mathcal{A}$.

...$\mathcal{A}_M$) — topological space which is Hausdorff & 2^{nd} countable; — smooth structure

...th atlas $\mathcal{A}$ induces a smooth structure $\mathcal{A}^{max}$

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— 2 —

Rk 2. Given manifold $(M, \mathcal{A}_M)$, any smooth atlas $\mathcal{A} \subseteq \mathcal{A}_M$ induces $\mathcal{A}_M$ ($\mathcal{A}^{max} = \mathcal{A}_M$!) Exercise

$\rightarrow$ Smooth maps between manifolds: $f: (M, \mathcal{A}_M) \rightarrow (N, \mathcal{A}_N)$

means $f: M \rightarrow N$ continuous & satisfying:

$(*)$ $(\forall)\, (U,x) \in \mathcal{A}_M$, $(\forall)\, (\tilde{U}, \tilde{x}) \in \mathcal{A}_N$ } one has that $\boxed{\tilde{x} \circ f \circ x^{-1}}$ is smooth in the classical sense — how f looks in coordinates

Rk 3: If we have smaller atlases $\mathcal{A}'_M \subseteq \mathcal{A}_M$, $\mathcal{A}'_N \subseteq \mathcal{A}_N$: suffices to check $(*)$ for $(U,x) \in \mathcal{A}'_M$, $(\tilde{U},\tilde{x}) \in \mathcal{A}'_N$

Rk 4: Also good to know: "for embeded in Euclidean spaces" $M \subseteq \mathbb{R}^L$, $N \subseteq \mathbb{R}^{L'}$:

$$\underset{\text{smooth}}{f: (M, \mathcal{A}_M^{max}) \longrightarrow (N, \mathcal{A}_N^{max})} \iff f \text{ is smooth in the classical sense}$$

— 3 —

Example: The classical smo... ... whenever ...

Condition !! ... Very useful cri...

Thm (the RVT in $\mathbb{R}^L$): Assume ...

$m := L - k$

Assume: f = submersion ...

Conclusion: M = m-dimensional ...

— 4 —

Example: $M = S^2 \subseteq \mathbb{R}^3$

Various possible ways to make it into a smooth 2-dimensional manifold

E.g.:

① Appeal to the RVT: $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$, $f(x,y,z) = x^2 + y^2 + z^2 - 1$

$(Df)_p = (2x \;\; 2y \;\; 2z)$ rank 1 $(\forall)$ $p = (x,y,z) \in S^2$ (since $0 \notin S^2$)

② use open half-spheres $\Rightarrow$ a smooth atlas made of 6 charts

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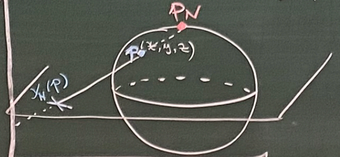


(U_N, x_N) with

$U_N = S^2 \setminus \{p_N\}$, $x_N: U_N \rightarrow \mathbb{R}^2$

$x_N(p) = \overline{p p_N} \cap$ plane $\mathbb{R}^2 \times \{0\}$

$= \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$ for $p = (x,y,z)$

Ex: all give rise to the same smooth structure on S^2 !!

Good to have at hand: The inverse of x_N: $x_N^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$

— 5 —

Some examples of how to represe...

Ex 1: $f: S^2 \longrightarrow \mathbb{R}$, $f(p)$ = the 1^{st} ...

Represent it w.r.t. (U_N, x_N) for S^2 ...

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & \mathbb{R} \\ \cup\!\!\mid & & \parallel \\ U_N & & \mathbb{R} \\ \downarrow x_N & & \downarrow id_{\mathbb{R}} \\ \mathbb{R}^2 & \dashrightarrow & \mathbb{R} \end{array}$$

p, (u,v), $\frac{2u}{u^2+v^2+1}$, $id_{\mathbb{R}}$

Ex 2: $id: S^2 \longrightarrow S^2$ w.r.t. (U_N, x_N) ...

$$\begin{array}{ccc} S^2 & \longrightarrow & S^2 \\ \cup\!\!\mid & & \cup\!\!\mid \\ U_N & & U_S \\ \downarrow x_N & & \downarrow x_S \\ \mathbb{R}^2 & \dashrightarrow & \mathbb{R}^2 \end{array}$$

$x_S \circ id \circ x_N^{-1}$

— 6 —

...constructions:

...es: $(M, \mathcal{A}_M)$, $U_0 \subseteq M$ open } $\rightsquigarrow$ form the induced smooth structure on U_0

$\{(U,x) \in \mathcal{A}_M \mid U \subseteq U_0\}$.

$(M, \mathcal{A}_M), (N, \mathcal{A}_N)$ given manifolds

$M \times N = \{(x,y) \mid x \in M, y \in N\}$

... } $\rightsquigarrow$ can form $x \times x': U \times U' \longrightarrow x(U) \times x'(U')$

chart of $M \times N$: $U \times U' \subseteq M \times N$, $x(U) \times x'(U') \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$

... atlas on $M \times N \Rightarrow$ a smooth structure (product smooth st.)

...start forming $\mathbb{R} \times M$, $S^1 \times M$, $S^1 \times S^1 \times S^2$

$\boxed{-7-}$

Similar examples: $M = S^n$ for $n \in \mathbb{N}$

$$M = \{ (x,y,z) \in \mathbb{R}^3 \,/\, x^2 + y^4 + z^6 = 1 \} \subseteq \mathbb{R}^3$$

$$M_{2,1} = \{ (x,y,z) \in \mathbb{R}^3 \,/\, \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 = 1 \}$$

a model for the torus

Similar examples $\boxed{\text{in disguise}}$:

$M = \mathbb{C}^n$ (a copy of $\mathbb{R}^{2n}$)

$M = \mathcal{M}_{n\times n}(\mathbb{R})$ = collection of all $n\times n$ matrices

$$x : M \longrightarrow \mathbb{R}^{n^2}$$
$$A \longmapsto (a_{11}, a_{12}, \dots a_{1n}, a_{21}, a_{22}, \dots$$

$$M = GL_n(\mathbb{R}) = \{ A \in \mathcal{M}_{n\times n}(\mathbb{R}) \,/\, \underbrace{A = \text{invertible}}_{\text{i.e. } \det(A) \neq 0} \}$$

$\det : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ continuous

open in $\mathcal{M}_{n\times n}(\mathbb{R})$

$$M = O(n) = \{ A \in \mathcal{M}_{n\times n}(\mathbb{R}) \,/\, A \cdot A^t = I_n \} \subseteq \mathcal{M}_{n\times n}(\mathbb{R})$$

embedded manif. by the RVT.

$\mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{\pi} P^n$

$(x_0, x_1, \dots, x_n) \longmapsto [x_0 : x_1 : \dots]$

endow P^n with the quotient topology i.e.:

$U \subseteq P^n$ open iff $\pi^{-1}(U)$ - open in $\mathbb{R}^{n+1}$

Exam

$\mathbb{R} \cdot x = [$

$\Rightarrow$

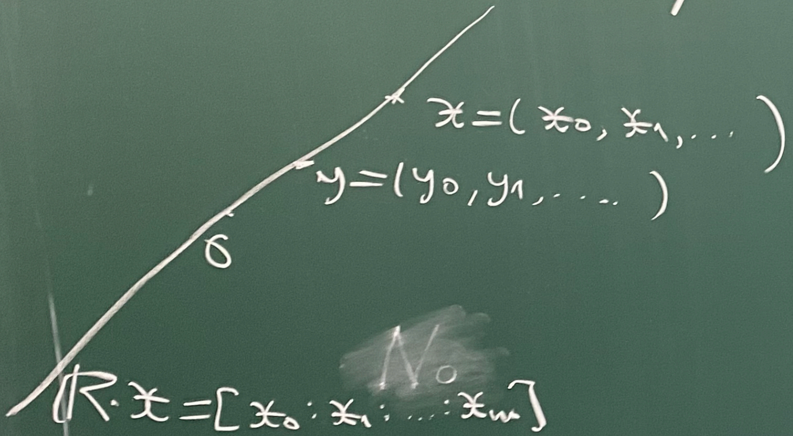
Rk

-8-

Example: the real projective space $\mathbb{P}^n$

$$\mathbb{P}^n := \{ \ell \ / \ \ell \subseteq \mathbb{R}^{n+1} \text{ is a line passing through the origin} \}$$

$$= \{ \ell \ / \ \ell \subseteq \mathbb{R}^{n+1} \text{ a } \underbrace{\text{1-dimensional vector subspace of } \mathbb{R}^{n+1}}_{\text{hence of type}} \}$$



$$\mathbb{R}\cdot x = \{(\lambda x_0, \lambda x_1, \dots, \lambda x_n) / \lambda \in \mathbb{R}\}$$

for some $x=(x_0,x_1,\dots,x_n)\in\mathbb{R}^{n+1}\setminus\{0\}$

Notation: $[\underbrace{x_0 : x_1 : \dots : x_n}_{\text{"some kind of coordinates"}}] = \mathbb{R}\cdot x$

$$\Rightarrow \mathbb{P}^n = \{ [x_0 : x_1 : \dots : x_n] \ / \ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}\setminus\{0\} \}$$

where: $[x_0 : x_1 : \dots : x_n] = [y_0 : y_1 : \dots : y_n] \iff \begin{cases} y_0 = \lambda x_0 \\ y_1 = \lambda x_1 \\ \vdots \end{cases}$ for some $\lambda \in \mathbb{R}$.

Rk: Hence: $[x_0 : x_1 : \dots : x_n] = [1 : \underbrace{\frac{x_1}{x_0}}_{y_1} : \dots : \underbrace{\frac{x_n}{x_0}}_{y_n}]$

OK if $x_0 \neq 0$!

$\boxed{-\, g \,-}$

Solve this problem: $U_0 := \{ [x_0 : x_1 : \dots : x_m] \in \mathbb{P}^m \;/\; x_0 \neq 0 \}$

$x_0 : U_0 \longrightarrow \mathbb{R}^m$, $x_0([x_0 : x_1 : \dots : x_m]) = \left(\dfrac{x_1}{x_0}, \dots, \dfrac{x_m}{x_0} \right)$

Similarly use x_1, or x_2, or etc $\qquad x_0^{-1}(y_1, \dots, y_m) = [1 : y_1 : \dots : y_m]$

$\Rightarrow$ topological charts $(U_0, x_0), (U_1, x_1), \dots, (U_m, x_m)$

$U_0, \dots, U_m$ cover $\mathbb{P}^m$

(... look at changes of coordinates, see they are smooth)

$\Rightarrow$ a smooth atlas on $\mathbb{P}^m$

$\Rightarrow$ a smooth structure on $\mathbb{P}^m$

$\Rightarrow$ $\mathbb{P}^m$ is a m-dimensional manifold!

$[1 : 0 : \dots : 0] \in U_0$!!!

$\notin U_1$

$\notin U_2$

$\vdots$

$\notin U_m$

Special smooth maps:

- diffeomorphisms : f = bijective, f, f^{-1} are smooth
- immersion (submersions) : if all $\tilde{x}\circ f\circ x^{-1}$ (for each $(U,x)\in \mathcal{A}_M$, $(\tilde{U},\tilde{x})\in\mathcal{A}_N$) are immersions (submersion) in the classical sense

The immersion/subm. theorems in $\mathbb{R}^n$ $\Longrightarrow$

$\Rightarrow$ for general immersions $f: M\to N$, $\forall\, p\in M$ one can find chart $(\underset{\ni p}{U},x)$, $(\tilde{U},\tilde{x})$ s.t.:

$$\tilde{x}\circ f\circ x^{-1} \text{ is given by } (x_1,\dots,x_m)\longmapsto(x_1,\dots,x_m,0,\dots,0)$$

Similarly for submersions: $(\underbrace{x_1,\dots,}_{\text{first } n}\cancel{x_m}) \longmapsto (x_1,\dots,x_n)$

- local diffeomorphisms : $(\forall)\, p\in M$, $\exists\ U\subseteq M$ open $p\in U$, $V\subseteq N$ — " — $f(p)\in V$ s.t. $f|_U: U\to V$ a diffeomorph.

Ex. $\mathbb{R}\longrightarrow S^1$
$t\longmapsto e^{2\pi i t}$

-2-

...ifold $(M, \mathcal{A}_M)$, any smooth atlas $\mathcal{A} \subseteq \mathcal{A}_M$ induces $\mathcal{A}_M$

Errase ($\mathcal{A}^{max} = \mathcal{A}_M$!)

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how f looks in coordinates

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$\ldots, \mathcal{A}_N^{max})$ $\Longleftrightarrow$ f is Smooth in the classical sense

-10-

Special Smooth maps:

- diffeomorphisms : f = bijective, f, f^{-1} are sm...
- immersion (submersions) : if all $\tilde{x} \circ f \circ x^{-1}$ (for each $(U, \ldots$ $(\tilde{U}, \ldots$

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one can find chart $\underset{p}{(U, x)}, (\tilde{U}, \tilde{x})$ s.t. :

$\tilde{x} \circ f \circ x^{-1}$ is given by $(x_1, \ldots, x_m) \longmapsto (x_1 \ldots$

Similarly for submersions: $(\underbrace{x_1, \ldots,}_{\text{first } n} \cancel{x_m}) \longmapsto (x_1, \ldots$

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Ex. $\mathbb{R} \to S^1$

$t \longmapsto e^{2\pi i t}$ s.t. $f|_U : U \to V \ldots$

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