-4- | $m=\dim M$

...charts, coordinate changes,
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smooth manifolds: $(M,\mathcal{A}_M)$ contains "the smooth charts of the manifold M"

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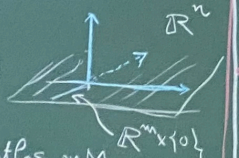
$\tilde{\Omega} \subseteq \mathbb{R}^n$

...in M (open in M!)
$\tilde{\Omega}\cap\mathbb{R}^m\times\{0\}$ (open in $\mathbb{R}^m$!)
$=\Omega$.



...Ω) for M ⇒ a smooth atlas on M ⇒
~~the classical smooth str on M~~ the smooth str on M induced from N.

-5-

... Smooth maps $f:(M,\mathcal{A}_M)\longrightarrow(N,\mathcal{A}_N)$: continuous & satisfying:

$$\left.\begin{array}{l}(\forall)(U,x)\in\mathcal{A}_M\\ (\forall)(\tilde{U},\tilde{x})\in\mathcal{A}_N\end{array}\right\} \Rightarrow \underbrace{\tilde{x}\circ f\circ x^{-1}}_{\text{"f in coordinates"}} : \underset{\mathbb{R}^m}{\overset{\text{open}}{\cap}} \longrightarrow \underset{\mathbb{R}^n}{\overset{\text{open}}{\cap}} \text{ smooth in classical sense } (*)$$

Particular (distinguished) classes of smooth maps:

- diffeomorphisms & local diffeomorphisms
- immersions / submersions : if all (*) are so (in classical sense)

Criteria (RVT): Assume that $M=\{x\in \cancel{\mathbb{R}^n}\,N \;/\; f(x)=\cancel{0}\,y\}$ where $f:\cancel{\mathbb{R}^n}\,N\longrightarrow\cancel{\mathbb{R}^k}\,K$ is a smooth map between manifolds N, K s.t. ($y\in K$) f is a submersion at all $x\in M=f^{-1}(y)$

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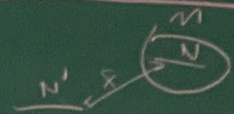
Pf: it follows ~~from the~~ similar statement in Euclidean space

→ call y a regular value of f

Next: smooth embeddings

Topological embedding of ...
f: ...
which, as a map ...
with the original ~~topology~~ smooth structure

Def: N = an n-dimensi...
An embedded (m-dimensi...
s.t. ...



S M ...

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$\longrightarrow f(M)$ it is a diffeomorphism (in the sense above)
the inclusion map $i:M\longrightarrow N$, $i(x)=x$ is a smooth embedding
...mbeddings of M_0 in another N

-7-

Immersions: $f: M\longrightarrow N$ smooth map (between two manifolds), $p\in M$.

Say that f is an immersion at p if $\dim M\le\dim N$ and:

(i) $(\forall)(\underset{p}{U},x),(\underset{f(p)}{\tilde{U}},\tilde{x})$ as above, the Jacobi... $x(p)$ has rank m

(ii)

(iii)

s.t. $\tilde{x}\circ$ ...

-1-

Reminder: ... topological charts, coordinate changes, smooth compatibility (of top. charts), smooth atlases, smooth structures, smooth manifolds: $(M, \mathcal{A}_M)$ contains "the smooth charts of the manifold M"

Examples:

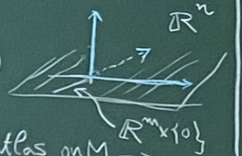
- "abstract": start with space M & exhibit a smooth atlas ($\mathbb{P}^n$, S^n, etc)
- "concrete": $\underbrace{\text{embeded manifolds}}$ $M \subseteq \mathbb{R}^n$ (m-dimensional ($m \le n$))

i.e: $(\forall)\, p \in M$, $(\exists)$ $\tilde{x} : \tilde{U} \longrightarrow \tilde{\Omega}$ chart of $\mathbb{R}^n$ around p, with $\tilde{U} \subseteq \mathbb{R}^n$ open, $\tilde{\Omega} \subseteq \mathbb{R}^n$ open,

s.t., denoting $U = \tilde{U} \cap M$ (open in M!)

$\Omega = \tilde{\Omega} \cap \mathbb{R}^m \times \{0\}$ (open in $\mathbb{R}^m$!)

one has $\tilde{x}(U) = \Omega$.



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-2-

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Then M is an embedded manifold in $\mathbb{R}^n$ of dimension $m = n - k$

Next: smooth embeddings

-4-

-5-

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-4-

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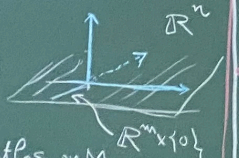

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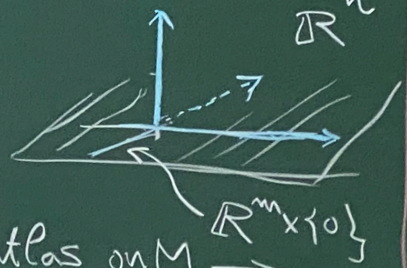

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... Smooth m...

$(\forall)\ (U, \ldots$

$(\forall)\ (\tilde{U}, \ldots$

Particular (disti...

- diffe...
- immers...

Criteria (RV)

$f: \cancel{\mathbb{R}^n}\ N \longrightarrow \cancel{\mathbb{R}^k}\ K$ ...

Then M is ...

∄: it follow...

Next: smooth e...

-6-

Def: A (smooth) embedding is a map $f: M \longrightarrow N$ s.t.:

-5-

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$f : \mathbb{R}^n$ ...

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(2) as a map $f : M \to f(M)$ it is a diffeomorphism (in the sense above)

Ex 1: $\left.\begin{array}{l} M \subseteq N \\ \text{embedded submanifold} \end{array}\right\} \Longrightarrow$ the inclusion map $i : M \to N$, $i(x) = x$ is a smooth embedding

If M_0 = given, looking for embeddings of M_0 in another N.

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Immersions:

Say that

(i) $(\forall)$

(ii)

(iii)

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$\Updownarrow$ ($U \ni p$, $\tilde{U} \ni f(p)$)

(ii) $(\exists)$ — " —

$\Updownarrow$ the immersion thm.

(iii) $(\exists)$ — " — s.t. $\tilde{x} \circ f \circ x^{-1}$ is of type $(x_1, \dots, x_m) \to (x_1, \dots, x_m, \underbrace{0, \dots, 0}_{n-m})$

Rk: (1) Any diffeomorphism (or local diff around p) is an immersion

(2) Composition of immersions is immersion

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{g} & P \\ \ni p & & \ni f(p) & & \end{array} \quad \xrightarrow{g \circ f}$$

f = immersion at p, g = immersion at $f(p)$ $\Rightarrow$ $g \circ f$ = immersion at p

since, in Euclidean spaces, $Jac(g \circ f) = Jac(g) \cdot Jac(f)$

Rk: (ii) $\Rightarrow$ (i) If ok for $(U_0, x_0), (\tilde{U}_0, \tilde{x}_0)$ then, for $(\forall)\ (U, x), (\tilde{U}, \tilde{x})$ as above $\Rightarrow$ $\tilde{x} \circ f \circ x^{-1} = (\tilde{x} \circ \tilde{x}_0^{-1}) \circ (\tilde{x}_0 \circ f \circ x_0^{-1}) \circ (x_0 \circ x^{-1})$

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— 6 —

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Rk: $\underbrace{(1)\ \&\ (2)}_{\text{smooth embedding}} \Longrightarrow f = \text{immersion}$

$$\begin{tikzcd} M \arrow[rr, "f", "\text{embedding}"'] \arrow[dr, "\text{given by } f"', "\text{diffeo}"] & & N \\ & f(M) \arrow[ur, "\text{inclusion}"'] & \end{tikzcd}$$

f:

Imm

Say

the → inverse thm.

Rk

Rk: for $(\forall$

...al sense)

where

$x \in M = f^{-1}(y)$

$m = n - k$

$= \dim N$

original ~~topology~~ smooth structure

with the ~~topology~~ smooth str induced from the one on N

meaning what?

Def. N = an n dimensional manifold

An embedded (m-dimensional) submanifold of N is any $M \subseteq N$ s.t.

---

m.

and

$x(p)$ has rank m

$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_m, \underbrace{0, \dots, 0}_{n-m})$

immersion

of = immers at p

Euclidean spaces,

$\mathrm{Jac}(g \circ f) = \mathrm{Jac}(g) \cdot \mathrm{Jac}(f)$

---

setting: $M \subseteq N$ embedded submanifold

def

$$\begin{tikzcd}
M \arrow[rr, hook, "i"', "\text{inclusion}"] & & N \\
U = \tilde{U} \cap M \arrow[u, phantom, "\cup"] \arrow[d, "x = \tilde{x}|_U"'] & & \tilde{U} \arrow[u, phantom, "\cup"] \arrow[d, "\tilde{x} \text{ of } N"] \\
\Omega = \tilde{\Omega} \cap \mathbb{R}^m \times \{0\} & & \tilde{\Omega} \\
\mathbb{R}^m \arrow[u, phantom, "\cap"] \arrow[rr] & & \mathbb{R}^n \arrow[u, phantom, "\cap"]
\end{tikzcd}$$

$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_m, 0, \dots, 0)$

In particular

i = an immersion