Reminder:

-1-

- smooth manifolds $(M, \mathcal{A}_M)$
- smooth maps $f:(M,\mathcal{A}_M)\to(N,\mathcal{A}_N)$ $(\forall)\, x\in\mathcal{A}_M,\ \tilde{x}\in\mathcal{A}_N \Rightarrow \tilde{x} f x^{-1} = \text{smooth}$ ($f_x^{\tilde{x}}$)
- immersions ——— " ——— = immersion

$\iff$ locally, $(\exists)$ — " — given by $(x_1,\dots,x_m)\mapsto(x_1,\dots,x_m,0,\dots,0)$

- embedded submanifolds of N: subset $M\subseteq N$ satisfying: $(\forall)\, p\in M$, $\exists\ \tilde{x}: \tilde{U}\to\tilde{\Omega}$ chart of N around p s.t. ($\tilde{U}\subseteq N$, $\tilde{\Omega}\subseteq\mathbb{R}^n$)

$$\tilde{x}(\tilde{U}\cap N) = \tilde{\Omega}\cap(\mathbb{R}^m\times\{0\})$$

Then $\tilde{x}|_U : U\to\Omega$ serves as a chart for M (topological).
all such together form a smooth atlas on M ⇝ get a smooth str. on M "induced smooth str."

Rk: Inclusion $i: M\hookrightarrow N$, when looking in charts w.r.t. $\tilde{x}$ and $x=\tilde{x}|_U$

$\Rightarrow (\tilde{x}\circ i\circ x^{-1})(x_1,\dots,x_m) = (x_1,\dots,x_m,0,0,\dots,0) \Rightarrow i = \text{immersion}$

-2-

- Smooth embedding: $f: M\to N$ (smooth) s.t.
  (1) $f(M)$ = embedded submanifold of N (with s.s. induced from N)
  (2) as a map $(M)\to f(M)$, f is a diffeomorphism (with original smooth str.)

Rk1: smooth embedding $\Rightarrow$ topological embedding

Rk2: • $M\subseteq N$ embedded submanifold $\Rightarrow i: M\to N$ topological embedding

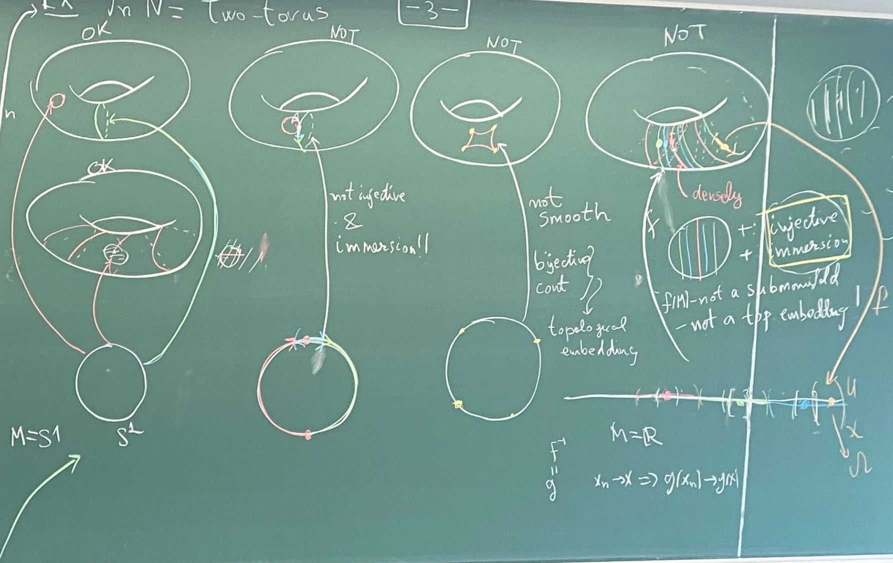
• $M \xrightarrow{f} N$ smooth embedding $\Rightarrow M \xrightarrow{f} N$ (by (2) diffeo ↘ $f(M)$ ↗ by (1) inclusion of embedded) $\Rightarrow$ f = immersion

Theorem: Given $f: M\to N$ smooth map between two manifolds:

$\{f$ = smooth embedding$\}$ $\iff$ $\{(1)\ f$ = topological embedding & $(2)\ f$ = immersion$\}$ $\overset{\text{when } M \text{ is compact}}{\Longleftarrow\!\Longrightarrow}$ $\{f$ = injective immersion$\}$

-3-

Ex: In N = two-torus



-4-

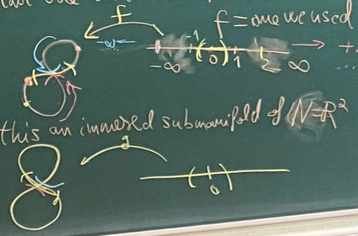
Immersed submanifolds of a given manifold N: "equivalent" viewpoints

- subset $M\subseteq N$ TOGETHER WITH a topology & smooth structure on M s.t. the inclusion $i: M\to N$ is an immersion. In practice, simply: "say $f(M_0)$ is an immersed submanifold"
- pairs (M_0, f) with M_0 = manifold and $f: M_0\to N$ injective immersion

$M := f(M_0)$; $M_0 = M$ WITH f = inclusion of M in M

Ex: last one on torus: $M_0=\mathbb{R}$, f = one we used

Ex:



is this an immersed submanifold of $N=\mathbb{R}^2$

Ex: with only one way ← $\mathbb{R}$ injective immersion

There may be more ways to turn a picture into an immersed submanifold!

Tangent spaces:

(I) For embedded ... $p\in M$: $T_p^{geom} M = \{ \dots$

Ex: for $M\subseteq\mathbb{R}^n$ embedded ... $M=\{x\in\mathbb{R}^n / f(x)=0\}$

-1-

Reminder:

- smooth manifolds $(M, \mathcal{A}_M)$
- smooth maps $f:(M,\mathcal{A}_M)\to(N,\mathcal{A}_N)$ $\left((\forall)x\in\mathcal{A}_M, \tilde{x}\in\mathcal{A}_N \Rightarrow \underbrace{\tilde{x}fx^{-1}}_{f_x^{\tilde{x}}}=\text{smooth}\right)$
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Then $\underbrace{\tilde{x}|_U : U\to\Omega}$ serves as a topological chart for M.

all such together form a smooth atlas on M $\rightsquigarrow$ get a smooth str. on M "induced smooth str."

Rk: Inclusion $i: M\hookrightarrow N$, when looking in charts w.r.t. $\tilde{x}$ and $x=\tilde{x}|_U$

$\Rightarrow(\tilde{x}\circ i\circ x^{-1})(x_1,\dots,x_m)=(x_1,\dots,x_m,0,0,\dots,0)$ $\Rightarrow$ i = immersion

-2-

- Smooth embedding: $f: M\to N$ (smooth) ...

Rk: inclusion $i: M \to N$; when looking in charts with ...

$$\Rightarrow (\tilde{x} \circ i \circ x^{-1})(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, 0, \dots, 0) \quad \Rightarrow i = \text{immersion}$$

- **Smooth embedding**: $f: M \to N$ (smooth) s.t.
  - (1) $f(M)$ = embedded submanifold of N (with s.s. induced from N)
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Rk1: Smooth embedding $\Rightarrow$ topological embedding

Rk2:
- $M \subseteq N$ embedded submanifold $\Rightarrow i: M \to N$ topological embedding
- $M \xrightarrow{f} N$ (smooth) embedding $\Rightarrow$ $M \xrightarrow{f} N$, with $M \to f(M)$ by (2): diffeo, and $f(M) \to N$ by (1): inclusion of embedded $\Rightarrow$ f = immersion

Theorem: Given $f: M \to N$ smooth map between two manifolds:

$$\left\{ f = \text{smooth embedding} \right\} \Leftrightarrow \left\{ \begin{array}{l} (1)\ f = \text{topological embedding} \\ \& \\ (2)\ f = \text{immersion} \end{array} \right\} \xLeftrightarrow[\text{is compact}]{\text{when } M} \left\{ f = \text{injective immersion} \right\}$$

$$(\forall)\, x \in \mathcal{A}_M,\ \tilde{x} \in \mathcal{A}_N \Rightarrow \underbrace{\tilde{x} f x^{-1}}_{f_x^{\tilde{x}}} = \text{smooth}$$

——————— = immersion

(∃) — " ——— given

by $(x_1,\dots,x_m) \mapsto (x_1,\dots,x_m,0,\dots,0)$

N around p s.t.

for M.

S ⇝ get a smooth str. on M

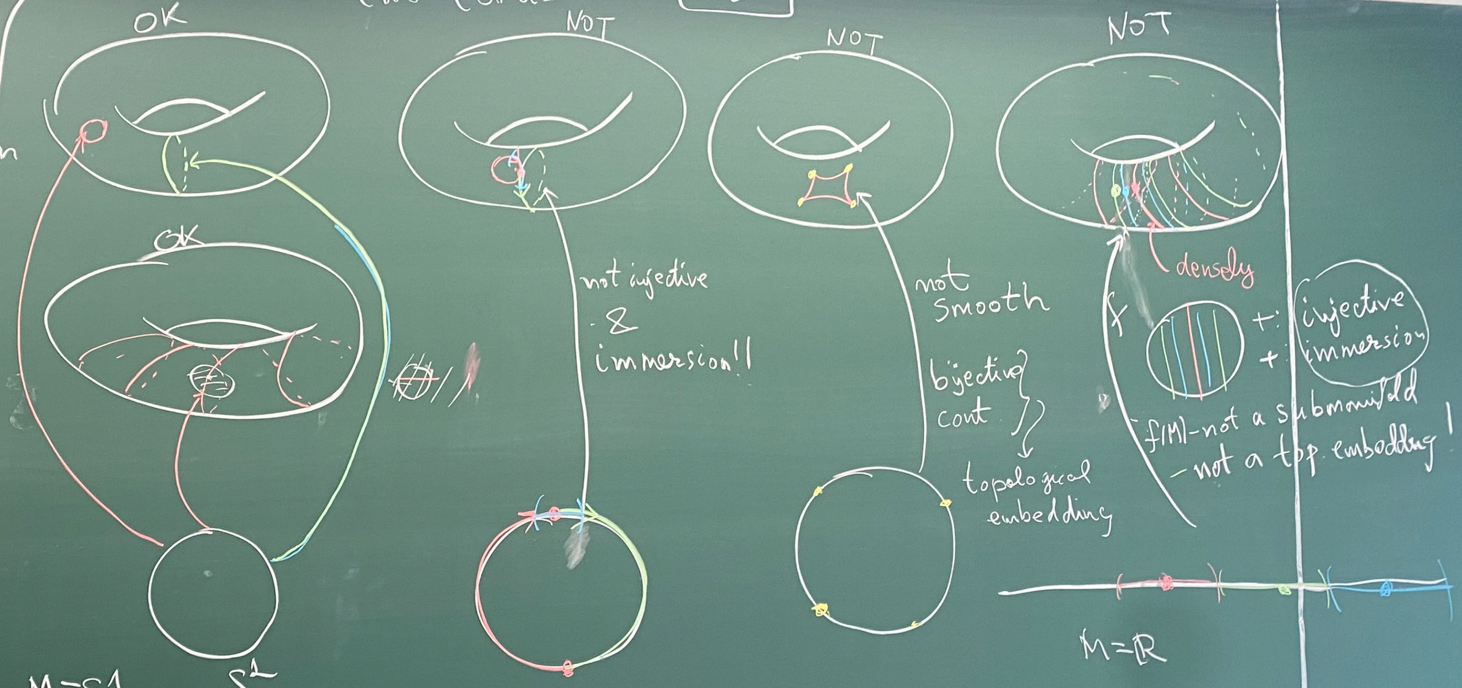
"induced smooth str."

charts w.r.t. $\tilde{x}$ and $x = \tilde{x}|_U$

,0) ⇒ i = immersion

Ex: In N = two-torus

-3-



-2-

X

morphism

with s.s. induced from N.

—1—

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—3—

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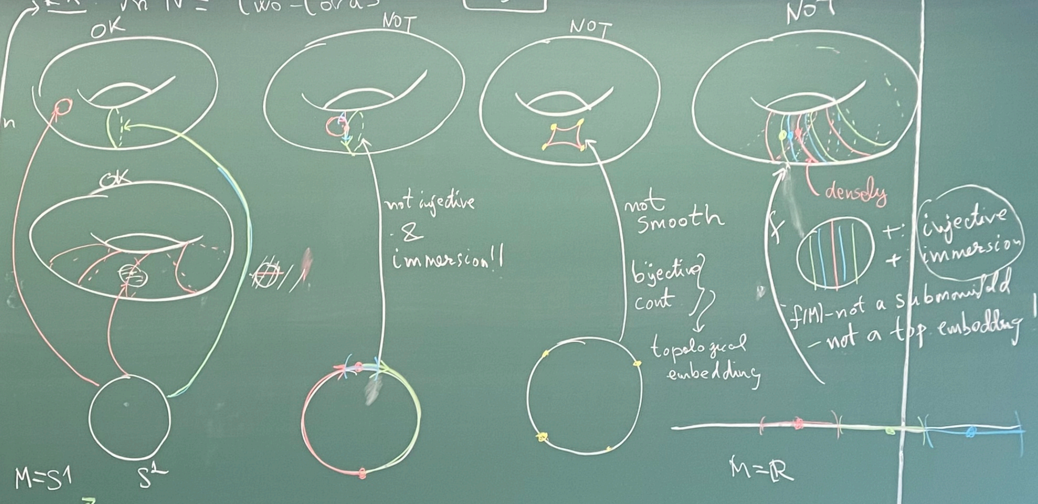

—2—

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Theorem: Given $f: M\to N$ smooth map between two manifolds:

$\{f = \text{smooth embedding}\} \Longleftrightarrow \{(1)\ f = \text{topological embedding}\ \&\ (2)\ f = \text{immersion}\} \underset{\text{is compact}}{\overset{\text{when } M}{\Longleftrightarrow}} \{f = \text{injective immersion}\}$

topological embedding

f(M) - not a submanifold
- not a top. embedding

-4-

Immersed submanifolds: "equivalent" viewpoints of a given manifold N:

$M := f(M_0)$ • subset $M \subseteq N$ TOGETHER WITH a topology & smooth structure on M s.t. the inclusion $i: M \hookrightarrow N$ is an immersion.

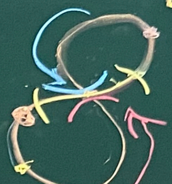
• pairs (M_0, f) with M_0 = manifold and $f: M_0 \to N$ injective immersion

In practice, sloppily: "Say $f(M_0)$ is an immersed submanif."

$M_0 := M$ WITH f = inclusion of M in M

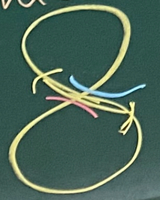
Ex: last one on torus: $M_0 = \mathbb{R}$, f = one we used

Ex:



f: $-\infty \leftarrow \;\; (\; 0 \;) \; 1 \;\; 2 \;\; \infty \;\; \dots \to +\infty$

is this an immersed submanifold of $N = \mathbb{R}^2$



g: $(\; 0 \;)$

Ex: with only one way:

$\leftarrow \mathbb{R}$

injective immersion ....

There may be more ways to turn a picture into an immersed submanifold!

Tangent

(I) F

$\forall p \in M$:

Ex: for $M \subseteq$

$M = \{x \in$

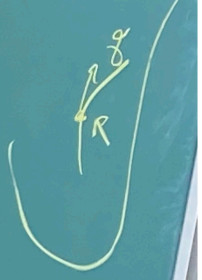
-5-

T_pM, $p \in M$

Tangent spaces:

(I) For embedded $M \subseteq \mathbb{R}^n$: the "classical" ones

$\forall p \in M$: $T_p^{geom} M = \left\{ \frac{d\gamma}{dt}(0) \in \mathbb{R}^n : \begin{array}{l} \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n \text{ smooth} \\ \text{s.t. } \gamma(0) = p,\ \gamma(t) \in M\ \forall t \end{array} \right\}$



$= \left\{ \frac{d\gamma}{dt}(0) : \gamma : (-\varepsilon, \varepsilon) \to M \text{ smooth}, \gamma(0) = p \right\}$

$= \left\{ \frac{d\gamma}{dt}(0) : \gamma \in \mathrm{Curves}_p(M) \right\} \subseteq \mathbb{R}^n$ vector subspace

$\mathbb{R}^n$?? do not make sense (YET) on arbitrary M

Ex: for $M \subseteq \mathbb{R}^n$ embedded ~~as~~ comes from applying the RVT
$M = \{x \in \mathbb{R}^n / f(x) = 0\}$, $f : \mathbb{R}^n \to \mathbb{R}^k$ submersion...
$\Rightarrow T_p^{geom}$ $(df)_p : \mathbb{R}^n \to \mathbb{R}^k$ linear map.

$\Rightarrow T_p^{geom} M = \{ v \in \mathbb{R}^n / (df)_p(v) = 0 \} = \mathrm{Ker}((df)_p)$

Intuition: $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, i.e.: $f(\gamma(t)) = 0\ \forall t$ $\left| \frac{d}{dt} \right|_{t=0}$
$\to (df)_p\left(\frac{d\gamma}{dt}(0)\right) = 0 \to \frac{d\gamma}{dt}(0) \in \mathrm{Ker}(df)_p$

$f_x^{\tilde{x}}$

$A_M, \tilde{x}\in A_N \Rightarrow \tilde{x} f x^{-1}$ = smooth

———— = immersion

— // ———— given

y $(x_1,\dots,x_m)\mapsto(x_1,\dots,x_m,0,\dots,0)$

around p s.t.

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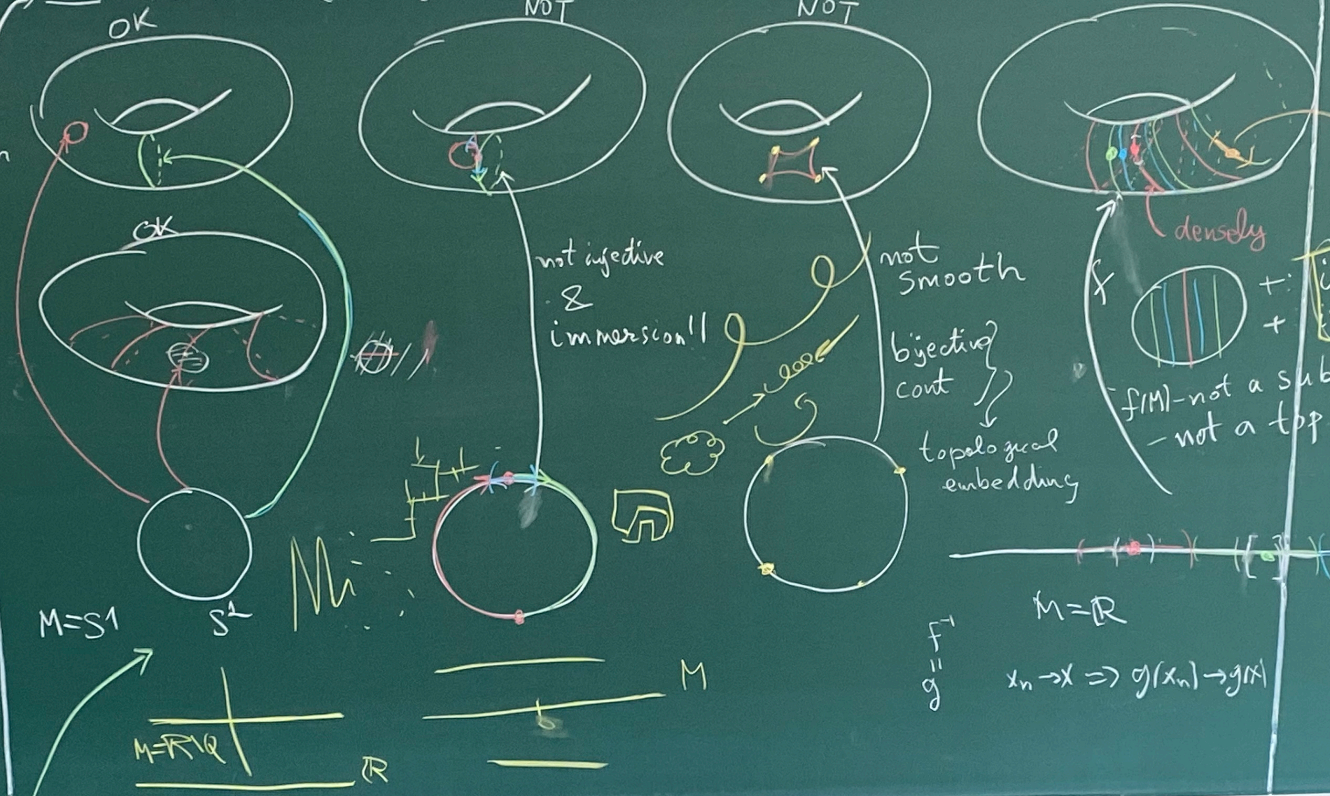

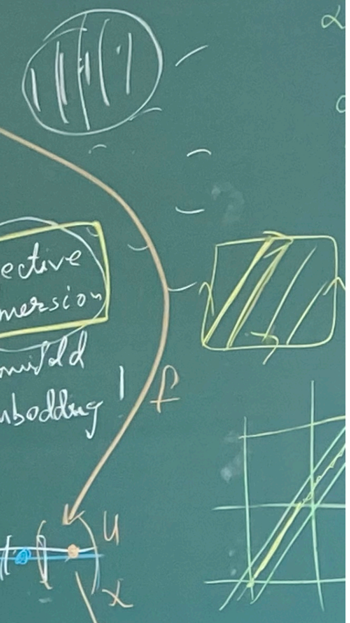

-2-

N

...morphism

with s.s. induced from N

→N topological embedding

⇒ f = immersion

...tween two manifolds:

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-4-

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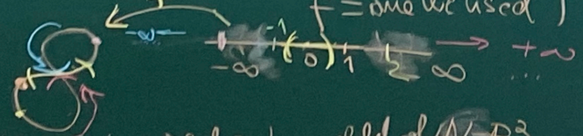

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$=\{\frac{d\gamma}{dt}(0)\}$

$=\{[\frac{d\gamma}{dt}(0)]\}$ comes fr...

Ex: for $M\subseteq\mathbb{R}^n$ embedded

$M=\{x\in\mathbb{R}^n / f(x)=0\}$, f

$\Rightarrow T_p^{geom}M=\{v\in\mathbb{R}^n/$

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