For embedded $\boxed{M \subseteq \mathbb{R}^L}$: the "classical" tangent spaces:

$$T_p^{class} M = \left\{ \frac{d\gamma}{dt}(0) \,\middle/\, \underbrace{\gamma : (-\varepsilon,\varepsilon) \to M \text{ smooth s.t. } \gamma(0)=p}_{\gamma \in \boxed{Curves_p(M)}} \right\}$$

$p \in M$ (intuitively: THE LINEAR APPROXIMATION OF M AROUND p)

For smooth maps $F: M \to N$ between $M \subseteq \mathbb{R}^L, N \subseteq \mathbb{R}^{L'}$:

$(d^{class}F)_p : T_p^{class} M \to T_{F(p)}^{class} N$ linear map.

(intuitively: THE LINEAR APPROXIMATION OF F AROUND p)

Main properties: $T_p^{class} M$ = vector spaces

$(d^{class}F)_p$ = linear maps satisfying the chain rule.

For arbitrary manifolds M we wish:

TW1 $(\forall)$ $\left.\begin{array}{l}\text{manifold } M\\ p\in M\end{array}\right\} \rightsquigarrow$ want a vector space $\boxed{T_pM}$

TW2 $(\forall)$ $\left.\begin{array}{l}F: M\to N \text{ smooth}\\ p\in M\end{array}\right\} \rightsquigarrow$ want a linear map $\boxed{(dF)_p : T_pM \to T_{F(p)}N}$

Such that:

- ⊛ For $F = \mathrm{Id}_M : M \to M$, $(dF)_p = \mathrm{Id}$ of T_pM
- ⊛ Chain rule: for smooth

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[rr, bend right] & N \arrow[r, "G"] & P \\ p \arrow[u, phantom, "\in"] & & \end{tikzcd}$$

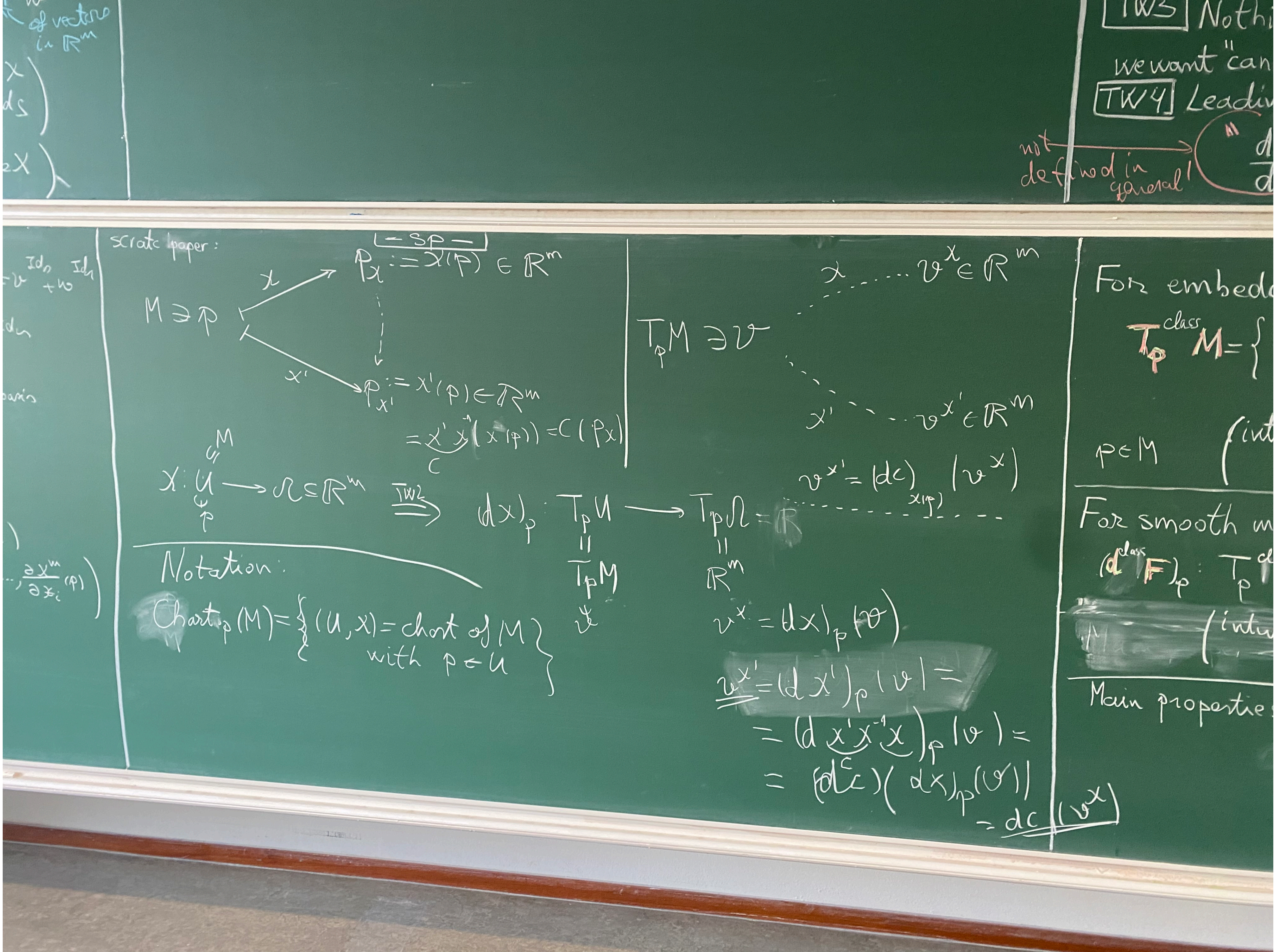
$$(dG\circ F)_p = (dG)_{F(p)} \circ (dF)_p$$

$$\begin{tikzcd} T_pM \arrow[r, "(dF)_p"] \arrow[rr, bend right] & T_{F(p)}N \arrow[r, "dG"] & TP \end{tikzcd}$$

TW3 Nothing new in $\mathbb{R}^k$: for embedded $M \subseteq \mathbb{R}^k$, we want "canonical" isomorphisms $T_pM \xrightarrow{\cong} T_p^{\mathrm{clas}}M$.

TW4 Leading slogan: T_pM is made of speeds

not defined in general! $\longrightarrow$ "$\dfrac{d\gamma}{dt}(0)$" with $\gamma \in \mathrm{Curves}_p(M)$



of vectors in $\mathbb{R}^m$

x)

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eX)

$= v^{Id_n} + w^{Id_n}$

Id_n

basis

$\dots, \frac{\partial x^m}{\partial x_i}(p))$

scratc paper:

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$$\begin{tikzcd}
 & p_x := x(p) \in \mathbb{R}^m \arrow[dd, dashed] \\
M \ni p \arrow[ur, mapsto, "x"] \arrow[dr, mapsto, "x'"'] & \\
 & p_{x'} := x'(p) \in \mathbb{R}^m
\end{tikzcd}$$

$$= \underbrace{x' x^{-1}}_{C}(x(p)) = C(p_x)$$

$$x : \underset{\underset{p}{\cup}}{\overset{\overset{M}{\cap}}{U}} \longrightarrow \Omega \subseteq \mathbb{R}^m \overset{\underline{TW2}}{\Longrightarrow} (dx)_p : \underset{\underset{v}{\cup}}{\overset{}{\underset{T_pM}{\overset{T_pU}{\parallel}}}} \longrightarrow \underset{\mathbb{R}^m}{\overset{T_p\Omega}{\parallel}} = \mathbb{R}$$

$$T_pM \ni v \quad \overset{x}{\cdots\cdots} \quad v^x \in \mathbb{R}^m$$

$$ \quad \overset{x'}{\cdots\cdots} \quad v^{x'} \in \mathbb{R}^m$$

$$v^{x'} = (dc)_{x(p)}(v^x)$$

$$v^x = (dx)_p(v)$$

$$\underline{\underline{v^{x'}}} = (dx')_p(v) =$$

$$= (d\,\underbrace{x'x^{-1}}_{c}\underbrace{x})_p(v) =$$

$$= (dc)((dx)_p(v))$$

$$= \underline{dc(v^x)}$$

Notation:

$\mathrm{Chart}_p(M) = \{ (U,x) = \text{chart of } M$ with $p \in U \}$

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For smooth m

$(d^{class}F)_p : T_p$

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Main propertie

M = fixed manifold, $p \in M$ $\boxed{-3-}$

Def 1 A tangent vector to M at p is a function

$$v : \mathrm{Chart}_p(M) \longrightarrow \mathbb{R}^m$$

denoted $\quad x \longmapsto v^x$ called "v in the chart x"

s.t. $(\forall)\, x, x' \in \mathrm{Chart}_p(M)$ one has $\boxed{v^{x'} = (dc)_{x(p)}(v^x) \quad (*)}$

where $c = x' \circ x^{-1}$ (change of coord)

The collection of such v's : denoted T_pM.

TW1: $\left.\begin{matrix} v, w \in T_pM \\ \lambda \in \mathbb{R} \end{matrix}\right\} \rightsquigarrow$ can we form $\begin{matrix} v+w \\ \lambda v \end{matrix} \in T_pM$? Yes:

$v+w : \mathrm{Chart}_p(M) \to \mathbb{R}^m$

$(v+w)^x := v^x \underset{\nwarrow}{+} w^x$ of vectors in $\mathbb{R}^m$

$(\lambda v)^x = \lambda \cdot v^x$

$\Longrightarrow T_pM$ = vector space

Rk: (To have / to specify $v \in T_pM$) $\Longleftrightarrow$ (to have $v^x \in \mathbb{R}^m$, one for each x & make sure that (*) holds)

$\Updownarrow$

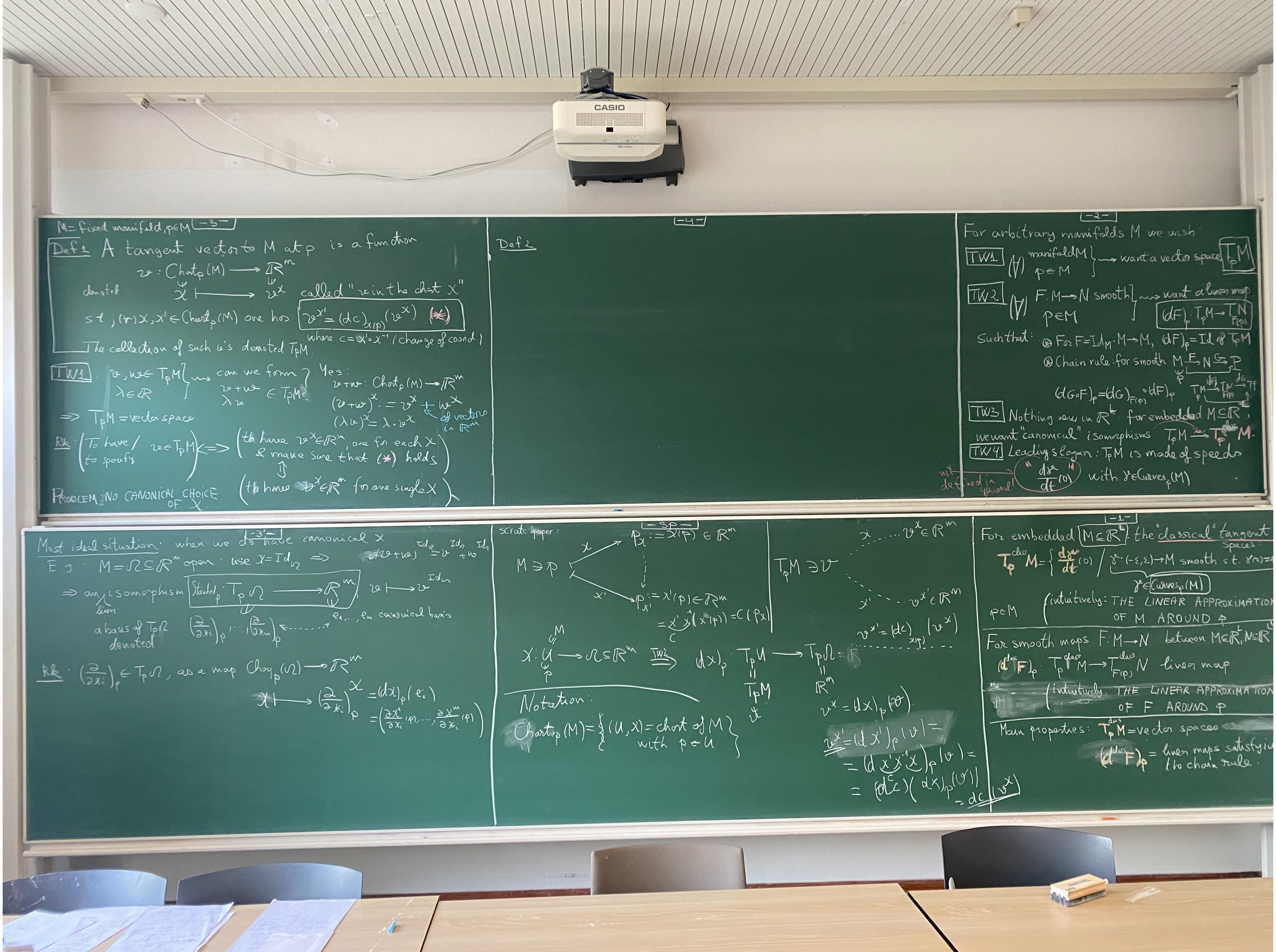
(to have $v^x \in \mathbb{R}^m$ for one single x)

PROBLEM: NO CANONICAL CHOICE OF x.

Def 2

$\boxed{-3'-}$

situation: when we do have canonical x.



-1-

For embedded $\boxed{M \subseteq \mathbb{R}^L}$: the "classical" tangent spaces:

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(intuitively: THE LINEAR APPROXIMATION OF F AROUND p

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$(d^{class} F)_p$ = linear maps satisfying the chain rule.

-2-

For arbitrary manifolds M we wish:

$\boxed{TW1}$ $(\forall)$ $\left.\begin{matrix} \text{manifold } M \\ p \in M \end{matrix}\right\} \rightsquigarrow$ want a vector space $\boxed{T_p M}$

$\boxed{TW2}$ $(\forall)$ $\left.\begin{matrix} F : M \to N \text{ smooth} \\ p \in M \end{matrix}\right\} \rightsquigarrow$ want a linear map $\boxed{(dF)_p : T_p M \to T_{F(p)} N}$

Such that:

- ⊛ For $F = Id_M : M \to M$, $(dF)_p = Id$ of $T_p M$
- ⊛ Chain rule: for smooth $\underset{\overset{\cup}{p}}{M} \xrightarrow{F} N \xrightarrow{G} P$

$$(dG \circ F)_p = (dG)_{F(p)} \circ (dF)_p \qquad T_p M \xrightarrow{(dF)_p} T_{F(p)} N \xrightarrow{dG} T P$$

$\boxed{TW3}$ Nothing new in $\mathbb{R}^k$: for embedded $M \subseteq \mathbb{R}^L$, we want "canonical" isomorphisms $T_p M \xrightarrow{\simeq} T_p^{class} M$.

$\boxed{TW4}$ Leading slogan: $T_p M$ is made of speeds "$\frac{d\gamma}{dt}(0)$" with $\gamma \in Curves_p(M)$

not defined in general!

-3-

M = fixed manifold, $p \in M$

Def 1 A tangent vector to M at p is a function

$$v : Chat_p(M) \longrightarrow \mathbb{R}^m$$
$$\overset{\psi}{x} \longmapsto \overset{\psi}{v^x}$$

denoted v^x, called "v in the chart x"

s.t. $(\forall) x, x' \in Chart_p(M)$ one has $\boxed{v^{x'} = (dc)_{x(p)}(v^x) \quad (*)}$

where $c = x' \circ x^{-1}$ (change of coord)

The collection of such v's denoted $T_p M$.

$\boxed{TW1}$. $\left.\begin{matrix} v, w \in T_p M \\ \lambda \in \mathbb{R} \end{matrix}\right\} \rightsquigarrow$ can we form $\left.\begin{matrix} v + w \\ \lambda v \end{matrix}\right. \in T_p M$? Yes:

$v + w : Chat_p(M) \to \mathbb{R}^m$

$(v+w)^x := v^x + w^x$ ← of vectors in $\mathbb{R}^m$

$(\lambda v)^x = \lambda \cdot v^x$

$\Rightarrow T_p M$ = vector space

Rk: $\left(\begin{matrix} \text{To have /} \\ \text{to specify} \end{matrix} \; v \in T_p M\right) \Longleftrightarrow \left(\begin{matrix} \text{to have } v^x \in \mathbb{R}^m, \text{ one for each } x \\ \text{\& make sure that } (*) \text{ holds} \end{matrix}\right)$

$\Updownarrow$

PROBLEM: NO CANONICAL CHOICE OF x. (to have $v^x \in \mathbb{R}^m$ for one single x)

-3'-

Most ideal situation: when we do have canonical x

E.g.: $M = \Omega \subseteq \mathbb{R}^m$ open: use $x = Id_\Omega$ $\Rightarrow$ $(v+w)^{Id_\Omega} = v^{Id_\Omega} + w^{Id_\Omega}$

$\Rightarrow$ an isomorphism (linear) $\boxed{Standard_p : T_p \Omega \longrightarrow \mathbb{R}^m}$, $v \longmapsto v^{Id_\Omega}$

a basis of $T_p \Omega$ denoted $\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p$ ←-------→ $e_1, \ldots, e_m$ canonical basis

Rk: $\left(\frac{\partial}{\partial x_i}\right)_p \in T_p \Omega$, as a map $Chay_p(\Omega) \to \mathbb{R}^m$

$$x \longmapsto \left(\frac{\partial}{\partial x_i}\right)_p^x = (dx)_p(e_i) = \left(\frac{\partial x^1}{\partial x_i}(p), \ldots, \frac{\partial x^m}{\partial x_i}(p)\right)$$

-4-

Def 2

-SP-

scratch paper:

$$M \ni p \;\overset{x}{\longmapsto}\; p_x := x(p) \in \mathbb{R}^m$$
$$ \;\overset{x'}{\longmapsto}\; p_{x'} := x'(p) \in \mathbb{R}^m = \underbrace{x' x^{-1}}_{c}(x(p)) = c(p_x)$$

$$T_p M \ni v \;\overset{x}{\cdots}\; v^x \in \mathbb{R}^m$$
$$ \;\overset{x'}{\cdots}\; v^{x'} \in \mathbb{R}^m$$
$$v^{x'} = (dc)_{x(p)}(v^x)$$

$$x : \underset{\overset{\cup}{p}}{\overset{\cap M}{U}} \longrightarrow \Omega \subseteq \mathbb{R}^m \overset{TW2}{\Longrightarrow} (dx)_p : \underset{\overset{\|}{T_p M}}{T_p U} \longrightarrow \underset{\overset{\|}{\mathbb{R}^m}}{T_p \Omega}$$

$v^x = (dx)_p(v)$

$\underline{\underline{v^{x'}}} = (dx')_p(v) =$

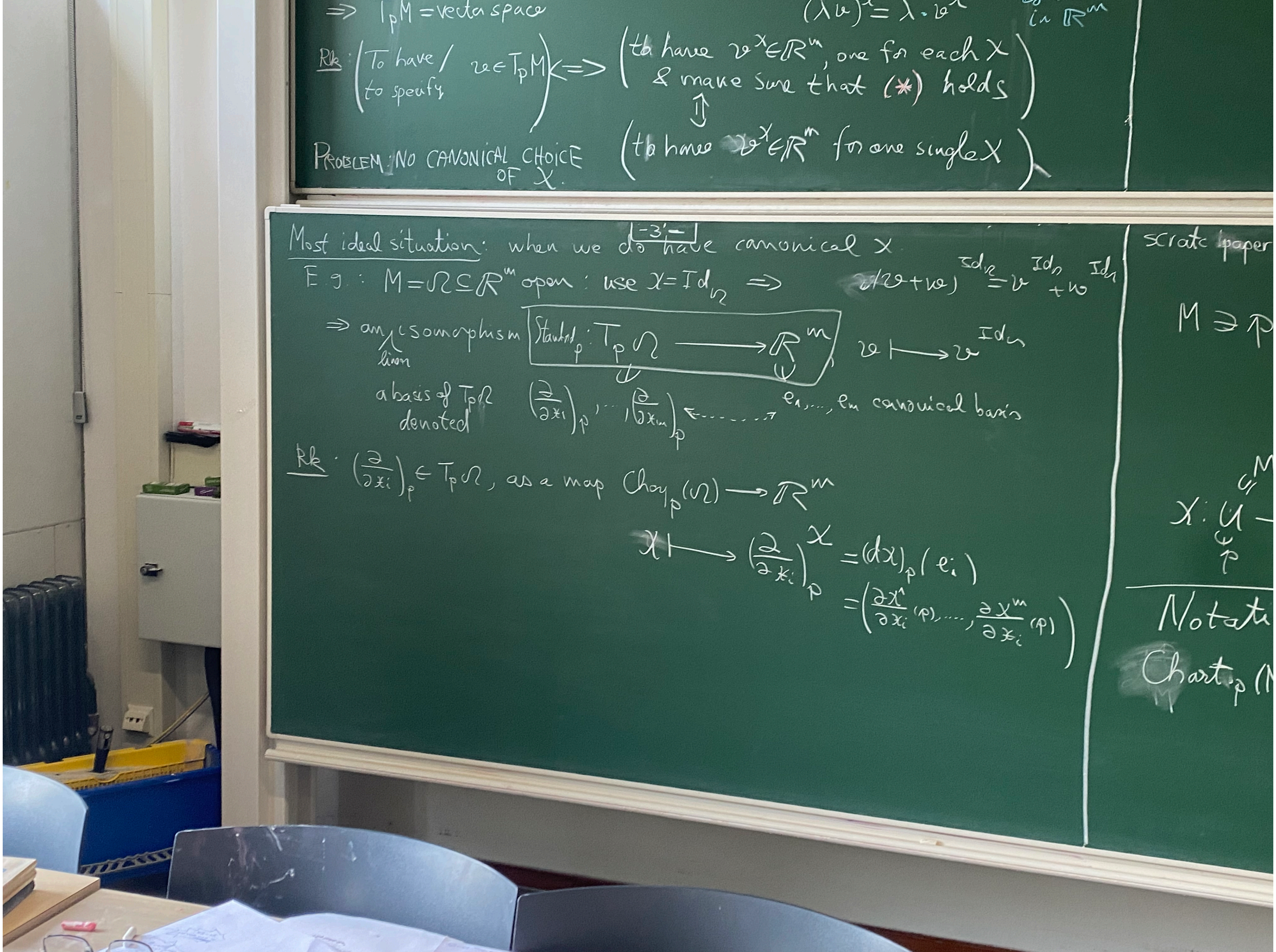
$= (d \underbrace{x' x^{-1}}_{c} \underbrace{x}) _p(v) =$

$= (dc)((dx)_p(v))$

$= \underline{\underline{dc(v^x)}}$

Notation:

$$Chart_p(M) = \left\{ \begin{matrix} (U, x) = \text{chart of } M \\ \text{with } p \in U \end{matrix} \right\}$$



⇒ T_pM = vector space $(\lambda v)^x = \lambda \cdot v^x$ in $\mathbb{R}^m$

Rk: $\left(\begin{array}{l}\text{To have /}\\ \text{to specify}\end{array}\ v \in T_pM\right) \Longleftrightarrow \left(\begin{array}{l}\text{to have } v^x \in \mathbb{R}^m\text{, one for each } x\\ \text{\& make sure that (*) holds}\end{array}\right)$

$\Updownarrow$

$\left(\text{to have } v^x \in \mathbb{R}^m \text{ for one single } x\right)$

PROBLEM: NO CANONICAL CHOICE OF x.

-3'-

Most ideal situation: when we do have canonical x.

E.g.: $M = \Omega \subseteq \mathbb{R}^m$ open: use $x = Id_\Omega$ ⇒ $(v+w)^{Id_\Omega} = v^{Id_\Omega} + w^{Id_\Omega}$

⇒ an (linear) isomorphism $\boxed{Standard_p : T_p\Omega \longrightarrow \mathbb{R}^m}$, $v \longmapsto v^{Id_\Omega}$

a basis of $T_p\Omega$ denoted $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \longleftarrow\dashrightarrow e_1, \dots, e_m$ canonical basis

Rk: $\left(\frac{\partial}{\partial x_i}\right)_p \in T_p\Omega$, as a map $Charts_p(\Omega) \longrightarrow \mathbb{R}^m$

$$x \longmapsto \left(\frac{\partial}{\partial x_i}\right)_p^x = (dx)_p(e_i) = \left(\frac{\partial x^1}{\partial x_i}(p), \dots, \frac{\partial x^m}{\partial x_i}(p)\right)$$

scratch paper

$M \ni p$

$x : U$

$U \subseteq M$, $p \in U$

Notati

$Chart_p(M$

M = fixed manifold, p ∈ M $\boxed{-3-}$

Def 1 A tangent vector to M at p is a function

$$v : \mathrm{Chart}_p(M) \longrightarrow \mathbb{R}^m, \quad \mathcal{X} \longmapsto v^{\mathcal{X}}$$

called "v in the chart $\mathcal{X}$"

denoted

s.t. $(\forall)\, \mathcal{X}, \mathcal{X}' \in \mathrm{Chart}_p(M)$ one has $\boxed{v^{\mathcal{X}'} = (dc)_{\mathcal{X}(p)}(v^{\mathcal{X}}) \quad (*)}$

where $c = \mathcal{X}' \circ \mathcal{X}^{-1}$ (change of coord)

The collection of such v's denoted T_pM.

define a linear map.

TW1 $v, w \in T_pM$, $\lambda \in \mathbb{R}$ ⇝ can we form $v + w$, $\lambda v \in T_pM$?

Yes: $v + w : \mathrm{Chart}_p(M) \longrightarrow \mathbb{R}^m$

$$(v+w)^{\mathcal{X}} := v^{\mathcal{X}} + w^{\mathcal{X}}$$

$$(\lambda v)^{\mathcal{X}} = \lambda \cdot v^{\mathcal{X}}$$

+ of vectors in $\mathbb{R}^m$

$\Rightarrow T_pM$ = vector space

(To define $v \in T_pM$) ⟺ (to specify $v^{\mathcal{X}} \in \mathbb{R}^m$, one for each $\mathcal{X} \in \mathrm{Chart}_p(M)$ & make sure (*) holds)

⇕

(To specify $v \in T_pM$) ⟺ (to specify $v^{\mathcal{X}} \in \mathbb{R}^m$ for one single $\mathcal{X} \in \mathrm{Chart}_p(M)$)

Ex: $\Omega \subseteq \mathbb{R}^m$ open ⇒ linear isomorphisms $\boxed{-6-}$ $\mathrm{Standard}_p : T_p\Omega \longrightarrow \mathbb{R}^m$, $v \longmapsto v^{\mathrm{Id}_\Omega}$.

Rk: More generally, $(\forall)$ M, p, $(\forall)\, \mathcal{X} \in \mathrm{Chart}_p(M)$ ⇒ we get:

- linear isomorphism $T_pM \longrightarrow \mathbb{R}^m$, $v \longmapsto v^{\mathcal{X}}$
- a basis of T_pM, denoted $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_pM$ uniquely characterised by:

$e_1, \dots, e_m$

$$\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)^{\mathcal{X}} = e_i = (0, \dots, 0, \underset{i\text{th}}{1}, 0, \dots, 0) \in \mathbb{R}^m$$

$$\left(\left(\frac{\partial}{\partial x_i}\right)_p\right)^{\mathcal{X}'} = \dots \text{ use } (*) \text{ exercise}$$

(In particular $\dim_{\mathbb{R}} T_pM = \dim M$ as a manifold)

TW4 Make sense of $\frac{d\gamma}{dt}(0) \in T_pM$ for $\gamma \in \mathrm{Curve}_p(M)$

$\mathcal{X} \in \mathrm{Chart}_p(M) \longrightarrow \mathbb{R}^m \ni$ what vector in $\mathbb{R}^m$?

$$\boxed{\left(\frac{d\gamma}{dt}(0)\right)^{\mathcal{X}} := \frac{d\gamma^{\mathcal{X}}}{dt}(0)}$$

classical $\frac{d}{dt}$ of curves in $\mathbb{R}^m$!

$\gamma^{\mathcal{X}} = \mathcal{X} \circ \gamma$ ← curve in $\mathbb{R}^m$

γ in chart $\mathcal{X}$. $(-\varepsilon, \varepsilon) \longrightarrow M \supseteq U$ $\xrightarrow{\mathcal{X}}$

Check (*):

$$\boxed{\left(\frac{d\gamma}{dt}(0)\right)^{\mathcal{X}'}} = \frac{d\gamma^{\mathcal{X}'}}{dt}(0) \overset{\text{pass to } \mathcal{X}:\ \gamma^{\mathcal{X}'} = \mathcal{X}' \circ \gamma = \mathcal{X}' \circ}{=\!=} \frac{d(c \circ \gamma^{\mathcal{X}})}{dt}(0) = (dc)_{\gamma^{\mathcal{X}}(0)}\left(\frac{d\gamma^{\mathcal{X}}}{dt}(0)\right) = \boxed{(dc)_p\left(\left(\frac{d\gamma}{dt}(0)\right)^{\mathcal{X}}\right)} \text{ i.e. } (*)$$

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Ex: For

$f : \Omega \to \mathbb{R}$

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Notation:

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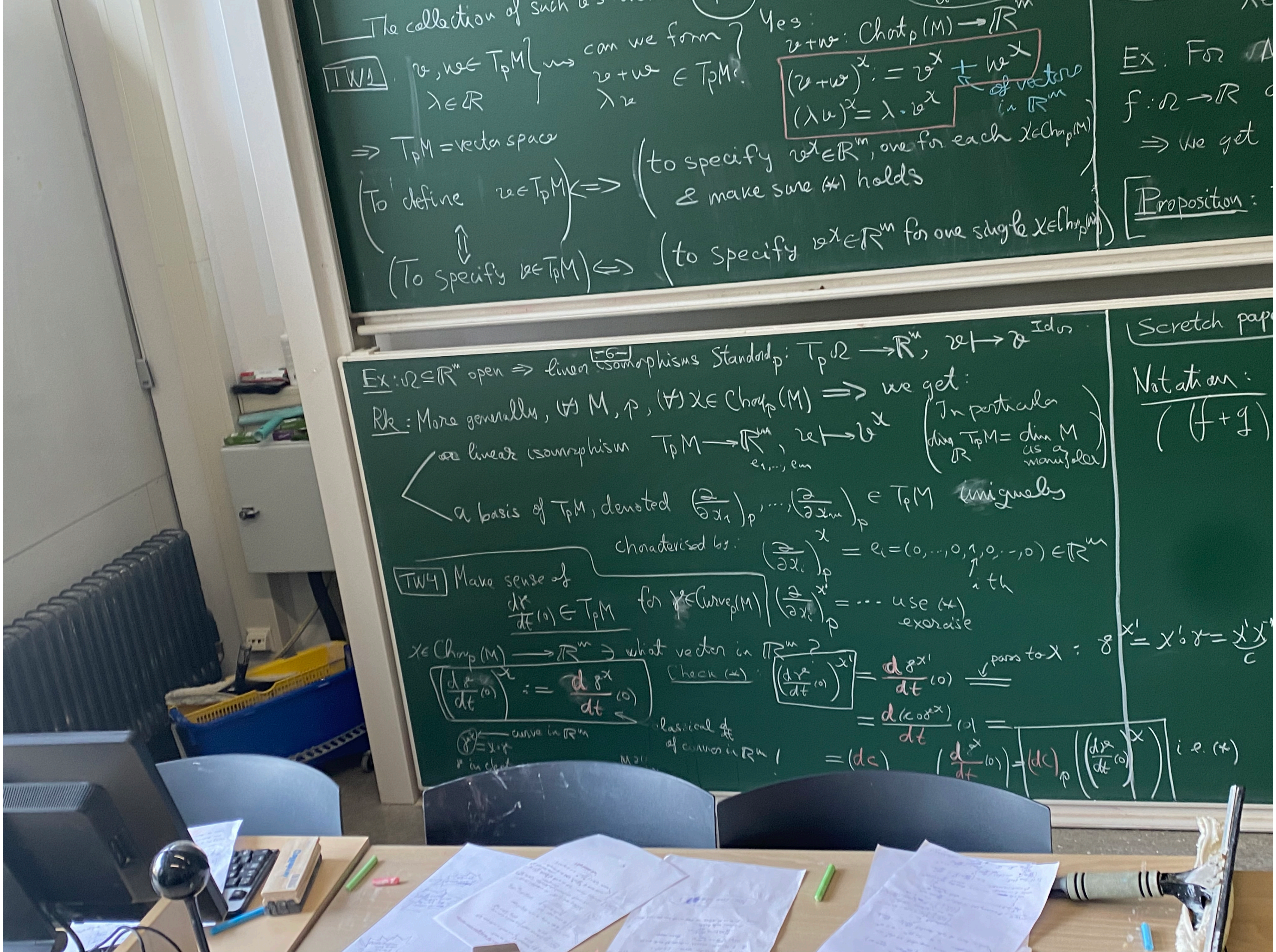


The collection of such

TW1 $v, w \in T_pM$, $\lambda \in \mathbb{R}$ $\rightsquigarrow$ can we form $v+w$, λv $\in T_pM$? Yes: $v+w : \mathrm{Chart}_p(M) \to \mathbb{R}^m$

$$\boxed{\begin{aligned}(v+w)^x &:= v^x \underset{\uparrow}{+} w^x \\ (\lambda v)^x &= \lambda \cdot v^x\end{aligned}}$$

of vectors in $\mathbb{R}^m$

$\Rightarrow$ T_pM = vector space

$\left(\text{To define } v \in T_pM\right) \Longleftrightarrow \left(\text{to specify } v^x \in \mathbb{R}^m\text{, one for each } x \in \mathrm{Chart}_p(M) \text{ \& make sure } (*) \text{ holds}\right)$

$\Updownarrow$

$\left(\text{To specify } v \in T_pM\right) \Leftrightarrow \left(\text{to specify } v^x \in \mathbb{R}^m \text{ for one single } x \in \mathrm{Chart}_p(M)\right)$

Ex. For

$f : \Omega \to \mathbb{R}$

$\Rightarrow$ we get

Proposition:

-6-

Ex: $\Omega \subseteq \mathbb{R}^m$ open $\Rightarrow$ linear isomorphisms $\mathrm{Standard}_p : T_p\Omega \to \mathbb{R}^m$, $v \mapsto v^{\mathrm{Id}_\Omega}$.

Rk: More generally, $(\forall)$ M, p, $(\forall)$ $x \in \mathrm{Chart}_p(M)$ $\Longrightarrow$ we get:

- linear isomorphism $T_pM \to \underset{e_1,\dots,e_m}{\mathbb{R}^m}$, $v \mapsto v^x$
- a basis of T_pM, denoted $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_pM$ uniquely characterised by: $\left(\frac{\partial}{\partial x_i}\right)_p^x = e_i = (0,\dots,0,\underset{\substack{\uparrow \\ i\text{th}}}{1},0,\dots,0) \in \mathbb{R}^m$

(In particular $\dim_{\mathbb{R}} T_pM = \dim M$ as a manifold)

$\left(\frac{\partial}{\partial x_i}\right)_p^{x'} = \cdots$ use $(*)$ exercise

TW4 Make sense of $\frac{d\gamma}{dt}(0) \in T_pM$ for $\gamma \in \mathrm{Curve}_p(M)$

$x \in \mathrm{Chart}_p(M) \longrightarrow \mathbb{R}^m \ni$ what vector in $\mathbb{R}^m$?

$$\boxed{\left(\frac{d\gamma}{dt}(0)\right)^x := \frac{d\gamma^x}{dt}(0)}$$

classical $\frac{d}{dt}$ of curves in $\mathbb{R}^m$

$\gamma^x \leftarrow$ curve in $\mathbb{R}^m$

$= x \circ \gamma$

γ in chart

Check $(*)$: $\boxed{\left(\frac{d\gamma}{dt}(0)\right)^{x'}} = \frac{d\gamma^{x'}}{dt}(0) \overset{\text{pass to } x}{=\!=}$

$= \frac{d(c \circ \gamma^x)}{dt}(0) = (dc)\left(\frac{d\gamma^x}{dt}(0)\right) = \boxed{(dc)_p\left(\left(\frac{d\gamma}{dt}(0)\right)^x\right)}$ i.e. $(*)$

pass to x: $\gamma^{x'} = x' \circ \gamma = \underbrace{x'x^{-1}}_{c}$

Scretch pap

Notation:

$\left((f+g)\right.$

$\boxed{\text{Scretch paper:}}$ $\boxed{-7-}$

Notation: $\mathcal{C}^{\infty}(M) = \{ f : M \to \mathbb{R} \,/\, f = \text{smooth} \}$

$\left((f+g)(x) = f(x) + g(x) \text{ etc} \right)$

$\boxed{\text{TW4}}$: Make sense, to $\boxed{-9-}$

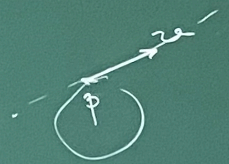
$\gamma \in \text{Curves}_p(M)$ of $\frac{d\gamma}{dt}(0) \in T_pM$

As a map:

$$\begin{aligned} \mathcal{C}^{\infty}(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto \frac{d\, f\circ\gamma}{dt}(0) \end{aligned}$$

$f\circ\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$

$$\gamma^{x'} = x'\circ\gamma = \underbrace{x' x^{-1}}_{c} \underbrace{x\,\gamma} = c\circ\gamma^{x}$$



$\boxed{\left(\frac{d\gamma}{dt}(0)\right)^{x}}$ i.e. (*)

for $F : \overset{\substack{\mathbb{R}^m \\ \cup}}{\Omega} \longrightarrow \mathbb{R}^n$

$(dF)_p : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$$(dF)_p(v) = \underbrace{\lim_{t\to 0} \frac{F(p+tv) - F(p)}{t}}_{\partial_v F}$$

for $F : \overset{\substack{\mathbb{R}^L \\ \cup}}{M} \longrightarrow \overset{\substack{\mathbb{R}^{L'} \\ \cup}}{N}$

$$(dF)_p(\underset{\substack{\| \\ \frac{d\gamma}{dt}(0)}}{v}) = \frac{d\, F\circ\gamma}{dt}(0)$$

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single $x \in \mathrm{Chap}_p(M)$)

-8-

Def 2 A derivation of M at p is any map

$$\partial : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R}$$

which: ⊛ ∂ = linear : $\begin{cases} \partial(f+g) = \partial(f) + \partial(g) \\ \partial(\lambda f) = \lambda\, \partial(f) \end{cases}$ $f, g \in \mathcal{C}^\infty(M),\ \lambda \in \mathbb{R}$

⊛ satisfies the (Leibniz) derivation rule $\partial(fg) = \underbrace{f(p)}_{\mathbb{R}} \cdot \underbrace{\partial(g)}_{\mathbb{R}} + g(p) \cdot \partial(f)$

The collection of such is denoted T_pM.

TW1: $\partial, \partial' \in T_pM$, $\lambda \in \mathbb{R}$ } ⟶ can define $\partial + \partial'$, $\lambda \cdot \partial \in T_pM$. } Yes: $(\partial + \partial')(f) = \partial(f) \overset{\text{from } \mathbb{R}}{+} \partial'(f)$

$(\lambda \partial)(f) = \lambda \cdot \partial(f)$

Ex: For $M = \Omega \subseteq \mathbb{R}^m$ open ⟹ the usual partial derivatives of function $f : \Omega \to \mathbb{R}$ at p are such maps $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$ ⟹

⟹ we get $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p\Omega$.

Proposition: These form a basis of $T_p\Omega$.

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Scretch paper:

-7-

TW4 : M

-9-

= smooth }

$\boxed{-9-}$

$\boxed{\text{TW4}}$: Make sense, fo

$\gamma \in \text{Curves}_p(M)$ of $\frac{d\gamma}{dt}(0) \in T_p M$.

As a map:

$$C^\infty(M) \longrightarrow \mathbb{R}$$

$$f \longmapsto \frac{d}{dt} f\circ\gamma\,(0)$$

$f\circ\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$

for $F : \Omega \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$(dF)_p : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

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