-1-

Def 1: $v \in T_pM$ means $\boxed{v : \mathrm{Charts}_p(M) \to \mathbb{R}^m,\ \ \boxed{x \mapsto v^x}}$ (notation)

such that, $(\forall)\, x, x'$: $\boxed{v^{x'} = (dc)_{x(p)}(v^x)}$ where $c = x' \circ x^{-1}$

$\boxed{\text{TW1}}$: T_pM = vector space : $(v+w)^x := v^x + w^x$ etc.

Rk : for $M = \Omega \subseteq \mathbb{R}^m$ open $\Rightarrow$ $\boxed{\mathrm{Standard}_p : T_p\Omega \xrightarrow{\sim} \mathbb{R}^m}$, $v \mapsto v^{\mathrm{Id}_\Omega}$

linear isomorphism

& basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ $\left(\xrightarrow[\mathrm{Standard}_p]{} e_1, \dots, e_m\right)$

Similarly any $x \in \mathrm{Charts}_p(M)$ induces

$T_pM \xrightarrow{\sim} \mathbb{R}^m$, $v \mapsto v^x$ $\Leftarrow$ a basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ of T_pM.

$\boxed{\text{TW4}}$ any $\gamma \in \mathrm{Curves}_p(M)$ has a speed $\frac{d\gamma}{dt}(0) \in T_pM$:

$$\left(\frac{d\gamma}{dt}(0)\right)^x := \frac{d\gamma^x}{dt}(0) \qquad (\gamma^x = x \circ \gamma, \text{ curve in } \mathbb{R}^m)$$

-2-

$\boxed{\text{TW2}}$: $(\forall)\ F : M \to N$ smooth, $p \in M$, we have a linear map

$$\begin{array}{rccc} (dF)_p : & T_pM & \longrightarrow & T_{F(p)}N \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & v & \longmapsto & \underbrace{(dF)_p(v)}_{\text{defined by}} \end{array}$$

$(dF)_p(v)$ ...

Corollary : F = immersion/submersion at $p \Leftrightarrow$

Exercise : Show that $(dF)_p\left(\frac{d\gamma}{dt}(0)\right) = \frac{dF\circ\gamma}{dt}(0)$ $\left((\forall)\ F : M \to N \text{ smooth}, p \in M,\ \gamma \in \mathrm{Curves}_p(M)\right)$

Exercise : Check the chain rule for dF's.

$\boxed{\text{TW3}}$ Nothing new in $\mathbb{R}^L$

General remark $M \subseteq N$ embedded $\Rightarrow$

-3-

-4-

Def 2: $\partial \in T_pM$ means that $\partial : C^\infty(M) \to \mathbb{R}$
(∂ = DERIVATION AT p) linear & derivation rule $\partial(fg) = f(p)\partial(g)$

$\boxed{\text{TW1}}$ T_pM = vector space : $(\partial + \partial')(f) := \partial(f) + \partial'(f)$

Rk : For $M = \Omega \subseteq \mathbb{R}^m$ open, the usual partial derivatives d

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_p\Omega$$

Prop. 1 : This is a basis of $T_p\Omega$.

More generally, any (U, x) around p induces $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in$

$(\forall)\ M$! $\left(\frac{\partial}{\partial x_i}\right)_p : C^\infty(M) \to \mathbb{R}$, $f \mapsto \frac{\partial f}{\partial x_i}(p) := \frac{\partial f_x}{\partial x_i}(x(p))$

$\boxed{\text{TW4}}$: any $\gamma \in \mathrm{Curves}_p(M)$ has a "speed" $\frac{d\gamma}{dt}(0) \in T_pM$ $\quad (f_x = f \circ x^{-1})$

namely the derivation at p:

$$\partial_{\frac{d\gamma}{dt}(0)} : C^\infty(M) \to \mathbb{R},\ f \mapsto \frac{d\, f\circ\gamma}{dt}(0) = (f \circ \gamma)$$

-5-

$\boxed{\text{TW2}}$ $(\forall)\ F : M \to N$ smooth, $p \in M$, we have a linear map

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defined by

$(dF)_p(\partial)$ ...

Exercise : Check that $(dF)_p\left(\frac{d\gamma}{dt}(0)\right) = \frac{dF\circ\gamma}{dt}(0)$

Exercise : Check the chain rule for dFs.

Proposition 2 : For any $U \subseteq M$ open, $(di)_p : T_pU \to T_pM$ is an isomorph
($i : U \to M$ the inclusion)

-6-

-1-

Def 1. $v \in T_pM$ means $\boxed{v : \mathrm{Charts}_p(M) \longrightarrow \mathbb{R}^m,\ \chi \longmapsto v^\chi}$ (notation)

such that, $(\forall)\, \chi, \chi'$: $\boxed{v^{\chi'} = (dc)_{\chi(p)}(v^\chi)}$ $\boxed{(*)}$ where $c = \chi' \circ \chi^{-1}$

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v, w

Rk : for $M = \Omega \subseteq \mathbb{R}^m$ open $\Rightarrow$ $\boxed{\mathrm{Standard}_p : T_p\Omega \xrightarrow{\sim} \mathbb{R}^m,\ v \longmapsto v^{\mathrm{Id}_\Omega}}$
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THE STANDARD IDENTIFICATION

linear isomorphism

& basis $\left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial x_m}\right)_p$ $\left(\xmapsto{\mathrm{Standard}_p} e_1, \dots, e_m \right)$

Similarly any $\chi \in \mathrm{Charts}_p(M)$ induces
M = arbitrary

$T_pM \xrightarrow{\sim} \mathbb{R}^m,\ v \mapsto v^\chi$ & a basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ of T_pM.
$e_1, \dots, e_m$

TW4 any $\gamma \in \mathrm{Curves}_p(M)$ has a speed $\frac{d\gamma}{dt}(0) \in T_pM$:

$\gamma : (-\varepsilon, \varepsilon) \to M$
$\gamma(0) = p$

$$\left(\frac{d\gamma}{dt}(0)\right)^\chi := \frac{d\gamma^\chi}{dt}(0) \quad \left(\gamma^\chi = \chi \circ \gamma, \text{ curve in } \mathbb{R}^m \quad \gamma^\chi : (-\varepsilon, \varepsilon) \to \mathbb{R}^m \right)$$

-2-

TW2 : $(\forall)\ F : M \to N$ smooth, $p \in M$, we have a linear map

$$(dF)_p : T_pM \longrightarrow T_{F(p)}N$$
$$v \longmapsto \underbrace{(dF)_p(v)}_{\text{defined by}} \,?$$

know χ, χ'

$$\begin{array}{ccc} \mathbb{R}^m & \xleftarrow[\sim]{\text{using } \chi} & T_pM \\ \downarrow (dF^{\chi'}_\chi) & & \downarrow (dF)_p \\ \mathbb{R}^n & \xleftarrow[\sim]{\text{using } \chi'} & T_{F(p)}N \end{array}$$

$\chi' \in \mathrm{Charts}_{F(p)}(N)$

$$\left((dF)_p(v)\right)^{\chi'} := \left(dF^{\chi'}_\chi\right)_{\chi(p)} (v^\chi) \qquad \left(F^{\chi'}_\chi = \chi' \circ F \circ \chi^{-1}\right)$$

choose $\chi \in \mathrm{Charts}_p(M)$

Corollary : F = immersion/submersion at p $\Longleftrightarrow$
$\Longleftrightarrow$ $(dF)_p$ = injective/surjective

Exercise : Show that $(dF)_p\left(\frac{d\gamma}{dt}(0)\right) = \frac{dF \circ \gamma}{dt}(0)$ $\left((\forall)\ F : M \to N \text{ smooth}, p \in M,\ \gamma \in \mathrm{Curves}_p(M) \right)$

Exercise : Check the chain rule for dF's.

TW3 Nothing new in $\mathbb{R}^L$

General remark : $M \subseteq N$ embedded $\Rightarrow$ ι induces an injection $(d\iota)_p : T_pM \hookrightarrow T_pN$
$\iota : M \to N$ inclusion $\qquad \gamma \in \mathrm{Curves}_p(M) \subseteq \mathrm{Curves}_p(N) \qquad \frac{d\gamma}{dt}(0) \longmapsto \frac{d\gamma}{dt}(0)$

-3-

For $M \subseteq \mathbb{R}^n$ embedded $\Rightarrow$ $\quad T_pM \subseteq T_p\mathbb{R}^n$

$$\begin{array}{ccc} T_pM & \subseteq & T_p\mathbb{R}^n \\ \vdots & & \downarrow \mathrm{Standard}_p \\ \mathrm{Standard}_p(T_pM) & \subseteq & \mathbb{R}^n \end{array}$$

Lemma : $\mathrm{Standard}_p(T_pM) = T_p^{\mathrm{class}}M$
In particular,

$$\underbrace{\mathrm{Standard}_p \big|_{T_pM}}_{\text{denoted } \mathrm{Standard}^M_p} \qquad \boxed{T_pM \longrightarrow T_p^{\mathrm{class}}M}$$

THE STANDARD IDENTIFICATION FOR M at p.

proof. $T_pM = \left\{ \frac{d\gamma}{dt}(0) \ \middle/\ \gamma \in \mathrm{Curves}_p(M) \right\}$

$$\mathrm{Standard}_p(T_pM) = \left\{ \underbrace{\mathrm{Standard}_p\left(\frac{d\gamma}{dt}(0)\right)}_{\left(\frac{d\gamma}{dt}(0)\right)^{\mathrm{Id}} = \frac{d\gamma}{dt}(0)} : \dots \right\} = \left\{ \frac{d\gamma}{dt}(0) : \gamma \in \mathrm{Curves}_p(M) \right\} = T_p^{\mathrm{class}}M$$

$T_pM \subseteq T_pN$

-4-

Def 2 : $\partial \in T_pM$ means that $\partial : C^\infty(M) \to \mathbb{R}$
(∂ = DERIVATION AT p) linear & derivation rule $\partial(fg) = f(p)\cdot\partial(g) + g(p)\cdot\partial(f)$

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Prop. 1 : This is a basis of $T_p\Omega$. $\qquad f_\chi = f \circ \chi^{-1}$

More generally, any (U, χ) around p induces $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p \in T_pM$
$(\forall)\ M$!

$$\left(\frac{\partial}{\partial x_i}\right)_p : C^\infty(M) \to \mathbb{R},\ f \mapsto \frac{\partial f}{\partial x_i}(p) := \frac{\partial f_\chi}{\partial x_i}(\chi(p))$$

TW4 : any $\gamma \in \mathrm{Curves}_p(M)$ has a "speed" $\frac{d\gamma}{dt}(0) \in T_pM$ $\quad (f_\chi = f \circ \chi^{-1})$
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$$\partial_{\frac{d\gamma}{dt}(0)} : C^\infty(M) \longmapsto \mathbb{R},\ f \longmapsto \frac{d\, f \circ \gamma}{dt}(0) = (f \circ \gamma)'(0)$$

-5-

TW2 $(\forall)\ F : M \to N$ smooth, $p \in M$, we have a linear map

$$(dF)_p : T_pM \longrightarrow T_{F(p)}N$$
$$\partial \longmapsto (dF)_p(\partial)$$

$\partial(f)$, $f \in C^\infty(M)$ $\qquad g \in C^\infty(N)$

defined by
$$(dF)_p(\partial)(g) := \partial(g \circ F).$$

Example : F = inclusion $\iota : M \hookrightarrow N$
$(d\iota)_p(\partial)(g) = \partial(g|_M)$, $\partial \in T_pM$

$M \xrightarrow{F} N \xrightarrow{g} \mathbb{R}$

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Exercise : Check the chain rule for dFs.

Proposition 2 : For any $U \subseteq M$ open, $(di)_p : T_pU \to T_pM$ is an isomorphism
($i : U \to M$ the inclusion)

-6-

$$T_pM \xrightarrow{\quad I_p \quad} T_pM$$
$$v \longmapsto \partial_v$$

$f : M \to \mathbb{R}$

where $\partial_v : C^\infty(M) \to \mathbb{R}$

$$\partial_v(f) := (df_\chi)_{\chi(p)}(v^\chi) = \partial_{v^\chi}(f_\chi)$$

Choose a/any $\chi \in \mathrm{Charts}_p(M)$ $\qquad f_\chi = f \circ \chi^{-1}$

Another χ' ?? $\qquad \boxed{c = \chi' \circ \chi^{-1}}$

$$(df_{\chi'})_{\chi'(p)}(v^{\chi'}) = (df_{\chi'})_{\chi'(p)}\left((dc)_{\chi(p)}(v^\chi)\right)$$
use $(*)$
$$= (d(\underbrace{f_{\chi'} \circ c}))_{\chi(p)}(v^\chi)$$
$$f \circ \underbrace{\chi'^{-1} \circ \chi'}_{\mathrm{Id}} \circ \chi^{-1} = f \circ \chi^{-1} = f_\chi$$
$$= (df_\chi)_{\chi(p)}(v^\chi)$$

Theorem : I_p is a linear isomorphism taking -1-, -2- to -4- and -5-

-1-

Def 1: $v \in T_pM$ means $\boxed{v : \mathrm{Charts}_p(M) \longrightarrow \mathbb{R}^m}$, $\chi \longmapsto v^\chi$ (notation)

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THE STANDARD IDENTIFICATION

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M = arbitrary

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$$v \longmapsto \underbrace{(dF)_p(v)}_{\text{defined by}} \,?$$

Know: v^χ

$$\begin{array}{ccc} \mathbb{R}^m & \xleftarrow[\sim]{\text{using } \chi} & T_pM \\ \Big\downarrow (dF^{\chi'}_\chi) & & \Big\downarrow (dF)_p \\ \mathbb{R}^n & \xleftarrow[\sim]{\text{using } \chi'} & T_{F(p)}N \end{array}$$

$\chi' \in \mathrm{Charts}_{F(p)}(N)$

$$\left((dF)_p(v)\right)^{\chi'} := \left(dF^{\chi'}_\chi\right)_{\chi(p)}(v^\chi)$$

choose $\chi \in \mathrm{Charts}_p(M)$

$$\left(F^{\chi'}_\chi = \chi' \circ F \circ \chi^{-1}\right)$$

Corollary: F = immersion/submersion at p $\Longleftrightarrow$ $(dF)_p$ = injective/surjective

Exercise: Show that $(dF)_p\left(\frac{d\gamma}{dt}(0)\right) = \frac{dF \circ \gamma}{dt}(0)$ $\left((\forall)\, F : M \to N \text{ smooth}, p \in M, \ \gamma \in \mathrm{Curves}_p(M) \right)$

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General remark: $M \subseteq N$ embedded $\Longrightarrow$ i induces an injection $(di)_p : T_pM \hookrightarrow T_pN$

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$$\gamma \in \mathrm{Curves}_p(M) \subseteq \mathrm{Curves}_p(N) \qquad \frac{d\gamma}{dt}(0) \longmapsto \frac{d\gamma}{dt}(0)$$

T_pM

The … is … to …

For …

Lemma …

In p…

denot…

proof.

$T_pM \subseteq T_p$…

$= (df_x)_{x(p)}(v^x)$

-3-

For $M \subseteq \mathbb{R}^n$ embedded $\Rightarrow$

$$\begin{array}{ccc} T_pM & \subseteq & T_p\mathbb{R}^n \\ \downarrow & & \downarrow \text{Standard}_p \\ \text{Standard}_p(T_pM) & \subseteq & \mathbb{R}^n \end{array}$$

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$\to T_pN$

$\mapsto \frac{d\gamma}{dt}(0)$

$T_pM \subseteq T_pN$

Exe

Exer

Proposit

$(i: U \to$

-4-

Def 2: $\partial \in T_pM$ means that $\partial : C^\infty(M) \to \mathbb{R}$ linear & derivation rule $\partial(fg) = f(p)\cdot\partial(g) + g(p)\cdot\partial(f)$

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$f_x = f\circ x^{-1}$

More generally, any (U,x) around p induces $\left(\frac{\partial}{\partial x_1}\right)_p,\dots,\left(\frac{\partial}{\partial x_m}\right)_p \in T_pM$

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$$\left(\frac{\partial}{\partial x_i}\right)_p : C^\infty(M)\to\mathbb{R},\quad f \mapsto \frac{\partial f}{\partial x_i}(p) := \frac{\partial f_x}{\partial x_i}(x(p))$$

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-5-

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defined by $(dF)_p(\partial)(g) := \partial(g\circ F)$. $\quad g\in C^\infty(N)$

$\partial(f)$, $f\in C^\infty(M)$

$$M \xrightarrow{F} N \xrightarrow{g} \mathbb{R}$$

Example: F = inclusion $i: M \hookrightarrow N$

$(di)_p(\partial)(g) = \partial(g|_M)$, $\partial \in T_pM$

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-6-

$$T_pM \ni v \xrightarrow{\quad I_p \quad} \partial_v \in T_pM$$

$f : M \to \mathbb{R}$

where $\partial_v : C^\infty(M) \to \mathbb{R}$

$$\partial_v(f) := (df_x)_{x(p)}(v^x) = \partial_{v^x}(f_x)$$

Choose a/any $x \in \mathrm{Chart}_p(M)$

$f_x = f \circ x^{-1}$

Another x' ⇝ ??

$$\boxed{c = x' \circ x^{-1}}$$

$$(df_{x'})_{x'(p)}(v^{x'}) = (df_{x'})_{x'(p)}\left((dc)_{x(p)}(v^x)\right)$$

use (*)

$$= \left(d(\underbrace{f_{x'} \circ c}_{})\right)_{x(p)}(v^x)$$

$$f \circ \underbrace{x'^{-1} \circ x'}_{Id} \circ x^{-1} = f \circ x^{-1} = f_x$$

$$= (df_x)_{x(p)}(v^x)$$

Theorem: I_p is a linear isomorphism taking -1-, -2- to -4- and -5-

-7-

$T_pM \xrightarrow{I_p} T_pM$

$v \mapsto \partial_v$

where $\partial_v : C^\infty(M) \to \mathbb{R}$

$f : M \to \mathbb{R}$

$$\boxed{\begin{array}{l}\partial_v(f) := (df_x)_{x(p)}(v^x) = \partial_{v^x}(f_x) \\ \text{choose } x \in \mathrm{Charts}_p(M)\end{array}}$$

for $f: M \to \mathbb{R}$ smooth

Is $\boxed{I_p \text{ linear}}$? Yes:

$\partial_{v+2v'}(f) = (\partial_v + \partial_{2v'})(f)$?

$\partial_{v+v'}(f) = \partial_v(f) + \partial_{v'}(f)$? $\Longleftrightarrow (df_x)_{x(p)}(\underbrace{(v+v')^x}_{v^x+v'^x}) = (df_x)_{x(p)}(v^x) + (df_x)_{x(p)}(v'^x)$

Does I_p for $M = \Omega \subseteq \mathbb{R}^m$ open take $\left(\frac{\partial}{\partial x_i}\right)_p$ to $\left(\frac{\partial}{\partial x_i}\right)_p$?

$f_x \circ \gamma^x = f \circ x^{-1} \circ x \circ \gamma = f \circ \gamma$

$\underbrace{I_p\left(\frac{d\gamma}{dt}(0)\right)}_{} = \frac{d\gamma}{dt}(0)$?

$\partial_v : C^\infty(M) \to \mathbb{R},\ f \longmapsto (df_x)_{x(p)}(v^x) = (df_x)_{x(p)}\left(\frac{d\gamma^x}{dt}(0)\right)$

$= (f_x \circ \gamma^x)'(0) = (f \circ \gamma)'(0)$

$$= \partial(f_x \circ \gamma^x)'(0) = (f \circ \gamma)'(0)$$

Lemma: $(dF)_p \circ I_p^M = I^N_{F(p)} \circ (dF)_p$

proof: unravel the definitions.

proof of THM: Choose $x: \underset{p}{U} \to \Omega \subseteq \mathbb{R}^m$ chart around p with $x(p) = 0$.

$$\begin{tikzcd}
M & T_pM \arrow[rr, "I_p^M"] & & T_pM \\
U \arrow[u, phantom, "\cup\!\shortmid", "i"'] \arrow[d, "x"] & T_pU \arrow[u, "(di)_p", "\cong"'] \arrow[rr, "I_p^U"] \arrow[d, "(dx)_p"', "\cong"] & & T_pU \arrow[u, "\cong"', "(di)_p"] \arrow[d, "\cong"', "(dx)_p"] \\
\Omega \arrow[d, phantom, "\cap\!\shortmid", "j"'] & T_0\Omega \arrow[rr, "I_0^\Omega"] \arrow[d, "(dj)_p"', "\cong"] & & T_0\Omega \arrow[d, "\cong"', "(dj)_p"] \\
\mathbb{R}^m & T_0\mathbb{R}^m \arrow[rr, "I_0^{\mathbb{R}^m}", "\cong\ \text{P1}"'] & & T_0\mathbb{R}^m
\end{tikzcd}$$

basis: $\left(\frac{\partial}{\partial x_i}\right)_0 \longmapsto \left(\frac{\partial}{\partial x_i}\right)_0$

P2 P2

$\cong$ QED

all squares are commutative

THEOREM: $I_p: T_pM \to T_pM$ is an isomorphism which takes the white notions to the green ones.

$T_pM \subseteq T_pN$

$\gamma(0) = p$. $\left(\frac{d\sigma}{dt}(0)\right) := \frac{d}{dt}(0)$ ... $\gamma = x \circ \gamma$, curve in $\mathbb{R}^m$

$\gamma^{x}: (-\varepsilon, \varepsilon) \to \mathbb{R}^m$

Proof of Prop 1 for $\Omega = \mathbb{R}^m$, $p = 0$: $\left(\frac{\partial}{\partial x_1}\right)_0, \dots, \left(\frac{\partial}{\partial x_m}\right)_0$ basis of $T_0\mathbb{R}^m$?

① Linearly indep? Assume $\lambda_1 \left(\frac{\partial}{\partial x_1}\right)_0 + \dots + \lambda_m \left(\frac{\partial}{\partial x_m}\right)_0 = 0$ in $T_0\mathbb{R}^m$

To prove: $\lambda_1 = \dots = \lambda_m = 0$

means $\boxed{(\forall) f \in C^\infty(\mathbb{R}^m)}$ we have $\lambda_1 \frac{\partial f}{\partial x_1}(0) + \dots + \lambda_m \frac{\partial f}{\partial x_m}(0) = 0$.

$f: \mathbb{R}^m \to \mathbb{R}$

$f(x) = x_1$ $\quad \lambda_1 1 + \lambda_2 0 + \dots + \lambda_m 0 = 0$

$\Rightarrow \lambda_1 = 0$

$x_2 \Rightarrow \lambda_2 = 0$ etc $\Rightarrow \lambda_1 = \dots = \lambda_m = 0$.

② Do they span the entire $T_0\mathbb{R}^m$? Start with any $\partial \in T_0\mathbb{R}^m$.

To prove: $\exists\, \lambda_1, \dots, \lambda_m$ s.t. $\partial = \lambda_1 \left(\frac{\partial}{\partial x_1}\right)_0 + \dots + \lambda_m \left(\frac{\partial}{\partial x_m}\right)_0$

Scratch paper: if we had $\lambda_1, \dots, \lambda_m \Rightarrow$

$\Rightarrow \lambda_1 = \partial(x_1)$

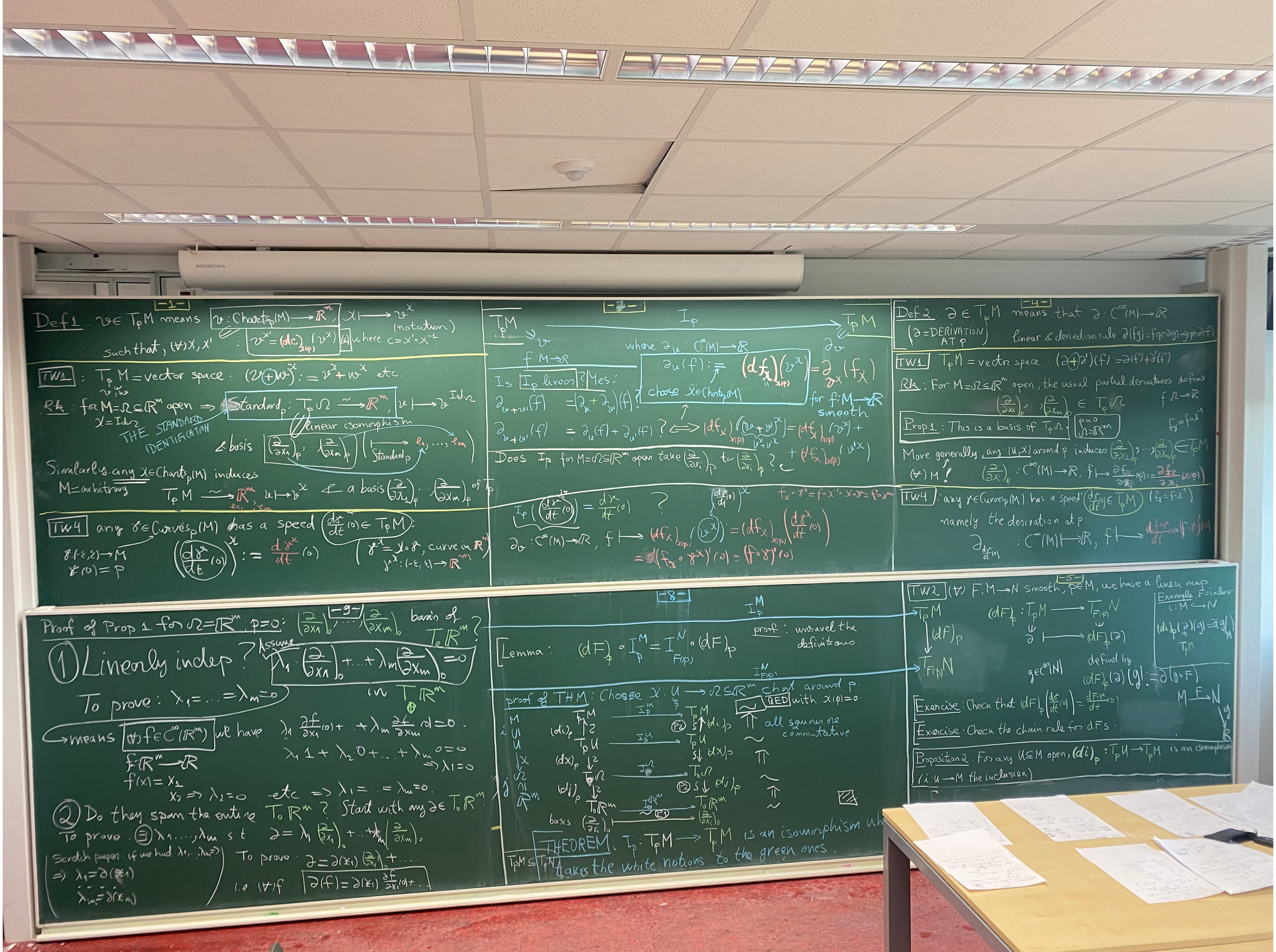
$\dots$

$\lambda_m = \partial(x_m)$

To prove: $\partial = \partial(x_1) \left(\frac{\partial}{\partial x_1}\right)_0 + \dots$

i.e. $(\forall) f$: $\boxed{\partial(f) = \partial(x_1) \frac{\partial f}{\partial x_1}(0) + \dots}$

$T_pM \subseteq$


-1-
Def1. v ∈ TpM means v : Chartsp(M) → ℝ^m, χ ↦ v^χ (notation)
such that, (∀)χ, χ' v^χ' = (dc)χ(p)(v^χ) where c = χ'∘χ^-1
TW1: TpM = vector space: (v + w)^χ := v^χ + w^χ etc.
Rk: for M = Ω ⊆ ℝ^m open ⇒ Standardp : TpΩ → ℝ^m, v ↦ v^IdΩ
THE STANDARD IDENTIFICATION
linear isomorphism
& basis
Similarly any χ ∈ Chartsp(M) induces
M = arbitrary
TpM → ℝ^m, v ↦ v^χ & a basis
TW4 any γ ∈ Curvesp(M) has a speed dγ/dt(0) ∈ TpM
γ^χ = χ∘γ, curve in ℝ^m
-7-
TpM
Ip
TpM
where ∂v : C^∞(M) → ℝ
f: M → ℝ
∂v(f) := (dfχ)χ(p)(v^χ) = ∂/∂v^χ(fχ)
choose χ ∈ Chartsp(M)
for f: M → ℝ smooth
Is Ip linear? Yes:
Does Ip for M = Ω ⊆ ℝ^m open take
Def2 ∂ ∈ TpM means that ∂ : C^∞(M) → ℝ
(∂ = DERIVATION AT P)
linear & derivation rule
TW1 TpM = vector space
Rk: For M = Ω ⊆ ℝ^m open, the usual partial derivatives define
Prop1: This is a basis of TpΩ
More generally, any (U, χ) around p induces
TW4: any γ ∈ Curvesp(M) has a speed
namely the derivation at p
-5-
TW2 (∀) F: M → N smooth, p ∈ M, we have a linear map
Example
defined by
Exercise: Check that
Exercise: Check the chain rule for dFs
Proposition2: For any U ⊆ M open, (di)p : TpU → TpM is an isomorphism
(i: U → M the inclusion)
-8-
Lemma: (dF)p ∘ Ip^M = I^N F(p) ∘ (dF)p
proof: unravel the definitions
proof of THM: Choose χ : U → Ω ⊆ ℝ^m chart around p
QED with χ(p) = 0
all squares are commutative
basis
THEOREM. Ip: TpM → TpM is an isomorphism
takes the white notions to the green ones.
-9-
Proof of Prop 1 for Ω = ℝ^m, p = 0: basis of T0ℝ^m?
1 Linearly indep? Assume
To prove: λ1 = ... = λm = 0
in T0ℝ^m
means (∀)f ∈ C^∞(ℝ^m) we have
f: ℝ^m → ℝ
f(x) = x1
etc ⇒ λ1 = ... = λm = 0.
2 Do they span the entire T0ℝ^m? Start with any ∂ ∈ T0ℝ^m
To prove: ∃ λ1, ..., λm s t
Scratch paper: if we had λ1, ..., λm?
To prove
i.e. (∀)f