-1-

Reminder:

- vector spaces, T_pM, linear maps $(dF)_p : T_pM \longrightarrow T_{F(p)}N$
- speeds $\frac{d\gamma}{dt}(0) \in T_pM$ for $\gamma \in \mathrm{Curves}_p(M)$
- working in charts (U,x): use basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p$ of T_pM

  [for another chart (U',x'): $\left(\frac{\partial}{\partial x'_i}\right)_p = \sum \underbrace{\frac{\partial c_j}{\partial x_i}(x(p))}_{\text{also denoted: } \frac{\partial x'_j}{\partial x_i}(p)} \left(\frac{\partial}{\partial x_j}\right)_p$ $\quad \left(c = x' x^{-1}\right)$

- any $v \in T_pM$ induces $\boxed{\partial_v : C^\infty(M) \to \mathbb{R}}$ derivation at p $\quad \boxed{\partial_v(fg) = f(p)\,\partial_v(g) + g(p)\,\partial_v(f)}$

X_p

X $\quad p \quad f$

$\partial_{X_p}(f) \in \mathbb{R}$

$\partial_v(f) = (df^x)_{x(p)}(v^x) = (df)_p(v)$

choose (U,x) around p

$= \sum \frac{\partial f^x}{\partial x_i}(x(p))\, v^x_i$

$= \sum \frac{\partial f}{\partial x_i}(p)\, v^x_i$

$C^\infty(M) = \{ f : M \to \mathbb{R} \,/\, f = \text{smooth} \}$

$(df)_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$, $\quad v \mapsto$

VECTO…

- a …
- Sm…

basi…

coef…

Def…

De…

Operations on $\mathfrak{X}(M)$ -3-

- addition: $X, Y \in \mathfrak{X}(M) \Rightarrow$ can form $X+Y \in \mathfrak{X}(M)$ by: … in $\mathfrak{X}(M)$ … in T_pM

-1-

Reminder:

- vector spaces T_pM, linear maps $(dF)_p : T_pM \longrightarrow T_{F(p)}N$
- speeds $\frac{d\gamma}{dt}(0) \in T_pM$ for $\gamma \in \mathrm{Curves}_p(M)$
- working in charts (U, x): use basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ of T_pM

  [for another chart (U', x'): $\left(\frac{\partial}{\partial x'_i}\right)_p = \sum \underbrace{\frac{\partial c_j}{\partial x_i}(x(p))}_{\text{also denoted: } \frac{\partial x'_j}{\partial x_i}(p)} \left(\frac{\partial}{\partial x_j}\right)_p$ $\left(\begin{array}{c} c = \\ = x' x^{-1} \end{array}\right)$

- any $v \in T_pM$ induces

$$\boxed{\partial_v : C^\infty(M) \longrightarrow \mathbb{R}} \quad \text{derivation at } p \quad \boxed{\partial_v(fg) = f(p)\,\partial_v(g) + g(p)\,\partial_v(f)}$$

X_p

X (p) (f)

$\partial_{X_p}(f) \in \mathbb{R}$

$$\partial_v(f) = (df^x)_{x(p)}(v^x) = (df)_p(v)$$

choose (U, x) around p

$$= \sum \frac{\partial f^x}{\partial x_i}(x(p))\, v^x_i$$

$$= \sum \frac{\partial f}{\partial x_i}(p)\, v^x_i$$

$C^\infty(M) = \{ f : M \to \mathbb{R} \,/\, f = \text{smooth} \}$

$(df)_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$, $v \in T_pM$

-2-

VECTOR FIELDS

M = manifold

- a set theoretical vector field X on M: any map

$$M \ni p \longmapsto X_p \in T_pM$$

- Smoothness of X: use charts (U, x) and the coresponding basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ for each $p \in U$, to write

~~coef X, x~~

$$X_p = \underbrace{\mathrm{coeff}_1(p)}\left(\frac{\partial}{\partial x_1}\right)_p + \dots + \underbrace{\mathrm{coeff}_m(p)}\left(\frac{\partial}{\partial x_m}\right)_p$$

$\mathrm{coeff}_i : U \longrightarrow \mathbb{R}$ ← require to be smooth $(\forall)$ (U, x) chart of M

[**Def**: A vector field on M is any X as above which is smooth. Denote by $\mathfrak{X}(M)$ the set of all vector fields on M.

-3-

Operations on $\mathfrak{X}(M)$

- addition: $X, Y \in \mathfrak{X}(M) \Rightarrow$ can form $X + Y \in \mathfrak{X}(M)$ by: $(X \underset{\text{in } \mathfrak{X}(M)}{+} Y)_p = X_p \underset{\text{in } T_pM}{+} Y_p$.
- multiplication by scalars $\lambda \in \mathbb{R}$: $(\lambda \cdot X)_p := \lambda \cdot X_p$

  } i.e. vector space structure on $\mathfrak{X}(M)$

- multiplication by $f \in C^\infty(M)$: $(f \cdot X)_p := f(p) \cdot X_p$

  i.e. an operation:

$$C^\infty(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M),\ (f, X) \longmapsto f \cdot X$$

} i.e. $\mathfrak{X}(M)$ = a module over $C^\infty(M)$

- action (by derivations) of $\mathfrak{X}(M)$ on $C^\infty(M)$:

$$\mathfrak{X}(M) \times C^\infty(M) \longrightarrow C^\infty(M),\ (X, f) \longmapsto \mathcal{L}_X(f)$$

$\mathcal{L}_X(f)(p) := \partial_{X_p}(f)$

The Lie derivative of f along X

also denoted $X(f)$, $X.f$

-4-

} enough to play with vector fields:

Ex: On $\mathbb{R}^m$: $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ are now vector fields, $\in \mathfrak{X}(\mathbb{R}^m)$

$$\left(p \longmapsto \left(\frac{\partial}{\partial x_i}\right)_p\right)$$

$\Rightarrow f_1 \frac{\partial}{\partial x_1} + \dots + f_m \frac{\partial}{\partial x_m} \in \mathfrak{X}(\mathbb{R}^m)$ $(\forall)$ $f_1, \dots, f_m$ smooth functions

These are all vector fields on $\mathbb{R}^m$!

Ex When $M \subseteq \mathbb{R}^L$ (when $T_pM \subseteq T_p\mathbb{R}^L$)

Careful: $\left(\frac{\partial}{\partial x_i}\right)_p$ may fail to be tangent to M!!

Nevertheless expressions of type $f_1 \frac{\partial}{\partial x_1} + \dots + f_L \frac{\partial}{\partial x_L}$ may define vector fields on M (i.f. $(\forall)$ $p \in M$, $f_1(p)\left(\frac{\partial}{\partial x_1}\right)_p + \dots + f_L(p)\left(\frac{\partial}{\partial x_L}\right)_p \in T_pM$)

Ex: $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$ defines a vector field on S^1 because

$(\forall)$ $p = (a, b) \in S^1$: $X_p = a\left(\frac{\partial}{\partial y}\right)_p - b\left(\frac{\partial}{\partial x}\right)_p \in T_pS^1$ | $x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathfrak{X}(S^1)$

$a \cdot b - b \cdot a = 0$

$\longrightarrow T_{F(p)} N$

$\left(\frac{\partial}{\partial x_m}\right)_p$ of $T_p M$

$\left(C = x' x^{-1} \right)$

$(fg) = f(p)\,\partial_u(g) + g(p)\,\partial_u(f)$

$\left\{ M \to \mathbb{R} \,/\, f = \text{smooth} \right\}$

$T_p M \to T_{f(p)} \mathbb{R}$

$\boxed{\text{Vector Fields}}$ M = manifold

$\boxed{-2-}$

- a <u>set theoretical vector field</u> X on M: any map

$$M \ni p \longmapsto X_p \in T_p M$$

- <u>Smoothness of X</u>: use charts (U, x) and the corresponding basis $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ for each $p \in U$, to write

$$X_p = \underbrace{\text{coeff}_1(p)}_{} \left(\frac{\partial}{\partial x_1}\right)_p + \dots + \underbrace{\text{coeff}_m(p)}_{} \left(\frac{\partial}{\partial x_m}\right)_p$$

$\text{coeff}_i : U \longrightarrow \mathbb{R}$ $\leftarrow$ require to be smooth $(\forall)$ (U, x) chart of M.

<u>Def</u>: A vector field on M is any X as above which is smooth. Denote by $\mathfrak{X}(M)$ the set of all vector fields on M.

---

in $T_p M$

i.e.

enough to play with vector fields: $\boxed{-4-}$

<u>Ex</u>: On $\mathbb{R}^m$: $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ are now vector fields, $\in \mathfrak{X}(\mathbb{R}^m)$

∂_{X_p} around p $= \sum \frac{\partial f}{\partial x_i}(p)\, v_i^{\lambda}$

Operations on $\mathfrak{X}(M)$ -3-

- **addition**: $X, Y \in \mathfrak{X}(M) \Rightarrow$ can form $X+Y \in \mathfrak{X}(M)$ by: $(X \underset{\text{in } \mathfrak{X}(M)}{+} Y)_p = X_p \underset{\text{in } T_pM}{+} Y_p$.
- **multiplication by scalars** $\lambda \in \mathbb{R}$: $(\lambda \cdot X)_p := \lambda \cdot X_p$.
- **multiplications by $f \in C^\infty(M)$**: $(f \cdot X)_p := f(p) \cdot X_p$.

  i.e. an operation:

  $$C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),\ (f, X) \longmapsto f \cdot X$$

  i.e. $\mathfrak{X}(M) =$ a module over C^∞
- **action (by derivations) of $\mathfrak{X}(M)$ on $C^\infty(M)$**:

  $$\mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M),\ (X, f) \longmapsto \mathcal{L}_X(f)$$

  $\mathcal{L}_X(f)(p) := \partial_{X_p}(f)$

  The Lie derivative of f along X, also denoted $X(f)$, $X.f$

i... ve... sp... str... on $\mathfrak{X}$... Cons... New... vec... Ex... $\forall p =$

Denote by $\mathfrak{X}(M)$ the set of all vector fields on M.

$=$ smooth

$T_pM \to T_{f(p)}\mathbb{R}$

in T_pM

$X_p + Y_p$.

i.e. vector space structure on $\mathfrak{X}(M)$

$f \cdot X$

i.e. $\mathfrak{X}(M) =$ a module over $C^\infty(M)$

$\boxed{-4-}$

enough to play with vector fields:

Ex: On $\mathbb{R}^m$: $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$ are now vector fields, $\in \mathfrak{X}(\mathbb{R}^m)$

$\left(p \longmapsto \left(\frac{\partial}{\partial x_i}\right)_p\right)$.

$\Rightarrow f_1 \frac{\partial}{\partial x_1} + \dots + f_m \frac{\partial}{\partial x_m} \in \mathfrak{X}(\mathbb{R}^m)$ ($\forall$ $f_1, \dots, f_m$ smooth functions

These are all vector fields on $\mathbb{R}^m$!

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Nevertheless expressions of type $f_1 \frac{\partial}{\partial x_1} + \dots + f_L \frac{\partial}{\partial x_L}$ may define vector fields on M (cf. $(\forall) p \in M$, $f_1(p)\left(\frac{\partial}{\partial x_1}\right)_p + \dots + f_L(p)\left(\frac{\partial}{\partial x_L}\right)_p \in T_pM$)

Ex $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$ defines a vector field on S^1 because $x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathfrak{X}(S^1)$

$\forall p = (a, b) \in S^1$: $X_p = a \left(\frac{\partial}{\partial y}\right)_p - b \left(\frac{\partial}{\partial x}\right)_p \in T_p S^1$

$a \cdot b - b \cdot a = 0$

$\to \mathcal{L}_X(f)$

(The Lie derivative of f along X)

also denoted

$X(f)$

Xf

— 5 —

For $X \in \mathfrak{X}(M)$: $\mathcal{L}_X : C^\infty(M) \longrightarrow C^\infty(M)$, $f \longmapsto \mathcal{L}_X(f)$ where $p \in M$ $\boxed{\mathcal{L}_X(f)(p) = \partial_{X_p}(f)}$ (*)

Example: $M = \mathbb{R}^n$, $X = \frac{\partial}{\partial x_i} \Rightarrow \mathcal{L}_X$ is the usual $\frac{\partial}{\partial x_i}$ interpreted as an operator.

$$X = \sum f_i \frac{\partial}{\partial x_i} \rightarrow \mathcal{L}_X(f) = \sum f_i \cdot \frac{\partial f}{\partial x_i}$$

Lemma: X = set theoretical vector field and, for $f \in C^\infty(M)$ define $\mathcal{L}_X(f)$ by (*) then

$$X = \text{vector field} \iff \mathcal{L}_X(f) \in C^\infty(M) \quad \forall f \in C^\infty(M)$$

Def: A derivation on $C^\infty(M)$: any linear map

$$L : C^\infty(M) \longrightarrow C^\infty(M)$$

s.t. $L(fg) = f \cdot L(g) + g \cdot L(f)$ $\forall f, g \in C^\infty(M)$

$\mathrm{Der}(C^\infty(M))$ collection of all such

Theorem: There is a following bijection

$$\mathfrak{X}(M) \longrightarrow \mathrm{Der}(C^\infty(M))$$
$$X \longmapsto \mathcal{L}_X$$

$L_1 \circ L_2$

proof of surjectivity $L : C^\infty(M) \longrightarrow C^\infty(M)$ derivation — 6 —

Fix $p \in M$. Look for $X_p \in T_pM$

$$\underbrace{\left(\partial_{X_p}(f) = L(f)(p) \quad \forall f \right)}_{\text{want}}$$

$\partial_{X_p}(f) = ev_p(L(f))$

$\partial_{X_p} = ev_p \circ L$

$C^\infty(M) \xrightarrow{ev_p} \mathbb{R}$

$ev_p \circ L : C^\infty(M) \longrightarrow \mathbb{R}$ is a derivation at p $= \partial_{X_p}$ for some $X_p \in T_pM$

$\Rightarrow$ defines X as a set theoretical vector field. X_p

Is it smooth? Yes! Because $\mathcal{L}_X(f) = L(f)$ and L takes smooth to smooth!

...y map

... corresponding

... write

$\frac{\partial}{\partial x_n} \Big)_p$

...e to be smooth

...,x) chart of M.

which is smooth

...s on M.

... fields, $\in \mathfrak{X}(\mathbb{R}^m)$

..., f_m

...oth functions

may define

$\ldots p)\left(\frac{\partial}{\partial x_i}\right)_p \in T_pM)$

... on S^1 becaus

$\ldots \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathfrak{X}(S^1)$

For $X \in \mathfrak{X}(M)$: $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$,

$f \longmapsto \mathcal{L}_X(f)$ where $p \in M$ $\boxed{\mathcal{L}_X(f)(p) = \partial_{X_p}(f)}$ $(*)$

Example: $M = \mathbb{R}^m$, $X = \frac{\partial}{\partial x_i} \Rightarrow \mathcal{L}_X$ is the usual $\frac{\partial}{\partial x_i}$ interpreted as an operator.

$$X = \sum f_i \frac{\partial}{\partial x_i} \to \mathcal{L}_X(f) = \sum f_i \cdot \frac{\partial f}{\partial x_i}$$

Lemma: X = set theoretical vector field and, for $f \in C^\infty(M)$ define $\mathcal{L}_X(f)$ by $(*)$ then:

$$X = \text{vector field} \iff \mathcal{L}_X(f) \in C^\infty(M) \ (\forall) f \in C^\infty(M).$$

Def: A derivation on $C^\infty(M)$: any linear map

$$L : C^\infty(M) \to C^\infty(M)$$

s.t. $L(fg) = f \cdot L(g) + g \cdot L(f)$ $(\forall) f, g \in C^\infty(M)$.

$Der(C^\infty(M))$ collection of all such

$L_1 \circ L_2$

Theorem: The following is a bijection

$$\mathfrak{X}(M) \longrightarrow Der(C^\infty(M))$$
$$X \longmapsto \mathcal{L}_X$$

$$\mathfrak{X}(M) \longrightarrow \mathrm{Der}(C^\infty(M))$$
$$X \longmapsto \mathcal{L}_X$$

<u>proof of surjectivity</u> $L : C^\infty(M) \to C^\infty(M)$ derivation $\boxed{-6-}$

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{L} & C^\infty(M) \\ & \searrow & \downarrow ev_p \\ & & \mathbb{R} \end{array}$$

Fix $p \in M$. Look for $X_p \in T_pM$

$$\big(\underbrace{\partial_{X_p}(f) = L(f)(p)}_{\text{want}} \quad (\forall f) \big)$$

$$\partial_{X_p}(f) = ev_p(L(f))$$

$$\partial_{X_p} = ev_p \circ L$$

$ev_p \circ L : C^\infty(M) \to \mathbb{R}$

$\Big($ is a derivation at p

$\Big)$ ∂_{X_p} for some $X_p \in T_pM$

$\Rightarrow$ defines X as a set theoretical vector field.

Is it smooth? Yes! Because $\mathcal{L}_X(f) = L(f)$ and L takes smooth to smooth!

... a derivation

$L_1, L_2 \in \mathrm{Der}(C^\infty(M))$ $\qquad L_i(fg) = f L_i(g) + L_i(f) g$

~6'~

$L_1 \circ L_2 \in \mathrm{Der}(C^\infty(M))$?

$$(L_1 \circ L_2)(fg) = L_1(L_2(fg)) = L_1(f L_2(g) + L_2(f) g) =$$

$$= L_1(f L_2(g)) + L_1(L_2(f) g) = L_1(f) L_2(g) + f L_1 L_2(g) + \dots =$$

$$= f \cdot (L_1 \circ L_2)(g) + (L_1 \circ L_2)(f) \cdot g + L_1(f) L_2(g) + L_1(g) L_2(f)$$

Theorem: The following is a bijection

$$\mathfrak{X}(M) \longrightarrow \mathrm{Der}(C^\infty(M))$$
$$X \longmapsto \mathcal{L}_X.$$

$L_1 \circ L_2$

But: looking at $(L_2 \circ L_1)(fg) \overset{-7-}{\Longrightarrow}$ same annoying red-term

$\Downarrow$

$L_1 \circ L_2 - L_2 \circ L_1 : C^\infty(M) \longrightarrow C^\infty(M)$

will be a derivation !

$-6'-$

$L_1, L_2 \in \mathrm{Der}(C^\infty(M))$ $\quad L_i(fg) = f L_i(g) + L_i(f) g$

$L_1 \circ L_2 \in \mathrm{Der}(C^\infty(M))$?

$X_p = \underbrace{\text{coeff}_1(p)} \left(\frac{\partial}{\partial x_1}\right)_p + \dots + \underbrace{\text{coeff}_m(p)} \left(\frac{\partial}{\partial x_m}\right)_p$

$\text{coeff}_i : U \to \mathbb{R}$ ← require to be smooth $(\forall)$ (U, χ) chart of M.

Def: A vector field on M is any X as above which is smooth

-8-

Def: For $X, Y \in \mathfrak{X}(M)$ the lie bracket of X and Y, denoted

$$[X,Y] \in \mathfrak{X}(M)$$

is the new vector field on M, the unique one s.t.

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X .$$

ex: $M = \mathbb{R}^m$, $X = \sum f_i \frac{\partial}{\partial x_i}$, $Y = \sum g_i \frac{\partial}{\partial x_i}$ compute $[X,Y]$.

$$\boxed{\mathcal{L}_{[X,Y]}(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))}$$

$\mathfrak{X}(M)$ is a lie algebra

$\dots, X] + [[Z, X], Y] = 0$

But:

$\Downarrow$

$L_1 \circ L_2 \dots$ will $\dots$

$L_1, L_2 \in \dots$

$L_1 \circ L_2 \in \dots$

$(L_1 \circ L_2)(f \dots$

• any $v \in T_pM$ induces $\partial_v : C^\infty(M) \to \mathbb{R}$ derivation at p

also denoted: $\frac{\partial X_i}{\partial x_i}(p)$

$\partial_v(fg) = f(p)\,\partial_v(g) + g(p)\,\partial_v(f)$

Def: A vector ...

Operations on $\mathfrak{X}(M)$

-9-

• addition: $X, Y \in \mathfrak{X}(M) \Rightarrow$ can form $\boxed{X+Y} \in \mathfrak{X}(M)$ by: $(X \underset{\text{in } \mathfrak{X}(M)}{+} Y)_p = X_p \underset{\text{in } T_pM}{+} Y_p$.

• multiplication by scalars $\lambda \in \mathbb{R}$: $(\boxed{\lambda \cdot X})_p := \lambda \cdot X_p$.

• multiplication by $f \in C^\infty(M)$: $(\boxed{f \cdot X})_p := f(p) \cdot X_p$.

i.e. an operation: $C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(f, X) \mapsto f \cdot X$

i.e. $\mathfrak{X}(M)$ = module ...

• action (by derivations) of $\mathfrak{X}(M)$ on $C^\infty(M)$: $\mathfrak{X}(M) \times C^\infty(M) \to C^\infty(M)$, $(X, f) \mapsto \mathcal{L}_X(f)$

• the Lie bracket operation $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X, Y) \mapsto [X, Y]$

• bilinear $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$

• skew symmetric $[Y, X] = -[X, Y]$

• Jacobi-identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

$\mathfrak{X}(M)$ is a Lie algebra

Also: $[X, f \cdot Y] = f \cdot [X, Y] + \mathcal{L}_X(f) \cdot Y$ ← Exercise

Def: For X ... denoted ... is the new ...

ex: $M = \mathbb{R}^m$, ...