Scratch paper:

$$x : U \longrightarrow \Omega \subseteq \mathbb{R}^m$$

$$X|_U \in \mathfrak{X}(U)$$

$$x_*(X|_U) \in \mathfrak{X}(\Omega)$$

$$x_*(X|_U) = \sum F_i^x \frac{\partial}{\partial x_i}$$

$$F_i^x \in C^\infty(\Omega)$$

$F : M \longrightarrow N$ diffeom induces

$$F_* : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(N)$$

$$\begin{array}{ccc} \Psi & & \Psi ? \\ X & & F_*(X) \\ (X_p \in T_pM & & \\ p \in M) & & \end{array}$$

for $q \in N$. $p := F^{-1}(q)$

$$F_*(X)_q = (dF)_p(X_p)$$

$X_p \in T_pM$, $(dF)_p(X_p) \in T_{F(p)}N = T_qN$

$$\alpha_{[X,Y]} = \underbrace{\alpha_X \circ \alpha_Y - \alpha_Y \circ \alpha_X}$$

All we know about $[\cdot,\cdot]$ is its definition:

$$\boxed{\alpha_{[X,Y]}(f) = \alpha_X(\alpha_Y(f)) - \alpha_Y(\alpha_X(f))} \quad \forall f \in C^\infty(M)$$

proof of Jacobi: The LHS is a vector field call it V. To show $V=0$

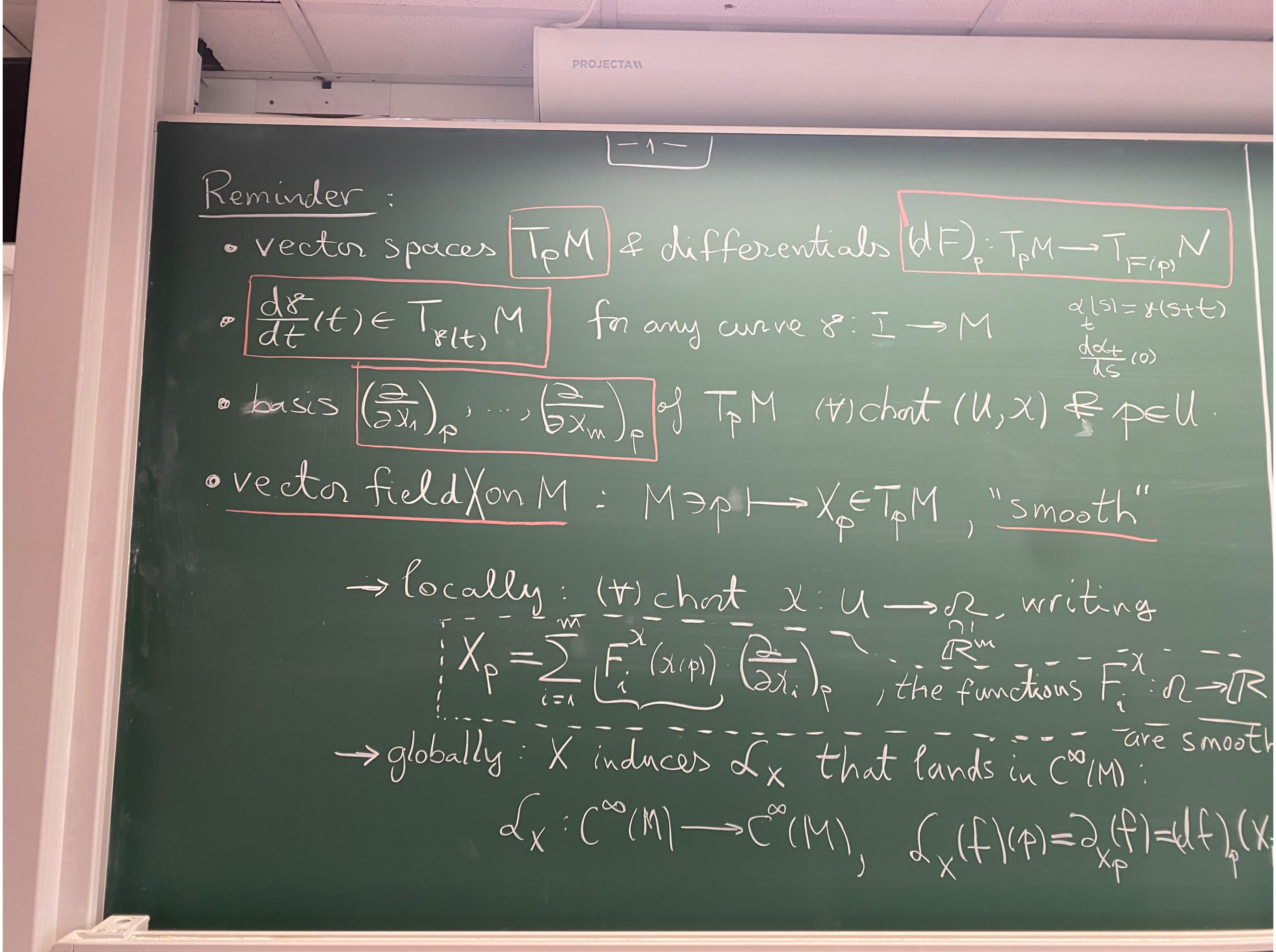
$\iff$ to show that $\alpha_V = 0$: Look at $\alpha_V(f) = \alpha_{[\boxed{[X,Y]},Z]}(f) + \dots$

$$= \alpha_{[X,Y]}(\alpha_Z(f)) - \alpha_Z(\alpha_{[X,Y]}(f)) + \dots = \alpha_X \alpha_Y \alpha_Z(f) - \alpha_Y \alpha_X \alpha_Z(f) - \alpha_Z \alpha_X \alpha_Y(f) + \alpha_Z \alpha_Y \alpha_X(f) + \dots = 0 \ \square$$

Operations:

- ADDITION: $(X+Y)_p =$
- MULTIPLICATION BY SC
- ——— " ——— SMOO
  i.e. operation $C^\infty(M) \times \mathfrak{X}$
- actions of $X \in \mathfrak{X}(M)$ by
- LIE BRACKET of vecto
  $\mathfrak{X}(M) \times \mathfrak{X}$
  - → skew symmetric $[Y$
  - → bilinear: $[X_1 + X_2, Y$
  - → Jacobi: $[[X,Y],Z] +$

Extra (all together): $[X, fY] =$



Reminder :

- vector spaces $\boxed{T_pM}$ & differentials $\boxed{(dF)_p : T_pM \longrightarrow T_{F(p)}N}$
- $\boxed{\frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M}$ for any curve $\gamma : I \longrightarrow M$

$\alpha_t(s) = \gamma(s+t)$

$\frac{d\alpha_t}{ds}(0)$

- basis $\boxed{\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p}$ of T_pM $(\forall)$ chart (U, x) & $p \in U$.
- vector field X on M : $M \ni p \longmapsto X_p \in T_pM$, "smooth"

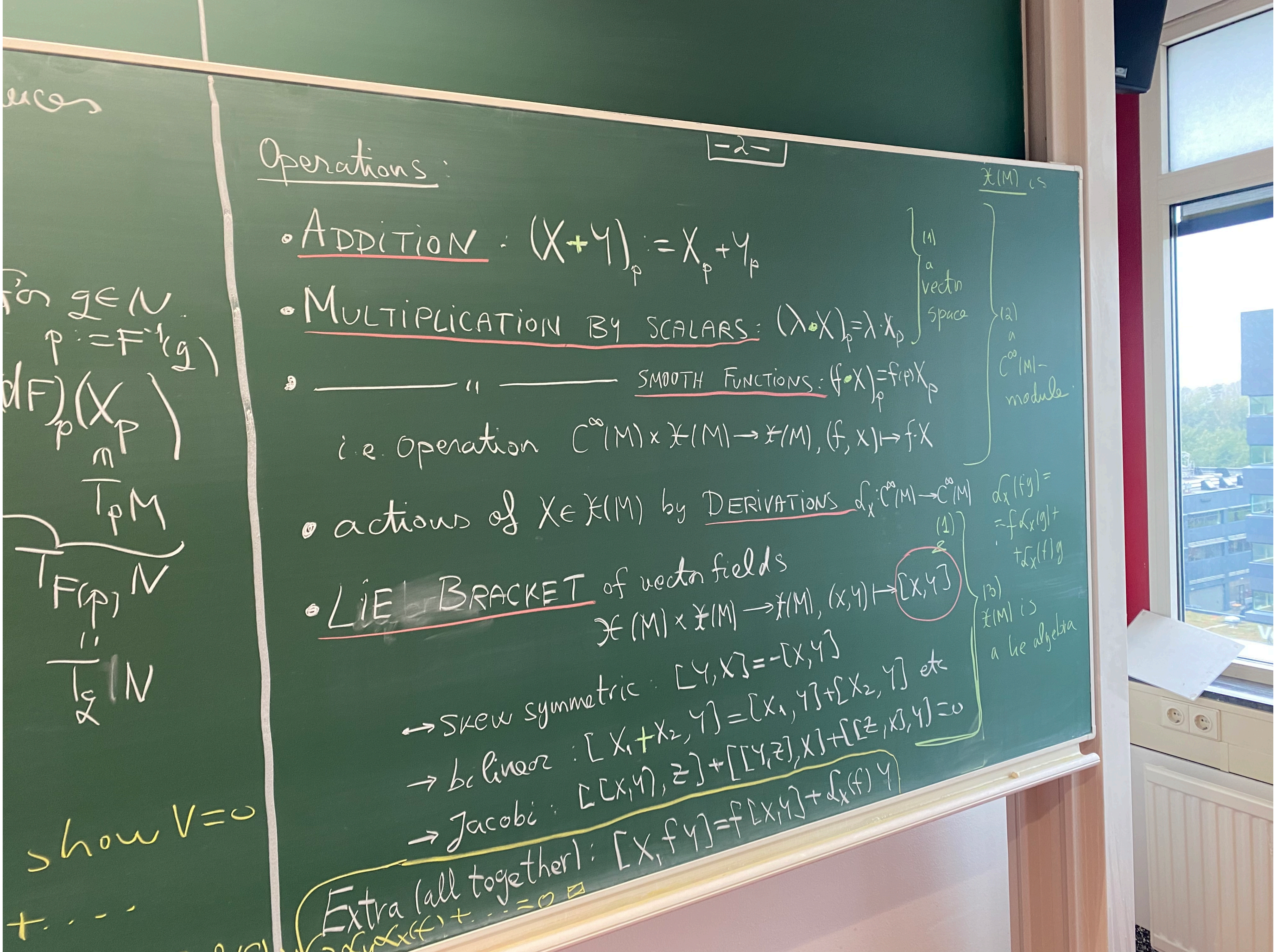
→ locally : $(\forall)$ chart $x : U \longrightarrow \underset{\mathbb{R}^m}{\overset{}{\Omega}}$, writing

$$\boxed{X_p = \sum_{i=1}^{m} \underbrace{F_i^x(x(p))}_{} \cdot \left(\frac{\partial}{\partial x_i}\right)_p}$$

, the functions $F_i^x : \Omega \to \mathbb{R}$ are smooth

→ globally : X induces $\mathcal{L}_X$ that lands in $C^\infty(M)$:

$$\mathcal{L}_X : C^\infty(M) \longrightarrow C^\infty(M), \quad \mathcal{L}_X(f)(p) = \partial_{X_p}(f) = (df)_p(X$$



ences

for $q \in N$

$p := F^{-1}(q)$

$$\underbrace{(dF)_p(\underset{\underset{T_pM}{\cap}}{X_p})}_{\underset{\underset{T_qN}{\shortparallel}}{T_{F(p)}N}}$$

show $V=0$

$+ \cdot \cdot \cdot$

$\boxed{-2-}$

Operations:

- ADDITION: $(X+Y)_p := X_p + Y_p$
- MULTIPLICATION BY SCALARS: $(\lambda \cdot X)_p = \lambda \cdot X_p$
- ——— " ——— SMOOTH FUNCTIONS: $(f \cdot X)_p = f(p) X_p$

i.e. Operation $C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),\ (f, X) \mapsto f \cdot X$

$\mathfrak{X}(M)$ is

(1) a vector space

(2) a $C^\infty(M)$-module.

- actions of $X \in \mathfrak{X}(M)$ by DERIVATIONS $\delta_X : C^\infty(M) \to C^\infty(M)$

$\delta_X(fg) = f\delta_X(g) + \delta_X(f)g$

(1)

2

- LIE BRACKET of vector fields

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),\ (X,Y) \mapsto [X,Y]$$

→ skew symmetric: $[Y,X] = -[X,Y]$

→ bilinear: $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$ etc

→ Jacobi: $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

(3) $\mathfrak{X}(M)$ is a Lie algebra

Extra (all together): $[X, fY] = f[X,Y] + \delta_X(f)Y$

$\delta_X \delta_X(f) + \dots = 0$ ☐

Scratch paper: $x : U \longrightarrow \Omega \subseteq \mathbb{R}^m$

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$x_*(X|_U) = \sum F_i^X \frac{\partial}{\partial x_i}$

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$$\alpha_{[X,Y]} = \underbrace{\alpha_X \circ \alpha_Y - \alpha_Y \circ \alpha_X}$$

All we know about $[\cdot,\cdot]$ is its definition:

$$\boxed{\alpha_{[X,Y]}(f) = \alpha_X(\alpha_Y(f)) - \alpha_Y(\alpha_X(f))} \quad (\forall) f \in C^\infty(M)$$

proof of Jacobi: The LHS is a vector field. Call it V. To show $V=0$

$\Longleftrightarrow$ to show that $\alpha_V = 0$: Look at $\alpha_V(f) = \alpha_{[[X,Y],Z]}(f) + \dots$

$= \alpha_{[X,Y]}(\alpha_Z(f)) - \alpha_Z(\alpha_{[X,Y]}(f)) + \dots = \alpha_X\alpha_Y\alpha_Z(f) - \alpha_Y\alpha_X\alpha_Z(f) - \alpha_Z\alpha_X\alpha_Y(f) + \alpha_Z\alpha_Y\alpha_X(f) + \dots = 0$

Ex: On $\mathbb{R}^m$: $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$

For $\underbrace{\text{arbitrary v. fields on } \mathbb{R}^m}_{\sum F_i \frac{\partial}{\partial x_i}}$

use bilinearity and the H.W.

Ex: $\left[\frac{\partial}{\partial x}, xy\frac{\partial}{\partial x} + e^{x+2y}\frac{\partial}{\partial y}\right] =$

$= \left[\underset{X}{\frac{\partial}{\partial x}}, \underset{f}{xy}\underset{Y}{\frac{\partial}{\partial x}}\right] + \left[\frac{\partial}{\partial x}, e^{x+2y}\frac{\partial}{\partial y}\right] =$

$= \underbrace{\underset{f}{xy}\left[\underset{X}{\frac{\partial}{\partial x}}, \underset{Y}{\frac{\partial}{\partial x}}\right]}_{0} + \alpha_{\underset{X}{\frac{\partial}{\partial x}}}(\overset{f}{xy})\overset{Y}{\frac{\partial}{\partial x}} + \dots$

$= \boxed{y\frac{\partial}{\partial x} + e^{x+2y}\frac{\partial}{\partial y}}$

Operations:

- ADDITION: (X
- MULTIPLICATION
- —— " ——

i.e. operation

- actions of X∈
- LIE BRACKET

→ skew symm

→ bilinear

→ Jacobi

Extra (all together)

are smooth

→ globally: X induces $\mathcal{L}_X$ that lands in $C^\infty(M)$:

$$\mathcal{L}_X : C^\infty(M) \longrightarrow C^\infty(M), \quad \mathcal{L}_X(f)(p) = \partial_{X_p}(f) = (df)_p(X_p)$$

RK2:

Exercise

DEF: Given $X \in \mathfrak{X}(M)$, an integral curve of X is any $\gamma : I \longrightarrow M$ smooth (with $I \subseteq \mathbb{R}$ open interval, $\cancel{0 \in I}$)

s.t. $$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad (\forall)\, t \in I$$

Say that γ starts at $p \in M$ if $0 \in I$, $\gamma(0) = p$.

Ex: $M = \mathbb{R}^2$, $X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$, $\lambda \in \mathbb{R}$ constant

Integral curve: $\gamma(t) = (x(t), y(t))$

$$\left.\begin{array}{ccc} & \frac{d\gamma}{dt}(t) = X_{\gamma(t)} & \\ \diagup\!\!\diagup & & \parallel \\ \dot x(t)\cdot\left(\frac{\partial}{\partial x}\right)_{\gamma(t)} + \dot y(t)\left(\frac{\partial}{\partial y}\right)_{\gamma(t)} & & \left(\frac{\partial}{\partial x}\right)_{\gamma(t)} + \lambda\left(\frac{\partial}{\partial y}\right)_{\gamma(t)} \end{array}\right\} \begin{cases} \dot x(t) = 1 \\ \dot y(t) = \lambda \end{cases}$$

$$\Longrightarrow \begin{cases} x(t) = t + a \\ y(t) = \lambda t + b \end{cases} \Longrightarrow \gamma_{a,b} \text{ one for each } a, b \in \mathbb{R}$$

$$\gamma_{a,b}(t) = (t+a, \lambda t + b)$$

Rk1: Prescribing $\gamma(0) =: (a,b) \Longrightarrow$ precisely one integral curve

Rk2: We can use $I = \mathbb{R}$, i.e. $\gamma_{a,b}$ makes sense as $\mathbb{R} \to \mathbb{R}^2$.

Exercise: $y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$

$\boxed{-7-}$

$M = \mathbb{R}^2$, $X = \underline{-x^2} \frac{\partial}{\partial x} \boxed{-y} \frac{\partial}{\partial y}$

$$\begin{cases} \dot{x}(t) = -x^2(t) \\ \dot{y}(t) = -y(t) \end{cases} \Rightarrow \dots \Rightarrow \gamma_{a,b}(t) = \left(\frac{a}{at+1}, b e^{-t} \right)$$

$\gamma(t) = (\textcircled{x(t)}, \textcircled{y(t)})$

$x = a$

$\dot{x} = 0$

$a, b \in \mathbb{R}$

$\dot{y} = -x^2 - y \Rightarrow \dot{y} = -y - a^2$

Rk1 : ✓

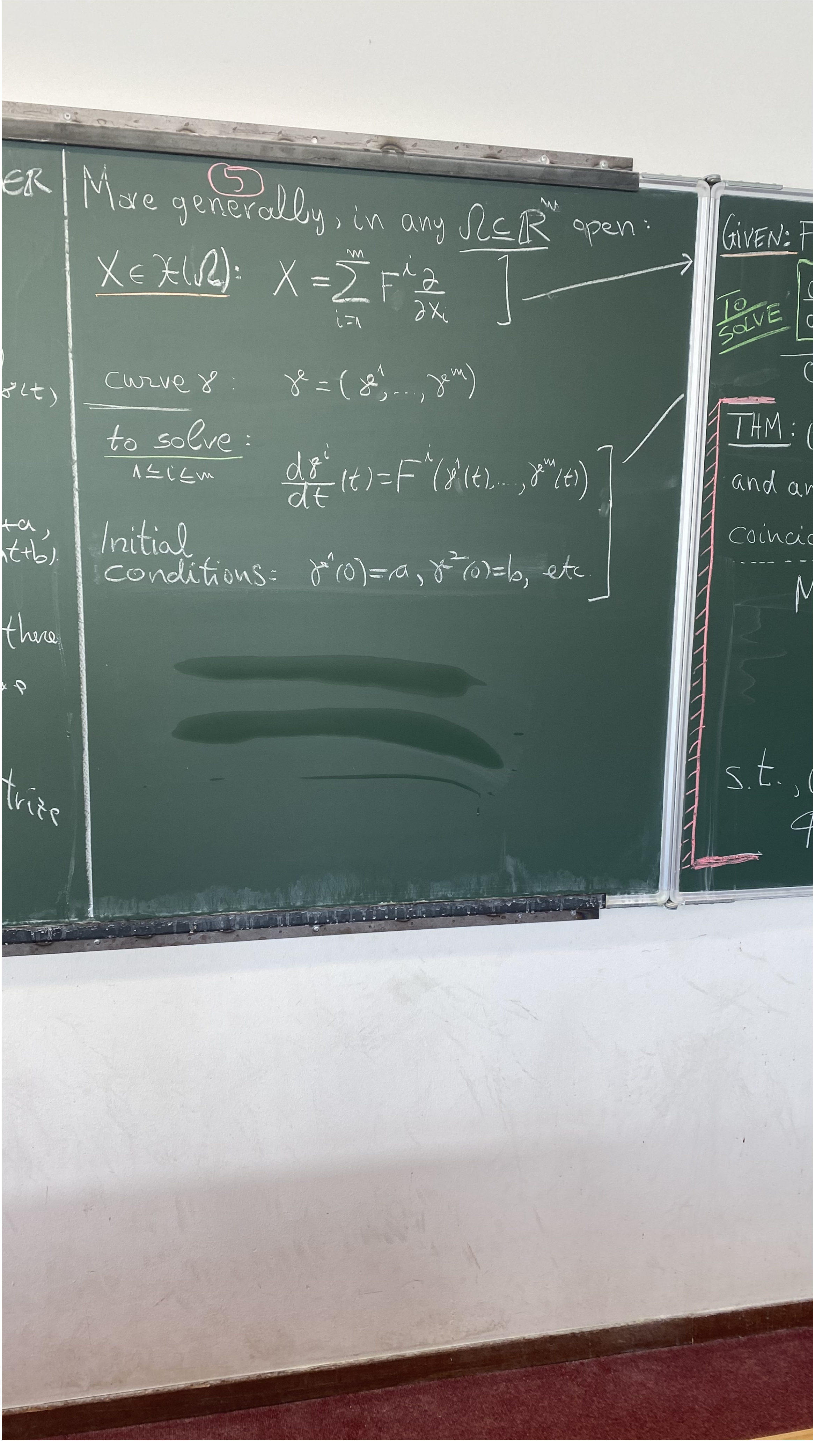
Rk2 : Makes sense on the entire $I = \mathbb{R}$ only when $a = 0$.

Fr $a < 0$ one could use $\underset{\substack{\Psi \\ 0}}{(-\infty, -\frac{1}{a})}$ or $(-\frac{1}{a}, \infty)$.

$y + a^2 = e^{-t+b}$

$y(t) = e^{-t+b} - a^2$

$\ln(y + a^2)' = \frac{\dot{y}}{y + a^2} = -1$



⑤

More generally, in any $\underline{\Omega \subseteq \mathbb{R}^m}$ open:

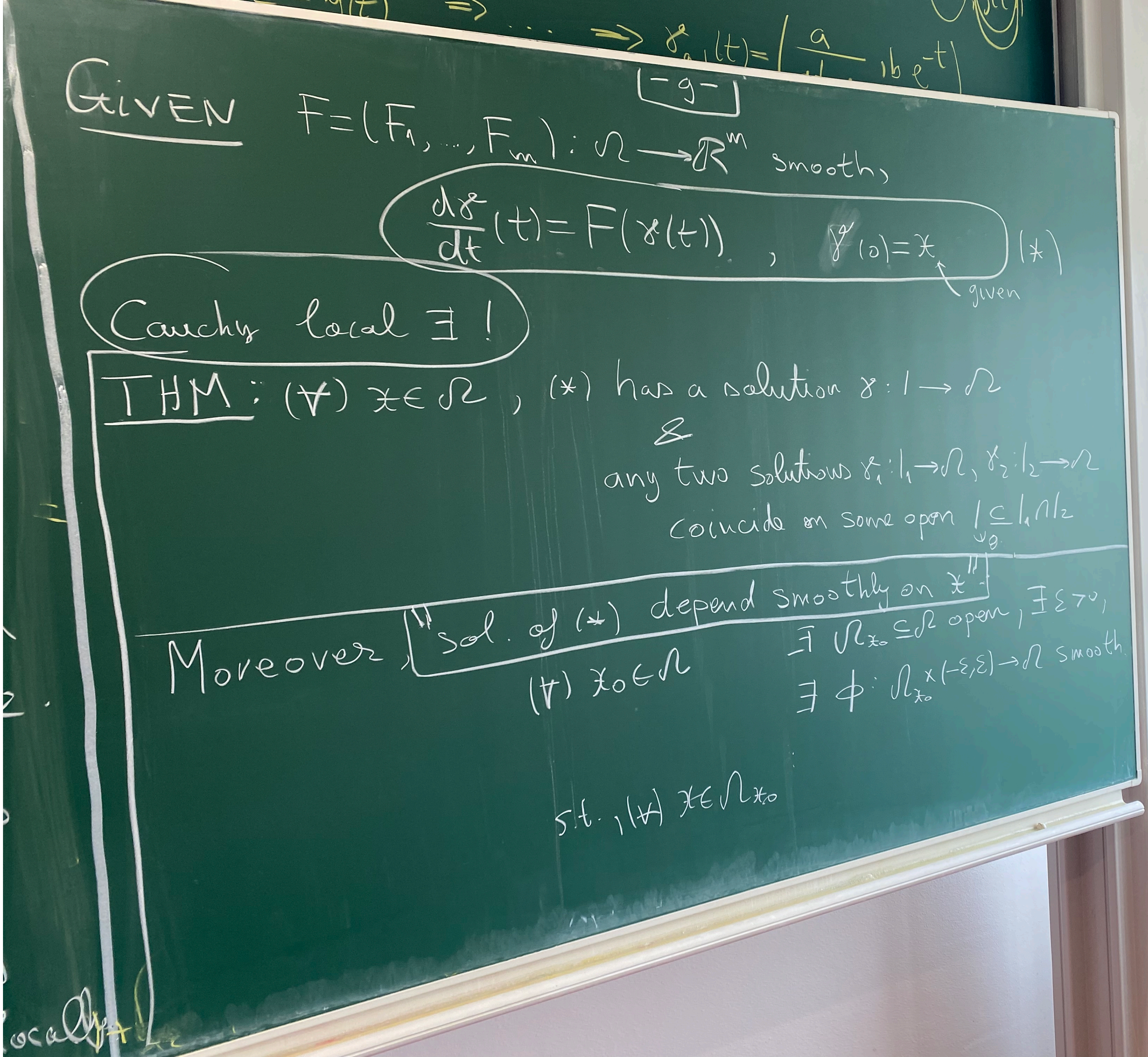
$\underline{X \in \mathfrak{X}(\Omega)}$: $X = \sum_{i=1}^{m} F^i \frac{\partial}{\partial x_i}$

$\underline{\text{curve } \gamma}$: $\gamma = (\gamma^1, \dots, \gamma^m)$

$\underline{\text{to solve}}$:
$1 \leq i \leq m$

$$\frac{d\gamma^i}{dt}(t) = F^i(\gamma^1(t), \dots, \gamma^m(t))$$

Initial
conditions: $\gamma^1(0) = a$, $\gamma^2(0) = b$, etc



$\dots \Rightarrow \gamma_a(t) = \left(\frac{a}{\dots}, b e^{-t}\right)$

−9−

GIVEN $F = (F_1, \dots, F_m) : \Omega \to \mathbb{R}^m$ smooth,

$$\boxed{\frac{d\gamma}{dt}(t) = F(\gamma(t)), \quad \gamma(0) = x} \quad (*)$$

x ← given

Cauchy local ∃!

THM: $(\forall)\ x \in \Omega$, $(*)$ has a solution $\gamma : I \to \Omega$

&

any two solutions $\gamma_1 : I_1 \to \Omega$, $\gamma_2 : I_2 \to \Omega$

coincide on some open $\underset{\ni 0}{I} \subseteq I_1 \cap I_2$

Moreover, "sol. of $(*)$ depend smoothly on x":

$(\forall)\ x_0 \in \Omega$ $\exists\ \Omega_{x_0} \subseteq \Omega$ open, $\exists\ \varepsilon > 0$,

$\exists\ \Phi : \Omega_{x_0} \times (-\varepsilon, \varepsilon) \to \Omega$ smooth

s.t., $(\forall)\ x \in \Omega_{x_0}$

ocally

Reminder:

- vector spaces $\boxed{T_pM}$ & differentials $\boxed{(dF)_p : T_pM \longrightarrow T_{F(p)}N}$

$d\gamma$ … T … M … $\alpha(s) = \gamma(s+t)$

Ex: $M = $ …

Integral cur…

DEF: Given $X \in \mathfrak{X}(M)$, an integral curve of X is any $\gamma : I \longrightarrow M$ smooth (with $I \subseteq \mathbb{R}$ open interval, ~~$0 \in I$~~) s.t.

$$\frac{d\gamma}{dt}(t) = X_{\gamma(t)} \quad (\forall)\, t \in I$$

Say that γ starts at $p \in M$ if $0 \in I$, $\gamma(0) = p$.

Prop: Given $X \in \mathfrak{X}(M)$:

1. $(\forall)\, p \in M$, $\exists$ int. curve $\gamma : I \longrightarrow M$ of X s.t. $\gamma(0) = p$
2. any two integral curves γ_1 and γ_2 of X which coincide at some t_0, must coincide on a $(t_0-\varepsilon, t_0+\varepsilon)$ for $\varepsilon > 0$.

Corollary: Given $X \in \mathfrak{X}(M)$, $(\forall)\, p \in M$ $\exists!$ $\gamma_p^X : I_p^X \longrightarrow M$ $\underbrace{\text{maximal integral curve of}}$ X starting at p!

($\nexists$ integral curve $\tilde{\gamma} : \tilde{I} \to M$ s.t. $\tilde{I} \supsetneq I$, $\tilde{\gamma}|_I = \gamma$.)

-13-

X is called complete if $I_p^X = \mathbb{R}$ $(\forall)\, p \in M$

Pf: {

the pro…

J just…

For the …

integr…

Pf: A

$\emptyset$ ✗

$A \neq \emptyset$

To prove:

Corollary:

$t\in I$ $\dot{x}(t)=x^2(t)$

$\dot{x}(t)=1$

$M=\mathbb{R}^2$, $X=-x^2\frac{\partial}{\partial}$

$\begin{cases}\dot{x}(t)=-x^2(t)\\ \dot{y}(t)=-y(t)\end{cases} \Rightarrow \dots$

-11-

Pf: $\begin{cases}\frac{d\gamma}{dt}(t)=X_{\gamma(t)}\\ \gamma(0)=p.\end{cases}$ It suffices to work in a <u>chart χ</u>

$$\begin{tikzcd} & \text{chart } \chi \arrow[dl] \arrow[dr] & \\ \chi_*(X)=\sum F_i^\chi \frac{\partial}{\partial x_i} & & \chi\circ\gamma \end{tikzcd}$$

the problem, transfered to $\Omega\subseteq\mathbb{R}^m$ is the one that I just erased ☹. Just apply classical Cauchy ∃!.

<u>For the corollary</u>: Remark: if $\gamma_1 : I_1\to M$, $\gamma_2 : I_2\to M$ integral curves, $\exists\, t_0\in I_1\cap I_2$ s.t. $\gamma_1(t_0)=\gamma_2(t_0)\Rightarrow\gamma_1=\gamma_2$ on $I_1\cap I_2$.

Pf: $\underset{\emptyset\neq}{A}=\{t\in I_1\cap I_2 : \gamma_1(t)=\gamma_2(t)\}\subseteq I_1\cap I_2$ (interval)

A = open in $I_1\cap I_2$ part 2 of Prop

$\overset{\in}{t_0}$

A = closed in $I_1\cap I_2$. γ_1,γ_2 = cont. ∎

$A\neq\emptyset$

To prove: $A=I_1\cap I_2$.

<u>Corollary</u>: just put together the int. curves that we know to exist locally.

<u>GIVEN</u> $F=(F_1,\dots,F$

$\frac{d\gamma}{dt}(t)$

Cauchy local ∃!

<u>THM</u>: $(\forall)\ x\in\Omega$,

Moreover "sol. of

$(\forall)$

s.t.

> precisely one integral curve

$\gamma_{a,b}$ makes sense as $\mathbb{R} \to \mathbb{R}^2$.

$p = (a,b)$ $\quad I_p = \mathbb{R}$

$\gamma_p : I_p \to \mathbr{R}^2$, $\gamma_p(t) = (t+a, \lambda t + b)$

one could use $(-\infty, -\frac{1}{a})$

$\Downarrow$

0

$y + a^2$

$y(t) =$

$\boxed{12}$

Ex: $p = (a,b)$, $I_p = \begin{cases} \mathbb{R} \\ (-\infty, -\frac{1}{a}) \\ (-\frac{1}{a}, \infty) \end{cases}$

γ_p = the maximal IC, on I_p

$a =$

$a <$

$a >$

$\boxed{-11-}$

s to work in a chart x

$$\mathcal{X}_*(X) = \sum F_i^x \frac{\partial}{\partial x_i} \qquad x \circ \gamma$$

$\mathbb{R}^m$ is the one that

pply classical Cauch $\exists !$

$\gamma_1 : I_1 \to M$, $\gamma_2 : I_2 \to M$

s.t. $\gamma_1(t_0) = \gamma_2(t_0) \Rightarrow \gamma_1 = \gamma_2$ on $I_1 \cap I_2$.

part 2 of

$\boxed{-9-}$

GIVEN $F = (F_1, \ldots, F_m) : \Omega \to \mathbb{R}^m$ smooth

$\frac{d\gamma}{dt}(t) = F(\gamma(t))$, $\gamma(0$

Cauchy local $\exists !$

THM: $(\forall)\ x \in \Omega$, $(*)$ has a solution $\gamma : I$

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Moreover, "sol. of $(*)$ depend smoothly on x

$(\forall)\ x_0 \in \Omega$ $\quad \exists\ \Omega_{x_0} \subseteq \Omega$ op

$\exists\ \phi : \Omega_{x_0} \times (-\varepsilon,$

$\longrightarrow M$

$\boxed{-13-}$

X is called **complete** if

$I_p^X = \mathbb{R}\ (\forall)\, p \in M$

$= \gamma.$)

A

To

Corolla