$\boxed{-1-}$

Reminder: M = m-dimensional manifold

VECTOR FIELD $X \in \mathfrak{X}(M)$: $M \ni p \longmapsto X_p \in T_pM$, "smooth in p"

LOCALLY, w.r.t. a chart $\underset{M}{\overset{\cap}{x:\mathcal{U}}} \to \underset{\mathbb{R}^m}{\overset{\cap}{\Omega}}$: $X_p = \sum_i \boxed{F_i^x}(x(p))\left(\frac{\partial}{\partial x_i}\right)_p$ for all $p \in \mathcal{U}$

$\left(\text{OR: } x_*(X) = \sum_i \boxed{F_i^x} \frac{\partial}{\partial x_i} , F_i \in C^\infty(\Omega) \right)$

INTEGRAL CURVE OF X: any $\gamma : I \to M$ s.t. $T_{\gamma(t)}M \ni \frac{d\gamma}{dt}(t) = X_{\boxed{\gamma(t)}}$ $(\forall) t \in I$ $(\underset{\text{interval}}{\subseteq} \mathbb{R})$

INITIAL CONDITIONS: say γ starts at $p \in M$ if $\gamma(0) = p$ $(0 \in I)$

MAXIMAL INTEGRAL CURVE: one which cannot be extended to a larger interval.

Rk. locally, w.r.t. $x : \mathcal{U} \to \Omega$ $\Rightarrow$ ends up with $\frac{d\gamma^x}{dt}(t) = F^x(\gamma^x(t))$, where $F^x = (F_1^x, \ldots, F_m^x)$

$\Rightarrow$ one can use Cauchy local $\exists!$ in $\mathbb{R}^m$ $\Longrightarrow$ similar local result on M $\Rightarrow$

$\Rightarrow \begin{cases} \text{for any } X \in \mathfrak{X}(M), \\ \text{for any } p \in M \end{cases}$ there exists, and is unique, maximal integral curve γ_p of X that starts at p

COMPLETE X: all γ_p's are defined on $\mathbb{R}$.

$\boxed{M=\mathbb{R}^2}$ $\boxed{-2-}$

Ex 1: $X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases} \dot{x} = 1 \\ \dot{y} = \lambda \end{cases} \Rightarrow$ for each $p = (a, b) \in \mathbb{R}^2$ the solution γ_p is $\gamma_p(t) = (t + a, \lambda t + b)$

$\left(\begin{array}{l} \gamma(t) = (x(t), y(t)) \\ \frac{d\gamma}{dt}(t) = \underline{\dot{x}(t)} \frac{\partial}{\partial x} + \underline{\underline{\dot{y}(t)}} \frac{\partial}{\partial y} \end{array} \right)$

$\Rightarrow X$ = complete

Ex 2: $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ ... for each $p = (a, b)$:

$\gamma_p(t) = (\underbrace{a\cos t + b\sin t}, \underbrace{b\cos t - a\sin t})$

"$(a + ib)(\cos t + i\sin t) = \ldots + i \ldots$"

$\Rightarrow X$ = complete

Ex 3 $X = -x^2\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \Rightarrow \ldots \Rightarrow$ for each $p = (a, b)$ $\gamma_p(t) = \left(\frac{a}{\boxed{at+1}}, be^{-t}\right)$ $\leftarrow$!! (can be 0)

$\Rightarrow X$ is not complete

FL

$\boxed{-3-}$

Consequence of $\exists!$: if $M \subseteq \mathbb{R}^n$ embedded submanifold and $X \in \mathfrak{X}(M)$ which happens to have an extension $\tilde{X} \in \mathfrak{X}(\mathbb{R}^n)$ then any integral curve of $\tilde{X}$ starting at $p \in M$ will be be an — " — X — " — for as long as it lands in M.

For instance:

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(S^2) \qquad S^2 \subseteq \mathbb{R}^3$$

$$\tilde{X} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^3)$$

$$\tilde{\tilde{X}} = \left(x + (x^2+y^2-1)e^{\cos(\ln xy)}\right)\frac{\partial}{\partial x} + e^{(x^2+y^2-1)x^{2023}}\frac{\partial}{\partial y}$$

$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$

: $M = m$-dimensional manifold $\boxed{-1-}$

VECTOR FIELD $X \in \mathfrak{X}(M)$: $M \ni p \longmapsto X_p \in T_pM$, "smooth in p"

LOCALLY, w.r.t. a chart $x : \underset{M}{\underset{\cap}{U}} \longrightarrow \underset{\mathbb{R}^m}{\underset{\cap}{\Omega}}$: $X_p = \sum_i \boxed{F_i^x}(x(p))\left(\frac{\partial}{\partial x_i}\right)_p$ for all $p \in U$

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$\Rightarrow$ one can use Cauchy local $\exists!$ in $\mathbb{R}^m$ $\Longrightarrow$ similar local result on

$\Rightarrow \begin{cases} \text{for any } X \in \mathfrak{X}(M), \\ \text{for any } p \in M \end{cases}$ there exists, and is unique, maximal integral curve γ_p of X that starts

COMPLETE X: all γ_p's are defined on $\mathbb{R}$.

$\boxed{M=\mathbb{R}^2}$

$\boxed{-2-}$

Ex 1: $X=\frac{\partial}{\partial x}+\underline{\lambda}\frac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases}\dot{x}=1\\ \dot{y}=\lambda\end{cases} \Rightarrow$ for each $p=(a,b)\in\mathbb{R}^2$ the solution γ_p is

$\gamma_p(t)=(t+a, \lambda t+b)$

$\left(\begin{array}{l}\gamma(t)=(x(t),y(t))\\ \frac{d\gamma}{dt}(t)=\underline{\dot{x}(t)}\frac{\partial}{\partial x}+\underline{\underline{\dot{y}(t)}}\frac{\partial}{\partial y}\end{array}\right)$

$\Rightarrow X=$ complete.

Ex 2: $X=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases}\dot{x}=y\\ \dot{y}=-x\end{cases}$ ... for each $p=(a,b)$:

$\gamma_p(t)=(\underbrace{a\cos t+b\sin t}, \underbrace{b\cos t-a\sin t})$

$" (a+ib)(\cos t+i\sin t)= \ldots + i \ldots$

$\Rightarrow X=$ complete

Ex 3: $X=-x^2\frac{\partial}{\partial y}-y\frac{\partial}{\partial x} \Rightarrow \ldots \Rightarrow$ for each $p=(a,b)$ $\gamma_p(t)=\left(\frac{a}{\boxed{at+1}}, be^{-t}\right)$

[illegible] (when $a\neq0$)

$\Rightarrow X$ is not complete.

[illegible]

Consequence of $\exists!$: if $M \subseteq \mathbb{R}^n$ $\boxed{-3-}$ embedded submanifold and $X \in \mathfrak{X}(M)$ which happens to have an extension $\tilde{X} \in \mathfrak{X}(\mathbb{R}^n)$ then any integral curve of $\tilde{X}$ stating at $p \in M$ will be be an —— " —— X —— " —— for as long as it lands in M.

For instance: $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(S^2)$ $\quad S^2 \subseteq \mathbb{R}^3$

$$\tilde{X} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^3)$$

$$\tilde{\tilde{X}} = (x + (x^2+y^2-1)e^{\cos(\ln xy)})\frac{\partial}{\partial x} + e^{(x^2+y^2-1)x^{2023}}\frac{\partial}{\partial y}$$

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$$

the solution γ_p is

$,t+b)$

$b)$:

$t, \underbrace{b\cos t - a\sin t}$

$\sin t) = \; + i$

$, be^{-t})$

!! (can be 0)

$\boxed{-4-}$

FLOWS: Given $X \in \mathfrak{X}(M)$ look at

$\boxed{(\forall) p \in M\text{, a function depending on } t} \longrightarrow \gamma_p(t_n) \in M$

M $I_p \subseteq \mathbb{R}$

& re-baptise them:

- interpret it as a function on two variables (p,t) ← THE FLOW OF X, Φ
- —〃— giving: for each t, view it as a function of p. ← THE FLOW OF X AT TIME t, Φ^t

$$\gamma_p(t) =: \Phi(p,t) = \Phi^t(p)$$

$\gamma_p : I_p \to M$
the maximal
IC of X starting
at p.

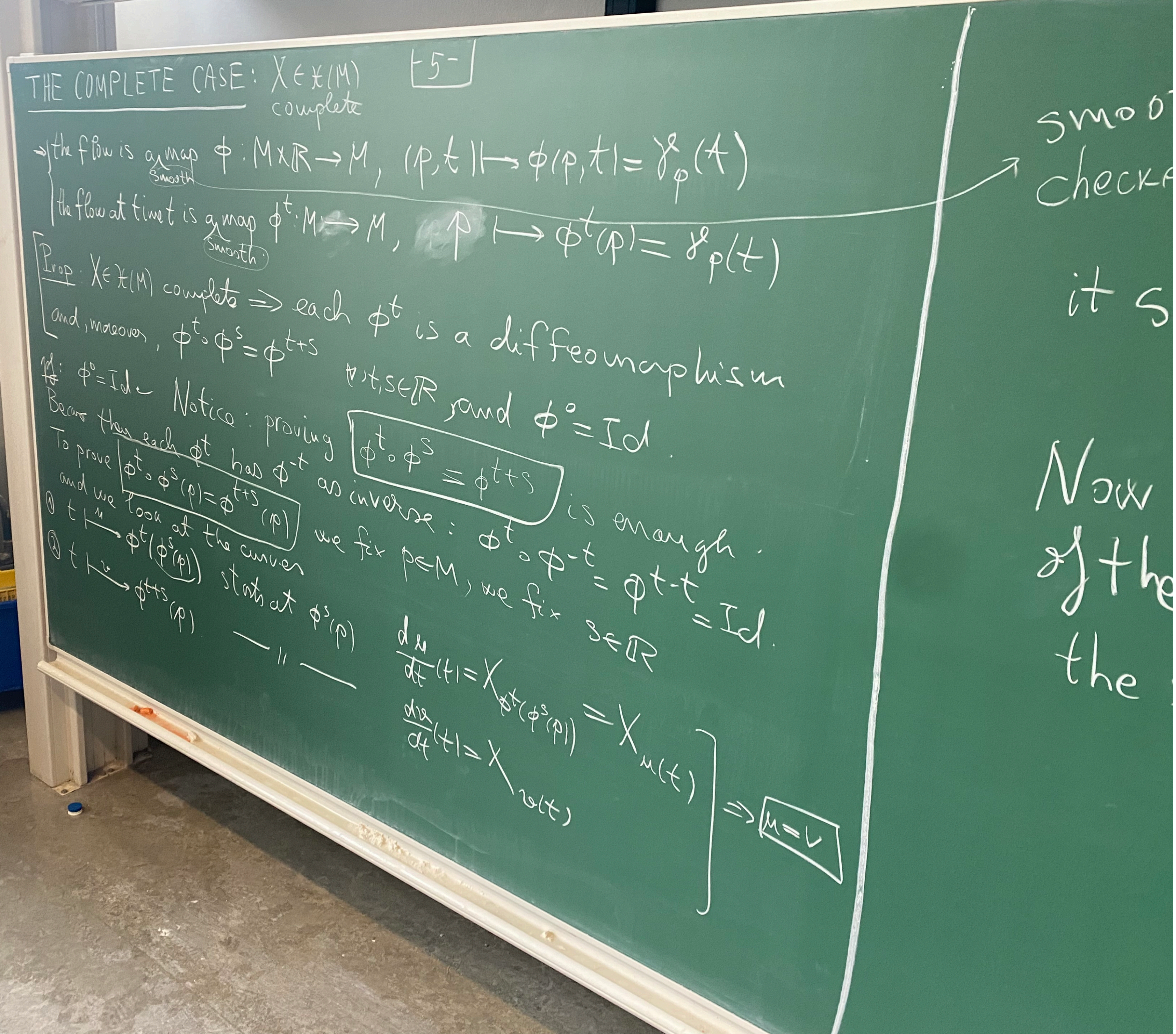
$$\begin{cases} \dfrac{d\gamma_p}{dt}(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = p. \end{cases}$$

$$\frac{d}{dt}\phi(p,t) = X_{\phi(p,t)}$$

$$\phi(p,0) = p.$$

$$\frac{d}{dt}\phi^t(\overset{\phi^s(p)}{p}) = X_{\phi^t(p)}$$

$$\phi^0(p) = p \quad \left(\phi^0 = \mathrm{Id}\right).$$



THE COMPLETE CASE: $X \in \mathfrak{X}(M)$ complete

$\boxed{-5-}$

→ the flow is a smooth map $\phi : M \times \mathbb{R} \to M$, $(p,t) \longmapsto \phi(p,t) = \gamma_p(t)$

the flow at time t is a smooth map $\phi^t : M \to M$, $p \longmapsto \phi^t(p) = \gamma_p(t)$

Prop: $X \in \mathfrak{X}(M)$ complete $\Rightarrow$ each ϕ^t is a diffeomorphism and, moreover, $\phi^t \circ \phi^s = \phi^{t+s}$ $\forall t,s \in \mathbb{R}$, and $\phi^0 = Id$.

Pf: $\phi^0 = Id$. Notice: proving $\boxed{\phi^t \circ \phi^s = \phi^{t+s}}$ is enough.

Because then each ϕ^t has ϕ^{-t} as inverse: $\phi^t \circ \phi^{-t} = \phi^{t-t} = Id$.

To prove $\boxed{\phi^t \circ \phi^s(p) = \phi^{t+s}(p)}$ we fix $p \in M$, we fix $s \in \mathbb{R}$ and we look at the curves

① $t \overset{u}{\longmapsto} \phi^t(\phi^s(p))$ starts at $\phi^s(p)$

② $t \overset{v}{\longmapsto} \phi^{t+s}(p)$ — " —

$$\frac{du}{dt}(t) = X_{\phi^t(\phi^s(p))} = X_{u(t)}$$

$$\frac{dv}{dt}(t) = X_{v(t)}$$

$$\Rightarrow \boxed{u = v}$$

smoo

check

it s

Now

of the

the

integral curve of X that starts at p.
defined on $\mathbb{R}$.

$\boxed{-5-}$

$M, \ (p,t) \longmapsto \phi(p,t) = \gamma_p(t)$

$M, \ p \longmapsto \phi^t(p) = \gamma_p(t)$

ϕ^t is a diffeomorphism

$t, s \in \mathbb{R}$ and $\phi^0 = Id$.

$\boxed{\phi^t \circ \phi^s = \phi^{t+s}}$ is enough.

inverse: $\phi^t \circ \phi^{-t} = \phi^{t-t} = Id$.

fix $p \in M$, we fix $s \in \mathbb{R}$

$$\left.\begin{array}{l} \frac{du}{dt}(t) = X_{\phi^t(\phi^s(p))} = X_{u(t)} \\ \frac{dv}{dt}(t) = X_{v(t)} \end{array}\right] \Rightarrow \boxed{u = v}$$

smoothness is a local property: can be checked in small enough opens

$\boxed{-5'-}$

$\Downarrow$

it suffices to look at the case

$M = \Omega \subset \mathbb{R}^n$ open.

Now: appeal to the second part of the Cauchy local $\exists !$:
the solutions γ_p depend smoothly also on p.

$p \in M$ if $\gamma(0) = p$ ($0 \in I$)

cannot be extended to a larger interval.

$\Rightarrow X$ is not complete.

$t) \longmapsto \phi(p,t) = \gamma_p(t)$

$p \longmapsto \phi^t(p) = \gamma_p(t)$

is a diffeomorphism

$\mathbb{R}$ and $\phi^0 = Id$.

$\boxed{\phi^s = \phi^{t+s}}$ is enough.

se : $\phi^t \circ \phi^{-t} = \phi^{t-t} = Id$.

$p \in M$, we fix $s \in \mathbb{R}$

$$\left.\begin{array}{l} \frac{du}{dt}(t) = X_{\phi^t(\phi^s(p))} = X_{u(t)} \\ \frac{dv}{dt}(t) = X_{v(t)} \end{array}\right] \Rightarrow \boxed{u = v}$$

—6—

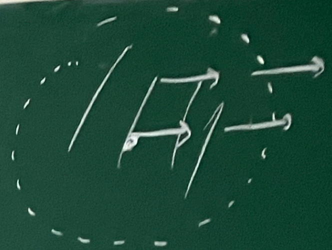
THE GENERAL CASE : $X \in \mathfrak{X}(M)$ arbitrary

$$\begin{cases} \mathcal{D}(X) := \{ (p,t) \in M \times \mathbb{R} \,/\, \gamma_p(t) \text{ is defined} \} \\ \phi : \mathcal{D}(X) \longrightarrow M . \end{cases}$$

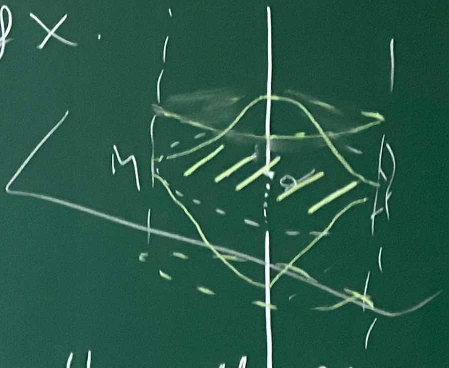
$\gamma_p(t)$ is defined $\Leftrightarrow t \in I_p$

$\mathcal{D}(X)$: the domain of the flow of X.

$$\begin{cases} \mathcal{D}^t(X) := \{ p \in M \,/\, \gamma_p(t) \text{ is defined} \} \\ \phi^t : \mathcal{D}^t(X) \longrightarrow M . \end{cases}$$



$X = \frac{\partial}{\partial x}$



Prop : (1) $\mathcal{D}(X)$ is open in $M \times \mathbb{R}$ and ϕ is smooth

(2) each $\mathcal{D}^t(X)$ open in M and ϕ^t is smooth, actually a diffeo : $\phi^t : \mathcal{D}^t(X) \longrightarrow \mathcal{D}^{-t}(X)$

(3) $\phi^t(\phi^s(p)) = \phi^{t+s}(p)$ "whenever things are defined"

$\rightarrow$ i.e. if t, s, p are so that : $\phi^s(p)$ is defined, $\phi^t(\phi^s(p))$ —"— $\Big\} \Rightarrow$ also $\phi^{t+s}(p)$ is defined and (3) holds.

Pf : local Cauchy $\Rightarrow$ (1) $\Longrightarrow$ (2)

(3) : same argument + careful as in the complete case

$\Rightarrow$

Us

$\Rightarrow$

cont

$\boxed{-7-}$

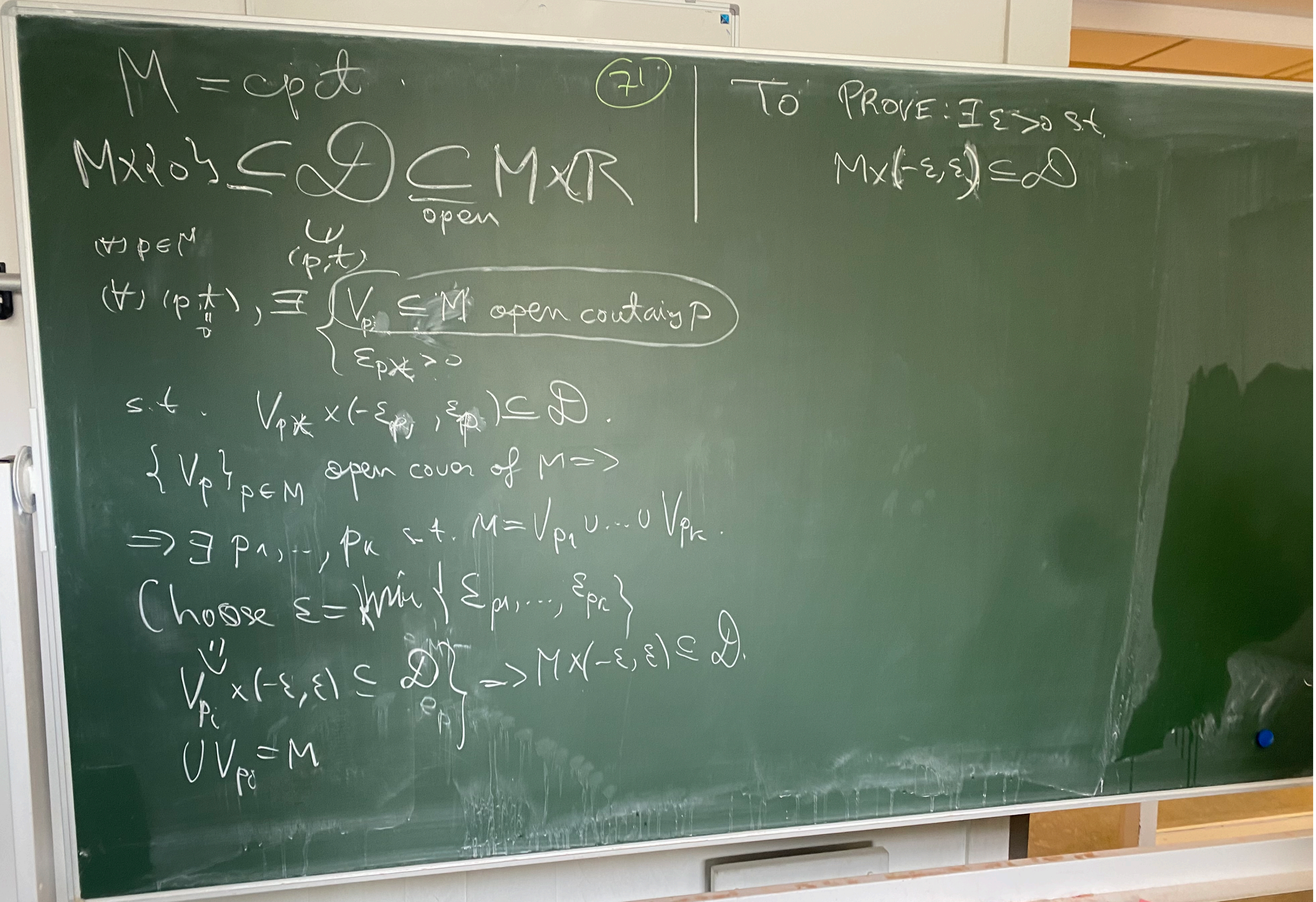
<u>THM</u>: If $M = \text{cpct} \Longrightarrow$ all $X \in \mathfrak{X}(M)$ are c

Pf: Start with $X \in \mathfrak{X}(M)$. Look at

$$\left.\begin{array}{c} M \times \{0\} \subseteq \mathcal{D}(X) \underset{\text{open}}{\subseteq} M \times \mathbb{R} \\ M = \text{compact} \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \exists\, \varepsilon > 0 \text{ s.t. } \underbrace{M \times [-\varepsilon, \varepsilon] \subseteq \mathcal{D}(X)}.$$

Use part 3 of prop



$M = \text{cpt}$.

⑦'

$M\times\{0\} \subseteq \mathcal{D} \underset{\text{open}}{\subseteq} M\times\mathbb{R}$

$\overset{\cup}{(p,t)}$

TO PROVE: $\exists\, \varepsilon > 0$ s.t. $M\times(-\varepsilon,\varepsilon) \subseteq \mathcal{D}$

$(\forall)\ p\in M$

$(\forall)\ (p, \underset{0}{\overset{\|}{t}}),\ \exists \begin{cases} \boxed{V_{p} \subseteq M \text{ open containg } p} \\ \varepsilon_{p\cancel{t}} > 0 \end{cases}$

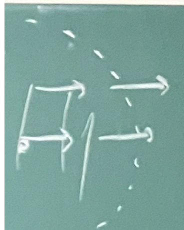
s.t. $V_{p\cancel{t}} \times (-\varepsilon_{p}, \varepsilon_{p}) \subseteq \mathcal{D}$.

$\{V_p\}_{p\in M}$ open cover of $M \Rightarrow$

$\Rightarrow \exists\ p_1, \dots, p_k$ s.t. $M = V_{p_1} \cup \dots \cup V_{p_k}$.

Choose $\varepsilon = \min\{\varepsilon_{p_1}, \dots, \varepsilon_{p_k}\}$

$$\left.\begin{array}{l} \overset{\cup}{V_{p_i}} \times (-\varepsilon,\varepsilon) \subseteq \mathcal{D} \\ \bigcup V_{p_i} = M \end{array}\right\} \Rightarrow M\times(-\varepsilon,\varepsilon) \subseteq \mathcal{D}.$$



actually a

$\mathcal{D}^t(X) \to \mathcal{D}^{-t}(X)$)

fined"

s defined

3) holds.

THM: If $M = \text{cpct} \Rightarrow$ all $X \in \mathfrak{X}(M)$ are complete.

Pf: Start with $X \in \mathfrak{X}(M)$. Look at

$$\left.\begin{array}{l} M \times \{0\} \subseteq \mathcal{D}(X) \underset{\text{open}}{\subseteq} M \times \mathbb{R} \\ M = \text{compact} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \exists\, \varepsilon > 0 \text{ s.t. } \underbrace{M \times [-\varepsilon, \varepsilon] \subseteq \mathcal{D}(X)}.$$

Use part 3 of prop:

$\phi^s(p)$ is defined for **all** p

—"— $s \in [-\varepsilon, \varepsilon]$

also $\phi^t(\phi^s(p))$ will

$\Rightarrow \phi^{t+s}(p)$ def $(\forall)\, t, s \in [-\varepsilon, \varepsilon] \Rightarrow \phi^s(p)$ defined $(\forall)\, s \in [-2\varepsilon, 2\varepsilon]$)

continue $\Rightarrow \phi^t(p)$ defined $\forall\, t \in \mathbb{R}$, $\forall\, p$ ▨

-8-

($M=\mathbb{R}^2$)

Ex 1: $X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases} \dot{x} = 1 \\ \dot{y} = \lambda \end{cases} \Rightarrow$ for each $p = (a,b) \in \mathbb{R}^2$ the solution γ_p is

$$\gamma_p(t) = (t+a, \lambda t + b)$$

$$\left(\begin{array}{l} \gamma(t) = (x(t), y(t)) \\ \frac{d\gamma}{dt}(t) = \underline{\dot{x}(t)} \frac{\partial}{\partial x} + \underline{\underline{\dot{y}(t)}} \frac{\partial}{\partial y} \end{array} \right)$$

$$\phi^t : \mathbb{R}^2 \to \mathbb{R}^2$$

$$\phi^t(a,b) = (t+a, t+b)$$

$\Rightarrow X =$ complete.

Ex 2: $X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \Rightarrow$ to solve $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ ... for each $p = (a,b)$:

$$\gamma_p(t) = (\underbrace{a\cos t + b\sin t}, \underbrace{b\cos t - a\sin t})$$

$$\overset{\shortparallel}{(a+ib)} \cdot (\cos t + i \sin t) = \quad + i$$

$$d^t : \mathbb{R}^2 \to \mathbb{R}^2$$

$\Rightarrow X =$ complete

$$\phi^t(a,b) =$$

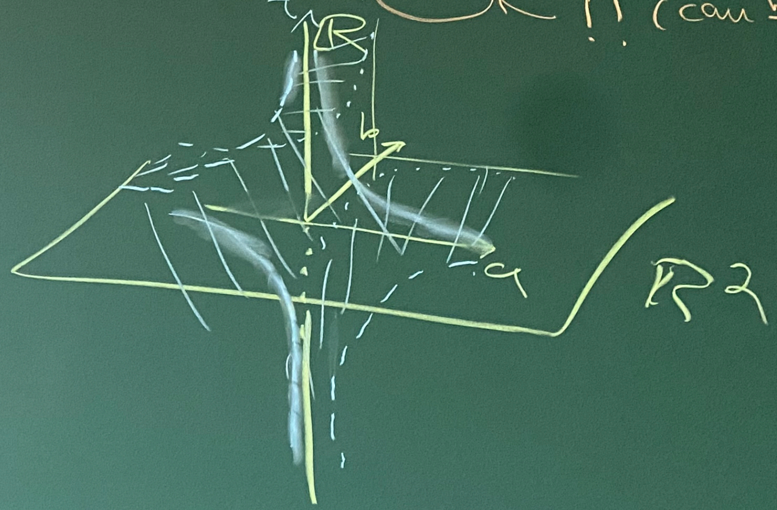
Ex 3: $X = -x^2 \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \Rightarrow \ldots \Rightarrow$ for each $p = (a,b)$: $\gamma_p(t) = \left(\frac{a}{\boxed{at+1}}, b e^{-t} \right)$ ← !! (can be 0)

$\Rightarrow X$ is not complete.

$$\mathcal{D}(X) = \{ (\underset{(a,b)}{\overset{\shortparallel}{p}}, t) \;/\; at+1 > 0 \}$$

$$\phi : \mathcal{D}(X) \longrightarrow \mathbb{R}^2$$

$$\phi(a,b,t) = \left(\frac{a}{at+1}, b e^{-t} \right)$$



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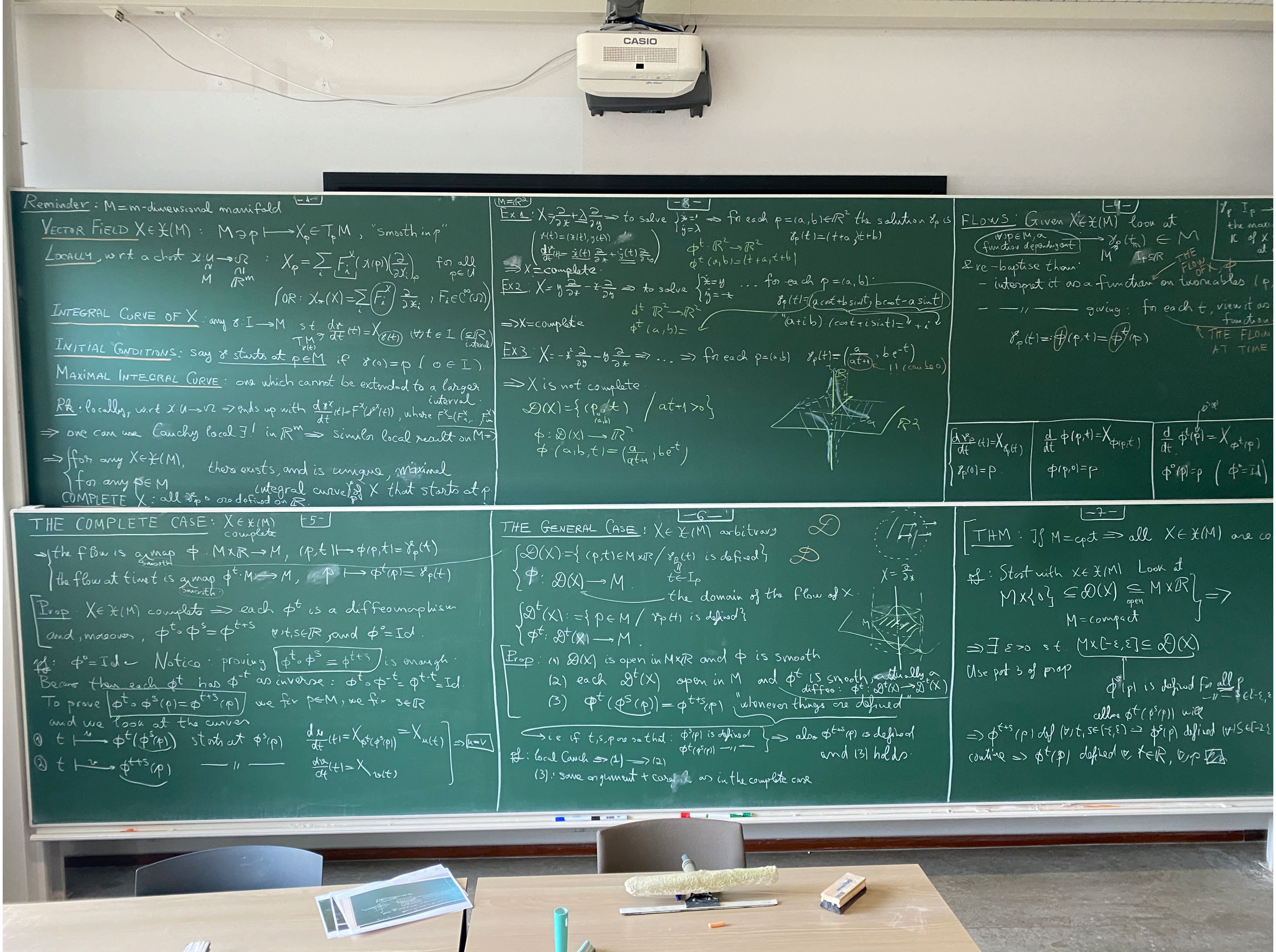

-1-

Reminder: M = m-dimensional manifold

VECTOR FIELD $X \in \mathfrak{X}(M)$: $M \ni p \longmapsto X_p \in T_pM$, "smooth in p"

LOCALLY, w.r.t. a chart $x: \underset{M}{\underset{\cap}{U}} \to \underset{\mathbb{R}^m}{\underset{\cap}{\Omega}}$: $X_p = \sum_i \boxed{F_i^X}(x(p))\left(\frac{\partial}{\partial x_i}\right)_p$ for all $p \in U$

$\left(\text{OR}: x_*(X) = \sum_i F_i^X \frac{\partial}{\partial x_i}, \; F_i \in C^\infty(\Omega)\right)$

INTEGRAL CURVE OF X: any $\gamma: I \longrightarrow M$ s.t. $T_{\gamma(t)}M \ni \frac{d\gamma}{dt}(t) = X_{\gamma(t)}$ $(\forall) t \in I$ ($\subseteq \mathbb{R}$ interval)

INITIAL CONDITIONS: say γ starts at p∈M if $\gamma(0) = p$ ($0 \in I$).

MAXIMAL INTEGRAL CURVE: one which cannot be extended to a larger interval.

Rk: locally, w.r.t. $x: U \to \Omega$ ⇒ ends up with $\frac{d\gamma^x}{dt}(t) = F^X(\gamma^x(t))$, where $F^X = (F_1^X, \dots, F_m^X)$

⇒ one can use Cauchy local ∃! in $\mathbb{R}^m$ ⇒ similar local result on M ⇒

⇒ $\begin{cases}\text{for any } X \in \mathfrak{X}(M), \\ \text{for any } p \in M\end{cases}$ there exists, and is unique, maximal integral curve γ_p of X that starts at p.

COMPLETE X: all γ_p's are defined on $\mathbb{R}$.

-8-

($M = \mathbb{R}^2$)

Ex 1: $X = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}$ ⇒ to solve $\begin{cases}\dot{x} = 1 \\ \dot{y} = \lambda\end{cases}$ ⇒ for each $p = (a,b) \in \mathbb{R}^2$ the solution γ_p is $\gamma_p(t) = (t+a, t+b)$

$\left(\begin{array}{l}\gamma(t) = (x(t), y(t)) \\ \frac{d\gamma}{dt}(t) = \dot{x}(t)\frac{\partial}{\partial x} + \dot{y}(t)\frac{\partial}{\partial y}\end{array}\right)$

$\phi^t: \mathbb{R}^2 \to \mathbb{R}^2$

$\phi^t(a,b) = (t+a, t+b)$

⇒ X = complete.

Ex 2: $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ ⇒ to solve $\begin{cases}\dot{x} = y \\ \dot{y} = -x\end{cases}$ ... for each $p = (a,b)$:

$\gamma_p(t) = (a\cos t + b\sin t, b\cos t - a\sin t)$

"$(a+ib)\cdot(\cos t + i\sin t) = \quad + i$"

$\phi^t: \mathbb{R}^2 \to \mathbb{R}^2$

$\phi^t(a,b) =$

⇒ X = complete

Ex 3: $X = -x^2\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ ⇒ ... ⇒ for each $p = (a,b)$ $\gamma_p(t) = \left(\frac{a}{\boxed{at+1}}, be^{-t}\right)$!! (can be 0)

⇒ X is not complete

$\mathcal{D}(X) = \{(\underset{(a,b)}{p}, t) \;/\; at+1 > 0\}$

$\phi: \mathcal{D}(X) \longrightarrow \mathbb{R}^2$

$\phi(a,b,t) = \left(\frac{a}{at+1}, be^{-t}\right)$

-4-

FLOWS: Given $X \in \mathfrak{X}(M)$ look at

$(\forall) p \in M$, a function depending on t $\longrightarrow \gamma_p(t) \in M$, $I_p \subseteq \mathbb{R}$

& re-baptise them:

- interpret it as a function on two variables (p, ← THE FLOW OF X, φ
- — " — giving: for each t, view it as function ← THE FLOW AT TIME

$\gamma_p(t) =: \phi(p,t) = \phi^t(p)$

$\gamma_p: I_p \longrightarrow$
the max
IC of X
at

$\begin{cases}\frac{d\gamma_p}{dt}(t) = X_{\gamma_p(t)} \\ \gamma_p(0) = p\end{cases}$

$\frac{d}{dt}\phi(p,t) = X_{\phi(p,t)}$

$\phi(p,0) = p$

$\frac{d}{dt}\phi^t(p) = X_{\phi^t(p)}$

$\phi^0(p) = p$ $(\phi^0 = \text{Id})$

-5-

THE COMPLETE CASE: $X \in \mathfrak{X}(M)$ complete

⇒ the flow is a smooth map $\phi: M \times \mathbb{R} \to M$, $(p,t) \longmapsto \phi(p,t) = \gamma_p(t)$

the flow at time t is a smooth map $\phi^t: M \longrightarrow M$, $p \longmapsto \phi^t(p) = \gamma_p(t)$

Prop: $X \in \mathfrak{X}(M)$ complete ⇒ each ϕ^t is a diffeomorphism and, moreover, $\phi^t \circ \phi^s = \phi^{t+s}$ $\forall t,s \in \mathbb{R}$, and $\phi^0 = \text{Id}$.

Pf: $\phi^0 = \text{Id}$ ✓ Notice: proving $\boxed{\phi^t \circ \phi^s = \phi^{t+s}}$ is enough.

Because then each ϕ^t has ϕ^{-t} as inverse: $\phi^t \circ \phi^{-t} = \phi^{t-t} = \text{Id}$.

To prove $\boxed{\phi^t \circ \phi^s(p) = \phi^{t+s}(p)}$ we fix p∈M, we fix s∈ℝ and we look at the curves

① $t \xmapsto{u} \phi^t(\phi^s(p))$ starts at $\phi^s(p)$ $\frac{du}{dt}(t) = X_{\phi^t(\phi^s(p))} = X_{u(t)}$

② $t \xmapsto{v} \phi^{t+s}(p)$ — " — $\frac{dv}{dt}(t) = X_{v(t)}$

⇒ $\boxed{u = v}$

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THE GENERAL CASE: $X \in \mathfrak{X}(M)$ arbitrary

$\begin{cases}\mathcal{D}(X) := \{(p,t) \in M \times \mathbb{R} \;/\; \gamma_p(t) \text{ is defined}\} \\ \phi: \mathcal{D}(X) \longrightarrow M\end{cases}$ $\quad \Leftrightarrow t \in I_p$

$\mathcal{D}(X)$: the domain of the flow of X.

$\begin{cases}\mathcal{D}^t(X) := \{p \in M \;/\; \gamma_p(t) \text{ is defined}\} \\ \phi^t: \mathcal{D}^t(X) \longrightarrow M\end{cases}$

$X = \frac{\partial}{\partial x}$

Prop: (1) $\mathcal{D}(X)$ is open in $M \times \mathbb{R}$ and φ is smooth

(2) each $\mathcal{D}^t(X)$ open in M and ϕ^t is smooth, actually a diffeo: $\phi^t: \mathcal{D}^t(X) \to \mathcal{D}^{-t}(X)$

(3) $\phi^t(\phi^s(p)) = \phi^{t+s}(p)$ "whenever things are defined"

i.e. if t, s, p are so that: $\begin{cases}\phi^s(p) \text{ is defined} \\ \phi^t(\phi^s(p)) \text{ — " —}\end{cases}$ ⇒ also $\phi^{t+s}(p)$ is defined and (3) holds.

Pf: local Cauch ⇒ (1) ⟶ (2)

(3): same argument + care as in the complete case

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THM: If M = cpct ⇒ all $X \in \mathfrak{X}(M)$ are co

Pf: Start with $X \in \mathfrak{X}(M)$. Look at

$\left.\begin{array}{l} M \times \{0\} \subseteq \mathcal{D}(X) \underset{\text{open}}{\subseteq} M \times \mathbb{R} \\ M = \text{compact}\end{array}\right\} \Rightarrow$

⇒ ∃ ε > 0 s.t. $M \times [-\varepsilon, \varepsilon] \subseteq \mathcal{D}(X)$

Use part 3 of prop

$\phi^s(p)$ is defined for **all** p
— " — $s \in [-\varepsilon, \varepsilon$

also $\phi^t(\phi^s(p))$ will

⇒ $\phi^{t+s}(p)$ def $(\forall) t, s \in [-\varepsilon, \varepsilon] \Rightarrow \phi^s(p)$ defined $(\forall) s \in [-2\varepsilon$

continue ⇒ $\phi^t(p)$ defined $\forall t \in \mathbb{R}$, $\forall p$ ☒