−1−

- for any M, p: the <u>tangent space</u> T_pM (a <u>vector space</u>!)
- any chart (U, x) around p gives rise to <u>a basis of our vector space</u> T_pM:

$$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$$

- <u>vector field</u> X on M:

$$M \ni p \longmapsto X_p \in T_pM$$

which is <u>smooth</u>: $(\forall)$ (U, x), the <u>coefficients</u> in writing $X_p = \sum_i \mathrm{coeff}_i(p) \left(\frac{\partial}{\partial x_i}\right)_p$ are smooth

$\mathfrak{X}(M)$:= the set of all vector fields on M

- $\mathfrak{X}(M)$ is a <u>vector space</u>: $(X+Y)_p := X_p + Y_p$, $(\lambda \cdot X)_p := \lambda X_p$
- $\mathfrak{X}(M)$ is a <u>$C^\infty(M)$-module</u>: $(f \cdot X)_p := f(p) X_p$ $\quad \left(f \in \mathcal{C}^\infty(M), \text{ i.e. } f : M \to \mathbb{R} \text{ smooth} \right)$
- globally, any $X \in \mathfrak{X}(M)$ induces $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$, $\left(\mathcal{L}_X(f)(p) = \boxed{(df)_p (X_p)} = \partial_{X_p}(f) \right)$

& we have $\mathfrak{X}(M) \xleftrightarrow{1-1} \left\{ L : C^\infty(M) \to C^\infty(M) \,/\, L \text{ derivation} \right\}$

−2−

- $\mathfrak{X}(M)$ is a <u>Lie algebra</u>: for $X, Y \in \mathfrak{X}(M) \Rightarrow$ we have $[X, Y] \in \mathfrak{X}(M)$ $\quad \left(\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X \right)$
- <u>in $\mathbb{R}^L$</u>, $X \in \mathfrak{X}(\mathbb{R}^L)$: $X = \sum_{i=1}^{L} F_i \frac{\partial}{\partial x_i}$ with $F_i \in C^\infty(\mathbb{R}^L)$

(&, for general $X \in \mathfrak{X}(M)$, what we see locally (in a chart): just m-functions!)

- <u>on submanifolds $M \subseteq \mathbb{R}^L$</u>: can use the ambient $\mathbb{R}^L$ and any $X \in \mathfrak{X}(M)$ can be written as

$$X_p = \sum_{i=1}^{L} F_i(p) \left(\frac{\partial}{\partial x_i}\right)_p \quad (\text{for } p \in M!)$$

but <u>not all</u> such expression are tangent to M.

<u>Ex</u>: For $M = S^1 \subseteq \mathbb{R}^2$, $x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ can be seen as a vector field on S^1, but $x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}$ not!

- for any M, p: the tangent space T_pM (a vector space!)
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- globally, any $X \in \mathfrak{X}(M)$ induces $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$, $\left(\mathcal{L}_X(f)(p) = \boxed{(df)_p (X_p)} = \partial_{X_p}(f)\right)$

& we have

$$\mathfrak{X}(M) \xleftrightarrow{1-1} \left\{ L : C^\infty(M) \to C^\infty(M) \,/\, L \text{ derivation} \right\}$$

... $(f \cdot X)_p := f(p)\, X_p$ $\quad \left(f \in \mathcal{C}^\infty(M),\ \text{i.e. } f : M \to \mathbb{R} \text{ smooth} \right)$

- globally, any $X \in \mathfrak{X}(M)$ induces $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$, $\left(\mathcal{L}_X(f)(p) = \boxed{(df)_p (X_p)} = \partial_{X_p}(f) \right)$

& we have $\mathfrak{X}(M) \xleftrightarrow{1-1} \left\{ L : C^\infty(M) \to C^\infty(M) \,/\, L \text{ derivation} \right\}$

-2-

- $\mathfrak{X}(M)$ is a Lie algebra: for $X, Y \in \mathfrak{X}(M) \Rightarrow$ we have $[X,Y] \in \mathfrak{X}(M)$ $\left(\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X \right)$

- in $\mathbb{R}^L$, $X \in \mathfrak{X}(\mathbb{R}^L)$: $X = \sum_{i=1}^{L} F_i \frac{\partial}{\partial x_i}$ with $F_i \in C^\infty(\mathbb{R}^L)$

(&, for general $X \in \mathfrak{X}(M)$, what we see locally (in a chart): just m-functions!)

- on submanifolds $M \subseteq \mathbb{R}^L$: can use the ambient $\mathbb{R}^L$ and any $X \in \mathfrak{X}(M)$ can be written as $X_p = \sum_{i=1}^{L} F_i(p) \left(\frac{\partial}{\partial x_i} \right)_p$ (for $p \in M$!)

but not all such expression are tangent to M.

Ex: For $M = S^1 \subseteq \mathbb{R}^2$, $x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ can be seen as a vector field on S^1, but $x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}$ not!

-4-

COTANGENT SPACES:

M = manifold, $p \in M$

Def: The cotangent space of M at p is: $T_p^*M := (T_pM)^*$ the dual of T_pM
$= \{ \xi_p : T_pM \to \mathbb{R} \,/\, \xi_p = \text{linear} \}$

Elts $\xi_p \in T_p^*M$ called: cotangent vectors.

Ex: $f \in C^\infty(M)$, $p \in M \Rightarrow$ we have $(df)_p : T_pM \to \mathbb{R}$ and this can now be seen as $(df)_p \in T_p^*M$

Ex: Whenever (U, x) = chart of M, apply previous to $f = x_i$

$$\Rightarrow \boxed{(dx_1)_p, \ldots, (dx_m)_p \in T_p^*M} \quad (1) \quad p \in U$$

This is a basis of T_p^*M !!!

(Actually:

$$\left\langle (dx_i)_p, \left(\frac{\partial}{\partial x_j}\right)_p \right\rangle = \delta^i_j \quad \Rightarrow (1) \text{ is the dual basis to } \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_m}\right)_p\text{)}$$

-3-

CO $\longleftrightarrow$ dual objects

DUALS/COTANGENT SPACES/1-FORMS

Duals: V = m-dimensional vector space. Its dual:

$$V^* := \{ \xi : V \to \mathbb{R} \,/\, \xi = \text{linear} \}$$

ξ called "co-vector for V".

$v \in V$ called "vector of V"

Remarks:

① V^* = vector space: $(\xi_1 + \xi_2)(v) := \xi_1(v) \overset{\text{in } \mathbb{R}!}{+} \xi_2(v)$

② covectors evaluate on vectors to give numbers. I.e. we have a map:

$$V^* \times V \longrightarrow \mathbb{R}, \quad (\xi, v) \longmapsto \xi(v) \overset{\text{notation}}{=\!=:} \langle \xi, v \rangle$$

③ V^* = m-dimensional. Actually any basis $e_1, \ldots, e_m$ of V induce a "dual basis" (of V^*) $e^1, \ldots, e^m \in V^*$

$\xi(\sum v^i e_i) = \sum v^i \boxed{\xi(e_i)}$ all info!

where each $e^i \in V^*$, as a map $V \xrightarrow{\xi} \mathbb{R}$ send $v = \sum v^i e_i \longmapsto v^i$ (OR: $\langle e^i, e_j \rangle = \delta_{ij}$).

④ Intuitively, think of $\xi \in V^*$ nonzero as:

- a hyperplane in V, namely $\underbrace{\mathrm{Ker}\,\xi = \{ v \in V \,/\, \xi(v) = 0 \}}_{\text{dimension } m-1} \subset V$
- a way to measure vectors of V that are orthogonal to thy hyperplane

After all... V^* is isomorphic to V BUT NOT CANONICALLY!

[1-FORMS]

- for any M, p : the cotangent space T_p^*M (a vector space!)
- any chart (U, x) around p gives rise to a basis of our vector space T_p^*M:

$$\cancel{\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p} \qquad (dx_1)_p, \dots, (dx_m)_p .$$

- ~~vector field~~ 1-FORMS $\cancel{X}$ ω on M :

$$M \ni p \longmapsto \cancel{X_p}\ \omega_p \in T_p^*M$$

which is smooth: $(\forall)$ (U, x), the coefficients in writing $\cancel{X_p}\ \omega_p = \sum_i \mathrm{coeff}_i(p) \cancel{\left(\frac{\partial}{\partial x_i}\right)_p}\ (dx_i)_p$ are smooth

$\Omega^1(M)$ $\cancel{\mathfrak{X}(M)}$:= the set of all ~~vector fields~~ 1-forms on M

- $\cancel{\mathfrak{X}(M)}$ $\Omega^1(M)$ is a vector space : $(\omega + \eta)_p := \omega_p + \eta_p$, $(\lambda \cdot \omega)_p := \lambda\, \omega_p$
- $\cancel{\mathfrak{X}(M)}$ is a $C^\infty(M)$-module : $(f \cdot \omega)_p := f(p)\, \omega_p$ ($f \in \mathcal{C}^\infty(M)$, i.e. $f : M \to \mathbb{R}$ smooth)
- globally, any $X \in \mathfrak{X}(M)$ induces $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$, $\mathcal{L}_X(f)(p) = \boxed{(df)_p(X_p)} = \partial_{X_p}(f)$

& we have $\mathfrak{X}(M) \xleftrightarrow{1-1} \left\{ L : C^\infty(M) \to C^\infty(M) \,/\, L \text{ derivation} \right\}$

For $\omega \in \Omega^1$

Thm

Hence

-7-

For $\omega \in \Omega^1(M)$ we have an operation still denoted ω

$$\mathfrak{X}(M) \longrightarrow C^\infty(M)$$
$$X \longmapsto \omega(X)$$

where $\underset{\in M}{\omega(X)(p)} := \underset{\in T_p^*M}{\omega_p}\big(\underset{\in T_pM}{X_p}\big)$

Thm: This gives a bijection

$$\Omega^1(M) \xleftrightarrow{1-1} \left\{ \xi : \mathfrak{X}(M) \longrightarrow C^\infty(M) \text{ which are } C^\infty(M)\text{-linear, i.e. } \begin{array}{ll} \xi(X+Y) = \xi(X) + \xi(Y) & (\forall)\, X, Y \in \mathfrak{X}(M) \\ \xi(f X) = f \cdot \xi(X) & f \in C^\infty(M) \end{array} \right\}$$

Hence . cotangent vector at p eats tangent vectors at p $\Longrightarrow$ real numbers

1-form eats vector fields $\Longrightarrow$ smooth functions

& we have $\mathfrak{X}(M) \xleftrightarrow{1-1} \left\{ L : C^\infty(M) \to C^\infty(M) \,/\, L \text{ derivation} \right\}$

$\alpha_X(f)(p) = \boxed{(df)_p}(X_p) = \partial_{X_p}(f)$

- $\boxed{\mathfrak{X}(M) \text{ is a Lie algebra: for } X, Y \in \mathfrak{X}(M) \Rightarrow \text{we have } [X,Y] \in \mathfrak{X}(M)}$ $\left(\alpha_{[X,Y]} = \alpha_X \circ \alpha_Y - \alpha_Y \circ \alpha_X \right)$

- in $\mathbb{R}^L$, $X \in \mathfrak{X}(\mathbb{R}^L)$: $X = \sum_{i=1}^{L} F_i \frac{\partial}{\partial x_i}$ with $F_i \in C^\infty(\mathbb{R}^L)$

(& , for general $X \in \mathfrak{X}(M)$, what we see locally (in a chart): just m-functions!)

- on submanifolds $M \subseteq \mathbb{R}^L$: can use the ambient $\mathbb{R}^L$ and any $X \in \mathfrak{X}(M)$ can be written as $X_p = \sum_{i=1}^{L} F_i(p) \left(\frac{\partial}{\partial x_i}\right)_p$ (for $p \in M$!)

but not all such expression are tangent to M.

Ex: For $M = S^1 \subseteq \mathbb{R}^2$, $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ can be seen as a vector field on S^1, but $x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$ not!

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[Exercise: Che

[1-FORMS] -5-

- for any M, p: the ~~cotangent space~~ T_p^*M (a vector space!)
- any chart (U, x) around p gives rise to a basis of our vector space T_p^*M:

$$\cancel{\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p} \qquad (dx_1)_p, \dots, (dx_m)_p.$$

- 1-FORMS ~~vector field~~ ω on M:

$$M \ni p \longmapsto \omega_p \in T_p^*M$$

which is smooth: $(\forall)\ (U, x)$, the coefficients in writing $\omega_p = \sum_i \mathrm{coeff}_i(p) (dx_i)_p$ are smooth

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- globally, any $X \in \mathfrak{X}(M)$ induces $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$, $\mathcal{L}_X(f)(p) = \boxed{(df)_p(X_p)} = \partial_{X_p}(f)$

& we have $\mathfrak{X}(M) \xleftrightarrow{1-1} \left\{ L : C^\infty(M) \to C^\infty(M) \,/\, L \text{ derivation} \right\}$

→ For $\omega \in \Omega^1(M)$ we have an operation still denoted ω $\quad$ $\mathfrak{X}(M) \longrightarrow C^\infty(M)$, $X \longmapsto \omega(X)$ $\quad$ -6-

where $\omega(X)(p) := \omega_p(X_p)$ ($p \in M$, $\omega_p \in T_p^*M$, $X_p \in T_pM$)

Thm This gives a bijection

$$\Omega^1(M) \xleftrightarrow{1-1} \left\{ \begin{array}{l} \xi : \mathfrak{X}(M) \to C^\infty(M) \text{ which are } C^\infty(M)\text{-linear} \\ \text{i.e. } \xi(X+Y) = \xi(X) + \xi(Y) \quad (\forall)\, X, Y \in \mathfrak{X}(M) \\ \qquad \xi(f \cdot X) = f\, \xi(X) \quad f \in C^\infty(M) \end{array} \right\}$$

Hence. cotangent vector at p eats tangent vectors at p ⟹ real number

1-form eats vector fields ⟹ smooth functions

COTANGENT S...

M = manifol...

Def: The c...

Elts $\xi_p \in T_p$...

Ex: $f \in C^\infty(\dots$

this can no...

Ex. Whenever ...

⟹

This is a bas...

(Actually

$\langle (dx_i)_p, ($...

-7-

- $\boxed{\mathfrak{X}(M) \text{ is a Lie algebra: for } X, Y \in \mathfrak{X}(M) \Rightarrow \text{we have } [X, Y] \in \mathfrak{X}(M)}$ $\quad$ $\left(\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X\right)$
- in $\mathbb{R}^L$, $X \in \mathfrak{X}(\mathbb{R}^L)$: $\quad X = \sum_{i=1}^{L} F_i \frac{\partial}{\partial x_i}$ $\quad$ with $F_i \in C^\infty(\mathbb{R}^L)$

(&, for general $X \in \mathfrak{X}(M)$, what we see locally (in a chart): just m-functions!)

- on submanifolds $M \subseteq \mathbb{R}^L$: can use the ambient $\mathbb{R}^L$ and any $X \in \mathfrak{X}(M)$ can be written as $\quad X_p = \sum_{i=1}^{L} F_i(p) \left(\frac{\partial}{\partial x_i}\right)_p$ $\quad$ (for $p \in M$!)

but not all such expression are tangent to M.

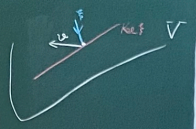
Ex: For $M = S^1 \subseteq \mathbb{R}^2$, $x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ can be seen as a vector field on S^1, but $x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}$ not!

→ • Operation $d : C^\infty(M) \to \Omega^1(M)$, $f \mapsto df$ $\quad$ given by $(df)_p := (df)_p$ $\quad$ -8-

(as a 1-form)

Exercise: Check d = linear and satisfies the "derivation rule" $d(f \cdot g) = f \cdot dg + g \cdot df$.

$$V \xrightarrow{\sim} V^*$$
$$\sum \lambda^i e_i \longmapsto \sum \lambda^i e^i$$



DUALS/CO...

Duals: ...

$v \in V$ called ...

Remarks:

① $V^{**} = V$...

② covect...

③ V...

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④ In ...

⑤ Af... al...

- Operation $d : C^\infty(M) \longrightarrow \Omega^1(M)$
  $f \mapsto df$ given by $(df)_p := (df)_p$

  as a 1-form

Exercise: Check: d = linear and satisfies the "derivation rule" $d(f \cdot g) = f \cdot dg + g \cdot df$.

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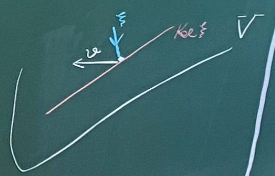

$\frac{\partial}{\partial x}$
not!

DUALS/COTANGENT SPACES/1-F

Duals: V = m-dimensional vec

$V^* := \{ \xi : V \to \mathbb{R}$

$v \in V$ called "vector of V"   called

Remarks:

1. V^* = vector space: $(\xi_1 + \xi_2)(v) := \xi_1($
2. covectors evaluate on vectors to
   $V^* \times V \longrightarrow \mathbb{R}$, $(\xi, v) \longmapsto$
3. $V^* = m$ Actually any
   induce

   where eac
4. Intuitivel
   - a hyp
   - a wo
5. After all...

restriction operation $_{|\partial N}$

$\Omega^1(N) \longrightarrow \Omega^1(M)$, $\omega \longmapsto$ denoted $\omega|_M$ OR: $i^*(\omega)$ (where $i: M \to N$ the inclusion)

$\langle (dx_i)_p, (\frac{\partial}{\partial x_j})_p \rangle = \delta_j^i$

- Operation $d: C^\infty(M) \longrightarrow \Omega^1(M)$, $f \mapsto df$ given by $(df)_p := (df)_p$ (as a 1-form)

Exercise: Check: d = linear and satisfies the "derivation rule" $d(f \cdot g) = f \cdot dg + g \cdot df$.

Rk: $df: \mathfrak{X}(M) \longrightarrow C^\infty(M)$ takes X to $\mathcal{L}_X(f)$: $df(X) = \mathcal{L}_X(f)$.

- in $\mathbb{R}^L$, $\omega \in \Omega^1(\mathbb{R}^L)$: $\omega = \boxed{\sum_{i=1}^{L} F_i \cdot dx_i}$ (*) with $F_i \in C^\infty(\mathbb{R}^L)$

Ex: $f \in C^\infty(\mathbb{R}^L)$: $df \in \Omega^1(\mathbb{R}^L)$, $df = \sum_i \frac{\partial f}{\partial x_i} \cdot dx_i$

- submanifolds $M \subseteq \mathbb{R}^L$: any expression can be "restricted to M" i.e. apply it only to vectors tangent to M. This is denoted $\left(\sum_{i=1}^{L} F_i \, dx_i\right)\Big|_M \in \Omega^1(M)$ also called "$\sum F_i \, dx_i$ on M" OR "$\sum F_i \, dx_i$ as 1-form on M".

CO-WARNING: Different looking formulas on $\mathbb{R}^L$ may describe the same 1-form on M!

Ex: $M = S^1$ and $\omega = (x + e^y)\,dx + (y + xy)\,dy$, $\eta = e^y\,dx + xy\,dy$ on S^1

$p = (x, y) \in S^1$, $X_p = a\left(\frac{\partial}{\partial x}\right)_p + b\left(\frac{\partial}{\partial y}\right)_p \in T_pS^1$

$\omega(X_p) = a \cdot (x + e^y) + b \cdot (y + xy)$
$\eta(X_p) = a \cdot e^y + b \cdot xy$

More elegantly: $x^2 + y^2 = 1$ on S^1; $2x\,dx + 2y\,dy = 0$ on S^1; $x\,dx + y\,dy = 0$ on S^1

DUALS/COTANGENT SPACES

Duals: V = m-dimensional ...
$V^* := \{\xi : V \to$ ... called ...
$v \in V$ called "vectors of V"

Remarks:
1. V^* = vector space $(\xi_1 + \xi_2)$ ...
2. covectors evaluate on vectors: $V^* \times V \longrightarrow \mathbb{R}$, ...
3. V^* = m-dimensional. Actually ... induce a "dual basis" (of V^*) $e^1, \dots, e^m \in V^*$ where each $e^i \in V^*$, as a map $V \to$ ...
4. Intuitively, think of $\xi \in V^*$ as ... Ker ...
   - a hyperplane in V, namely Ker ...
   - a way to measure vectors of V ...
5. After all ... V^* is isomorphic to V. BUT ...

... $\mathcal{L}_X$)

... com

$\frac{\partial}{\partial y} + y \frac{\partial}{\partial x}$ not!

$b = 0$.

For $\omega \in \Omega^1(M)$ we have an operation $\mathfrak{X}(M) \longrightarrow C^\infty(M)$
still denoted ω $\quad X \longmapsto \omega(X)$ where $\omega(X)(p) := \omega_p(X_p)$, $p \in M$, $\omega_p \in T_p^*M$, $X_p \in T_pM$

-6-

Thm: This gives a bijection

$$\Omega^1(M) \xleftrightarrow{1-1} \left\{ \xi : \mathfrak{X}(M) \longrightarrow C^\infty(M) \text{ which are } C^\infty(M)\text{-linear} \right\}$$

i.e. $\xi(X+Y) = \xi(X) + \xi(Y)$ $\quad (\forall) X, Y \in \mathfrak{X}(M)$

$\xi(f \cdot X) = f \cdot \xi(X)$ $\quad f \in C^\infty(M)$

Hence . cotangent vector at p eats tangent vectors at p $\Rightarrow$ real numbers

1-form eats Vector fields $\Rightarrow$ smooth functions

<u>Rk</u>: (continuation from page -8-) : more generally for

$M \subseteq N$ embedded

then:

① any $\xi_p \in T_p^*N$ can be restricted to $T_pM \subseteq T_pN$

② do it for all $p \Rightarrow$ restriction operation

$$\Omega^1(N) \longrightarrow \Omega^1(M), \quad \omega \longmapsto \text{denoted } \omega|M$$

OR: $i^*(\omega)$ (where $i : M \to N$ the inclusion)

-9-

k-forms: V = vector space, $k \in \mathbb{N}$ ($k=1$: 1-forms)

$$\Lambda^k V^* := \{ \xi : \underbrace{V \times \dots \times V}_{k} \to \mathbb{R} \;/\; \xi = \text{linear in each argument},\ \xi = \text{skew symmetric} \}$$

$$\xi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\, \xi(v_1, \dots, v_k), \quad \sigma \in S_k$$

$$T^k V^* := \{ \xi : V \times \dots \times V \to \mathbb{R} \;/\; \xi = \text{linear in each argument} \}$$

$\mathrm{Alt}: T^k V^* \to \Lambda^k V^*$

- Skew-symmetrization of $\xi \in T^k V^*$: $\underbrace{\mathrm{Alt}(\xi)}_{\in \Lambda^k V^*}(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, \xi(v_{\sigma(1)}, \dots, v_{\sigma(k)})$

  (Rk: $\xi \in \Lambda^k V^* \Leftrightarrow \mathrm{Alt}(\xi) = \xi$)

- For $\omega \in T^k V^*$, $\eta \in T^\ell V^*$ can define $\omega \otimes \eta \in T^{k+\ell} V^*$

$$(\omega \otimes \eta)(v_1, \dots, v_{k+\ell}) := \omega(v_1, \dots, v_k)\, \eta(v_{k+1}, \dots, v_{k+\ell})$$

- For $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^\ell V^*$ $\not\Rightarrow$ $\omega \otimes \eta$ is skew symmetric!

$$\omega \wedge \eta := \frac{(k+\ell)!}{k!\,\ell!} \mathrm{Alt}(\omega \otimes \eta)$$