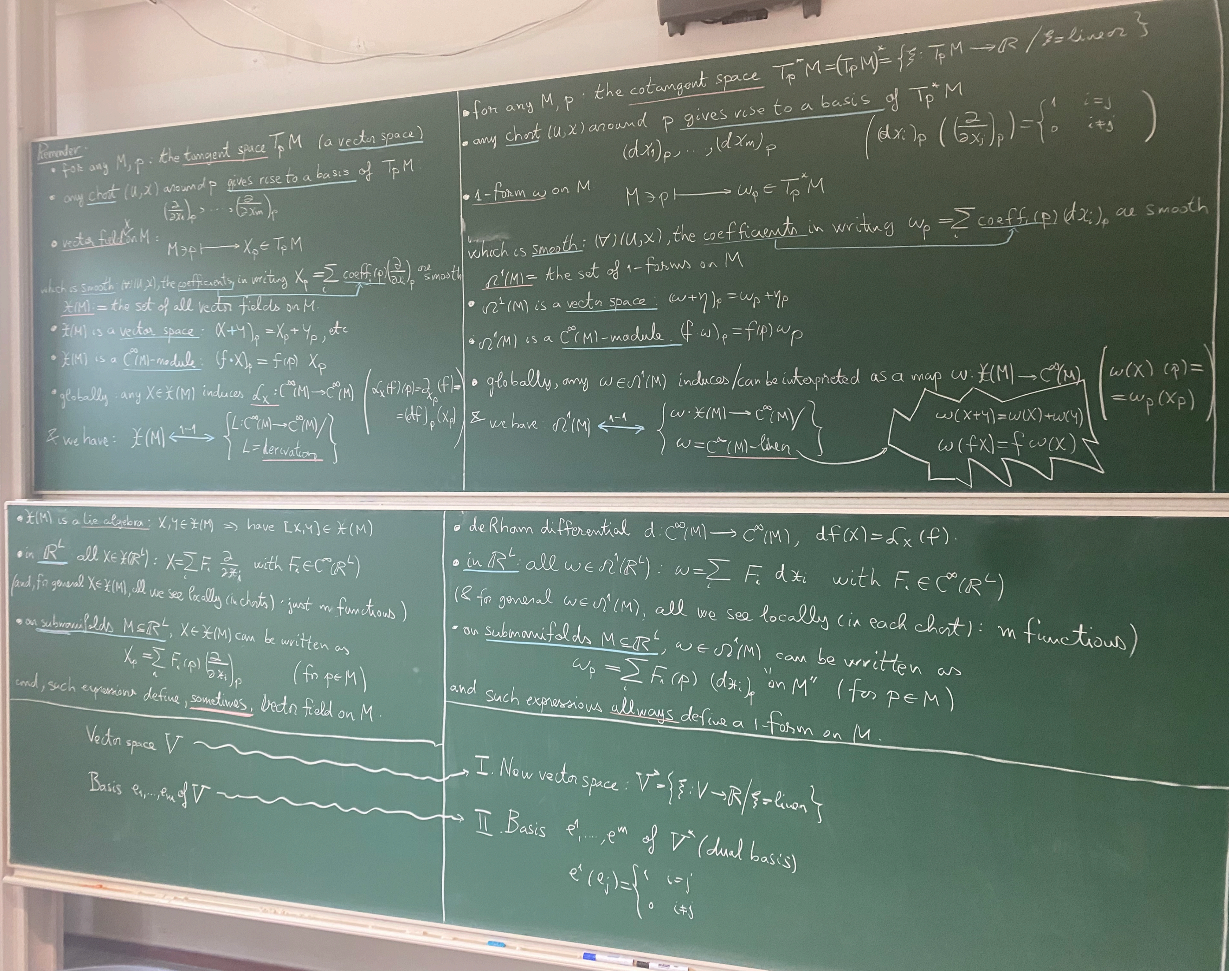


Remember:

- for any M, p: the tangent space T_pM (a vector space)
- any chart (U, x) around p gives rise to a basis of T_pM: $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$
- vector field X on M: $M \ni p \longmapsto X_p \in T_pM$

which is smooth: $(\forall)(U, x)$, the coefficients in writing $X_p = \sum_i \text{coeff}_i(p) \left(\frac{\partial}{\partial x_i}\right)_p$ are smooth

$\mathfrak{X}(M) :=$ the set of all vector fields on M.

- $\mathfrak{X}(M)$ is a vector space: $(X+Y)_p = X_p + Y_p$, etc
- $\mathfrak{X}(M)$ is a $C^\infty(M)$-module: $(f \cdot X)_p = f(p)\, X_p$
- globally: any $X \in \mathfrak{X}(M)$ induces $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$ $\left(\mathcal{L}_X(f)(p) = \partial_{X_p}(f) = (df)_p(X_p) \right)$

& we have: $\mathfrak{X}(M) \xleftrightarrow{1-1} \left\{ L : C^\infty(M) \to C^\infty(M) \,/\, L = \text{derivation} \right\}$

- for any M, p: the cotangent space $T_p^*M = (T_pM)^* = \{ \xi : T_pM \to \mathbb{R} \,/\, \xi = \text{linear} \}$
- any chart (U, x) around p gives rise to a basis of T_p^*M: $(dx_1)_p, \dots, (dx_m)_p$ $\left((dx_i)_p \left(\left(\frac{\partial}{\partial x_j}\right)_p \right) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \right)$
- 1-form ω on M: $M \ni p \longmapsto \omega_p \in T_p^*M$

which is smooth: $(\forall)(U, x)$, the coefficients in writing $\omega_p = \sum_i \text{coeff}_i(p)(dx_i)_p$ are smooth

$\Omega^1(M) =$ the set of 1-forms on M

- $\Omega^1(M)$ is a vector space: $(\omega + \eta)_p = \omega_p + \eta_p$
- $\Omega^1(M)$ is a $C^\infty(M)$-module: $(f \cdot \omega)_p = f(p)\, \omega_p$
- globally, any $\omega \in \Omega^1(M)$ induces/can be interpreted as a map $\omega : \mathfrak{X}(M) \to C^\infty(M)$ $\left(\omega(X)(p) = \omega_p(X_p) \right)$

& we have: $\Omega^1(M) \xleftrightarrow{1-1} \left\{ \omega : \mathfrak{X}(M) \to C^\infty(M) \,/\, \omega = C^\infty(M)\text{-linear} \right\}$

$\omega(X+Y) = \omega(X) + \omega(Y)$

$\omega(fX) = f\,\omega(X)$

- $\mathfrak{X}(M)$ is a Lie algebra: $X, Y \in \mathfrak{X}(M) \Rightarrow$ have $[X, Y] \in \mathfrak{X}(M)$
- in $\mathbb{R}^L$ all $X \in \mathfrak{X}(\mathbb{R}^L)$: $X = \sum_i F_i \frac{\partial}{\partial x_i}$ with $F_i \in C^\infty(\mathbb{R}^L)$

(and, for general $X \in \mathfrak{X}(M)$, all we see locally (in charts): just m functions)

- on submanifolds $M \subseteq \mathbb{R}^L$, $X \in \mathfrak{X}(M)$ can be written as

$$X_p = \sum_i F_i(p) \left(\frac{\partial}{\partial x_i}\right)_p \qquad (\text{for } p \in M)$$

and, such expressions define, sometimes, vector field on M.

- deRham differential $d : C^\infty(M) \to C^\infty(M)$, $df(X) = \mathcal{L}_X(f)$.
- in $\mathbb{R}^L$: all $\omega \in \Omega^1(\mathbb{R}^L)$: $\omega = \sum_i F_i\, dx_i$ with $F_i \in C^\infty(\mathbb{R}^L)$

(& for general $\omega \in \Omega^1(M)$, all we see locally (in each chart): m functions)

- on submanifolds $M \subseteq \mathbb{R}^L$, $\omega \in \Omega^1(M)$ can be written as

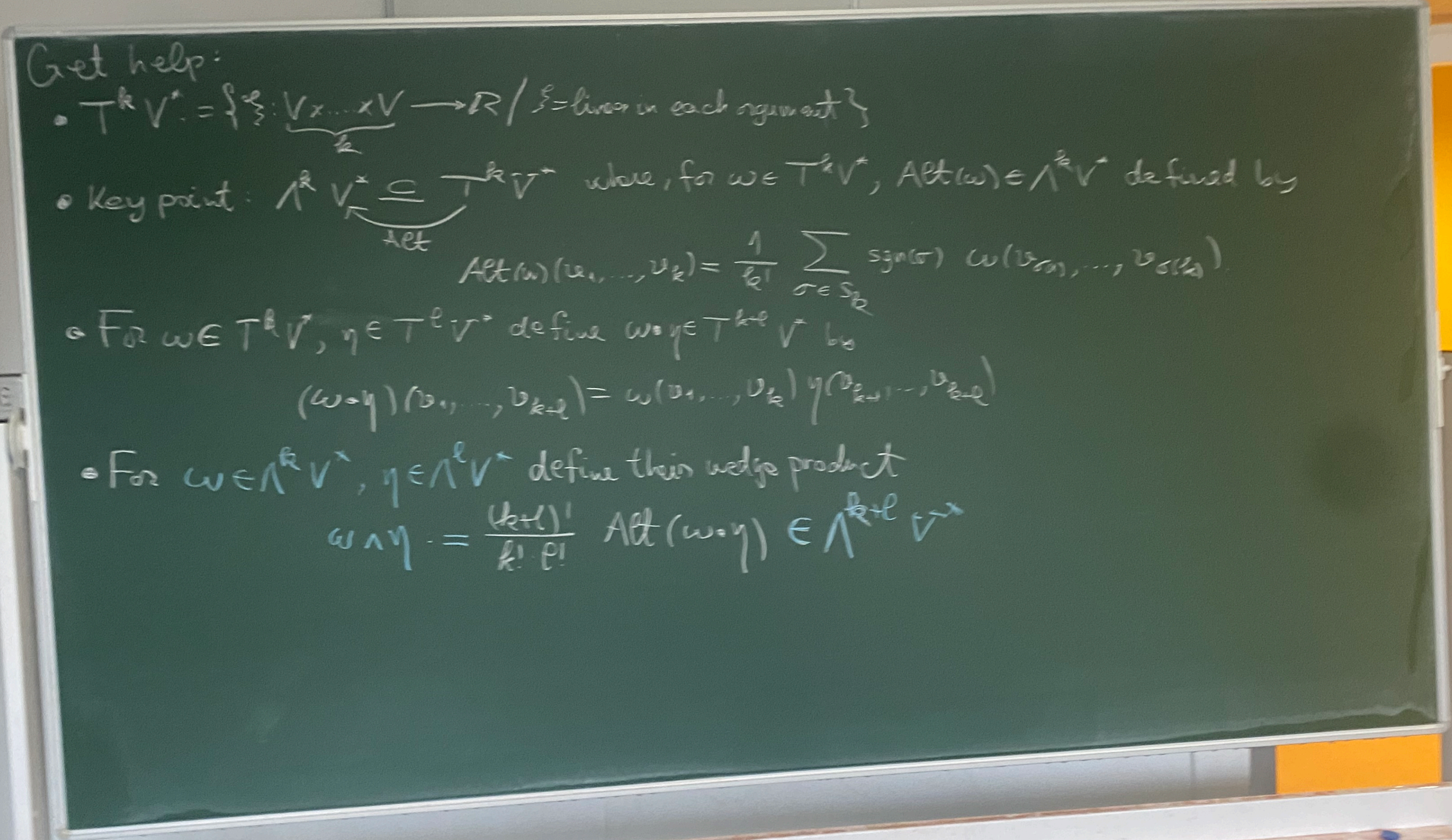
$$\omega_p = \sum_i F_i(p)\, (dx_i)_p \text{ "on } M\text{"} \quad (\text{for } p \in M)$$

and such expressions allways define a 1-form on M.

Vector space V $\rightsquigarrow$ I. New vector space: $V^* = \{ \xi : V \to \mathbb{R} \,/\, \xi = \text{linear} \}$

Basis $e_1, \dots, e_m$ of V $\rightsquigarrow$ II. Basis $e^1, \dots, e^m$ of V^* (dual basis)

$$e^i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$



Get help:

- $T^k V^* := \{\xi : \underbrace{V \times \dots \times V}_{k} \longrightarrow \mathbb{R} \,/\, \xi = \text{linear in each argument}\}$
- Key point: $\begin{tikzcd} \Lambda^k V^* \arrow[r, phantom, "\subseteq"] & T^k V^* \arrow[l, bend left, "\mathrm{Alt}"] \end{tikzcd}$ where, for $\omega \in T^k V^*$, $\mathrm{Alt}(\omega) \in \Lambda^k V^*$ defined by

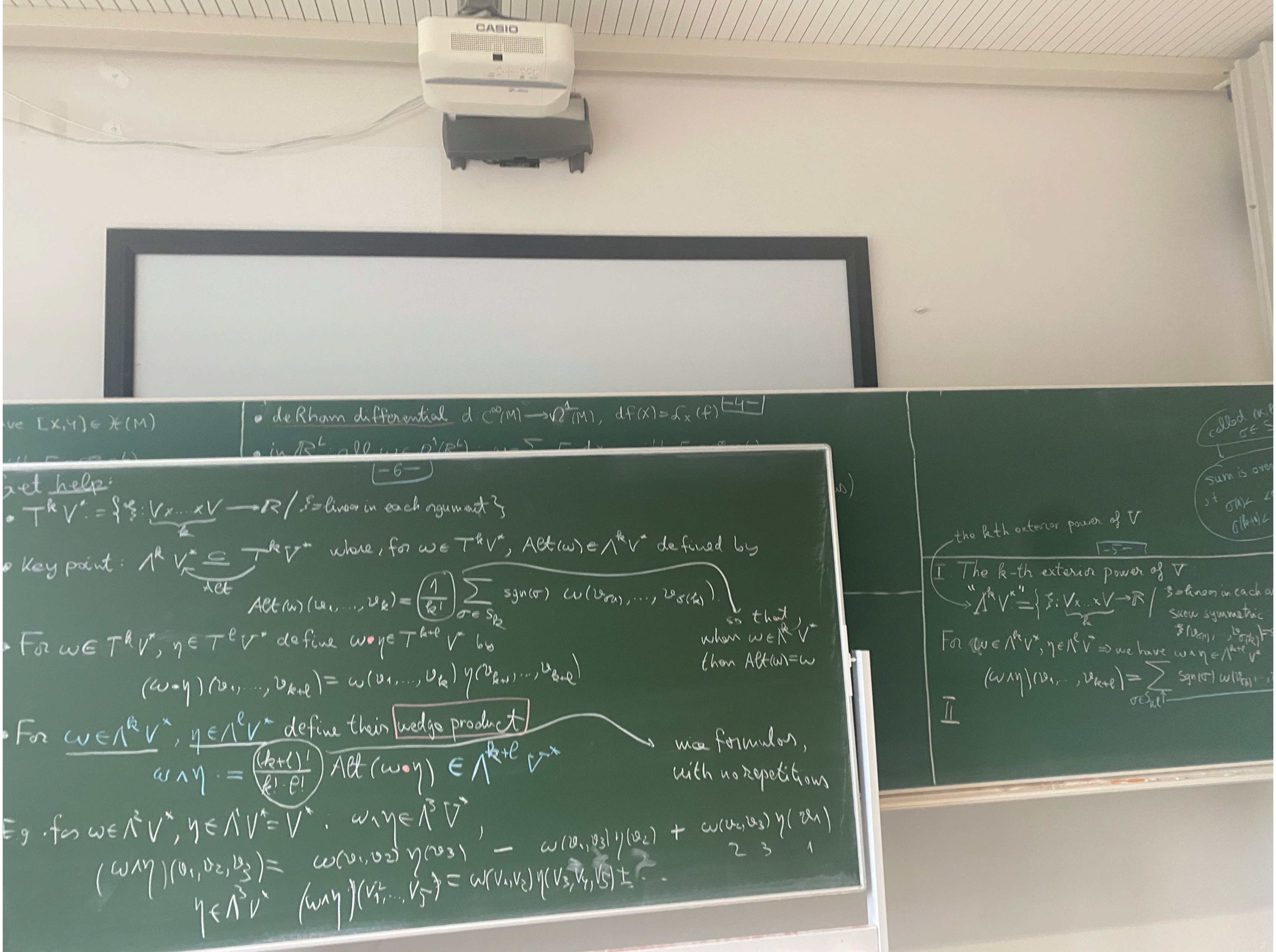
$$\mathrm{Alt}(\omega)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

- For $\omega \in T^k V^*$, $\eta \in T^\ell V^*$ define $\omega \circ \eta \in T^{k+\ell} V^*$ by

$$(\omega \circ \eta)(v_1, \dots, v_{k+\ell}) = \omega(v_1, \dots, v_k)\, \eta(v_{k+1}, \dots, v_{k+\ell})$$

- For $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^\ell V^*$ define their wedge product

$$\omega \wedge \eta := \frac{(k+\ell)!}{k! \cdot \ell!} \mathrm{Alt}(\omega \circ \eta) \in \Lambda^{k+\ell} V^*$$



ve $[X,Y] \in \mathfrak{X}(M)$

- de Rham differential $d: C^\infty(M) \to \Omega^1(M)$, $df(X) = \mathcal{L}_X(f)$ $\boxed{-4-}$

$\boxed{-6-}$

et help:

- $T^k V^* = \{ \xi : \underbrace{V \times \ldots \times V}_{k} \to \mathbb{R} \ / \ \xi = \text{linear in each argument} \}$
- Key point: $\Lambda^k V^* \underset{Alt}{\subseteq} T^k V^*$ where, for $\omega \in T^k V^*$, $Alt(\omega) \in \Lambda^k V^*$ defined by

$$Alt(\omega)(v_1, \ldots, v_k) = \boxed{\frac{1}{k!}} \sum_{\sigma \in S_k} sgn(\sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

so that, when $\omega \in \Lambda^k V^*$ then $Alt(\omega) = \omega$

- For $\omega \in T^k V^*$, $\eta \in T^\ell V^*$ define $\omega \bullet \eta \in T^{k+\ell} V^*$ by

$$(\omega \bullet \eta)(v_1, \ldots, v_{k+\ell}) = \omega(v_1, \ldots, v_k)\, \eta(v_{k+1}, \ldots, v_{k+\ell})$$

- For $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^\ell V^*$ define their $\boxed{\text{wedge product}}$

$$\omega \wedge \eta := \boxed{\frac{(k+\ell)!}{k! \cdot \ell!}} Alt(\omega \bullet \eta) \in \Lambda^{k+\ell} V^*$$

via formulas, with no repetitions

E.g. for $\omega \in \Lambda^2 V^*$, $\eta \in \Lambda^1 V^* = V^*$. $\omega \wedge \eta \in \Lambda^3 V^*$,

$$(\omega \wedge \eta)(v_1, v_2, v_3) = \omega(v_1, v_2)\, \eta(v_3) - \omega(v_1, v_3)\, \eta(v_2) + \underset{2\ \ 3}{\omega(v_2, v_3)}\, \underset{1}{\eta(v_1)}$$

$\eta \in \Lambda^3 V^*$ $(\omega \wedge \eta)(v_1, \ldots, v_5) = \omega(v_1, v_2)\, \eta(v_3, v_4, v_5) \pm \ldots$

the kth exterior power of V

$\boxed{-5-}$

I. The k-th exterior power of V

$$\text{“}\Lambda^k V^*\text{”} = \{ \xi : \underbrace{V \times \ldots \times V}_{k} \to \mathbb{R} \ / \ \xi = \text{linear in each a} \ldots$$

skew symmetric

$\xi(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \ldots$

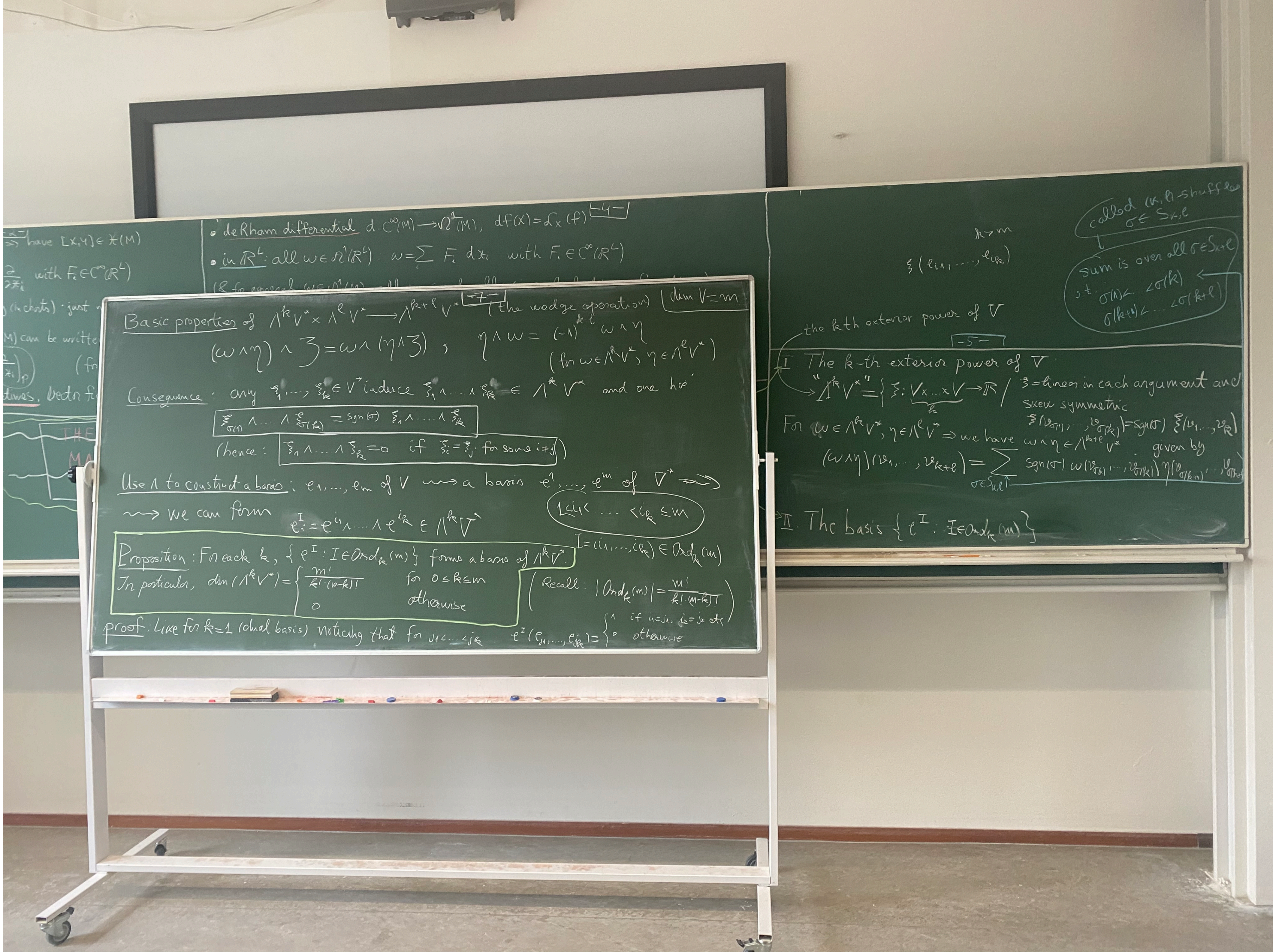
For $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^\ell V^*$ ⇒ we have $\omega \wedge \eta \in \Lambda^{k+\ell} V^*$

$$(\omega \wedge \eta)(v_1, \ldots, v_{k+\ell}) = \sum_{\sigma \in S_{k,\ell}} sgn(\sigma)\, \omega(v_{\sigma(1)}, \ldots$$

called (k, ℓ
$\sigma \in S$

sum is over
s.t. $\sigma(1) < \ldots <$
$\sigma(k+1) <$

II



⇒ have $[X,Y]\in\mathfrak{X}(M)$

$\frac{\partial}{\partial x_i}$ with $F_i\in C^\infty(\mathbb{R}^L)$

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-4-

- de Rham differential $d: C^\infty(M)\longrightarrow\Omega^1(M)$, $df(X)=\mathcal{L}_X(f)$
- in $\mathbb{R}^L$: all $\omega\in\Omega^1(\mathbb{R}^L)$: $\omega=\sum_i F_i\,dx_i$ with $F_i\in C^\infty(\mathbb{R}^L)$

(& for general $\omega\in\Omega^1(M)$

-7-

dim $V=m$

Basic properties of $\Lambda^k V^*\times\Lambda^\ell V^*\longrightarrow\Lambda^{k+\ell}V^*$ (the wedge operation)

$(\omega\wedge\eta)\wedge\xi=\omega\wedge(\eta\wedge\xi)$, $\eta\wedge\omega=(-1)^{k\cdot l}\,\omega\wedge\eta$

(for $\omega\in\Lambda^k V^*$, $\eta\in\Lambda^\ell V^*$)

Consequence: any $\xi_1,\dots,\xi_k\in V^*$ induce $\xi_1\wedge\dots\wedge\xi_k\in\Lambda^k V^*$ and one has

$$\boxed{\xi_{\sigma(1)}\wedge\dots\wedge\xi_{\sigma(k)}=\mathrm{sgn}(\sigma)\,\xi_1\wedge\dots\wedge\xi_k}$$

(hence: $\boxed{\xi_1\wedge\dots\wedge\xi_k=0 \text{ if } \xi_i=\xi_j \text{ for some } i\neq j}$)

Use $\wedge$ to construct a basis: $e_1,\dots,e_m$ of V $\rightsquigarrow$ a basis $e^1,\dots,e^m$ of V^*

$\rightsquigarrow$ we can form $e^I:=e^{i_1}\wedge\dots\wedge e^{i_k}\in\Lambda^k V^*$

$1\leq i_1<\dots<i_k\leq m$

$I=(i_1,\dots,i_k)\in\mathrm{Ord}_k(m)$

Proposition: For each k, $\{e^I: I\in\mathrm{Ord}_k(m)\}$ forms a basis of $\Lambda^k V^*$.
In particular, $\dim(\Lambda^k V^*)=\begin{cases}\frac{m!}{k!\cdot(m-k)!} & \text{for } 0\leq k\leq m\\ 0 & \text{otherwise}\end{cases}$

(Recall: $|\mathrm{Ord}_k(m)|=\frac{m!}{k!\cdot(m-k)!}$)

proof: Like for $k=1$ (dual basis) noticing that for $j_1<\dots<j_k$ $e^I(e_{j_1},\dots,e_{j_k})=\begin{cases}1 & \text{if } i_1=j_1,\ i_2=j_2 \text{ etc}\\ 0 & \text{otherwise}\end{cases}$

$k>m$

$\xi(e_{i_1},\dots,e_{i_k})$

called (k,l)-shuffles $\sigma\in S_{k,\ell}$

sum is over all $\sigma\in S_{k+\ell}$ s.t. $\sigma(1)<\dots<\sigma(k)$, $\sigma(k+1)<\dots<\sigma(k+\ell)$

the kth exterior power of V

-5-

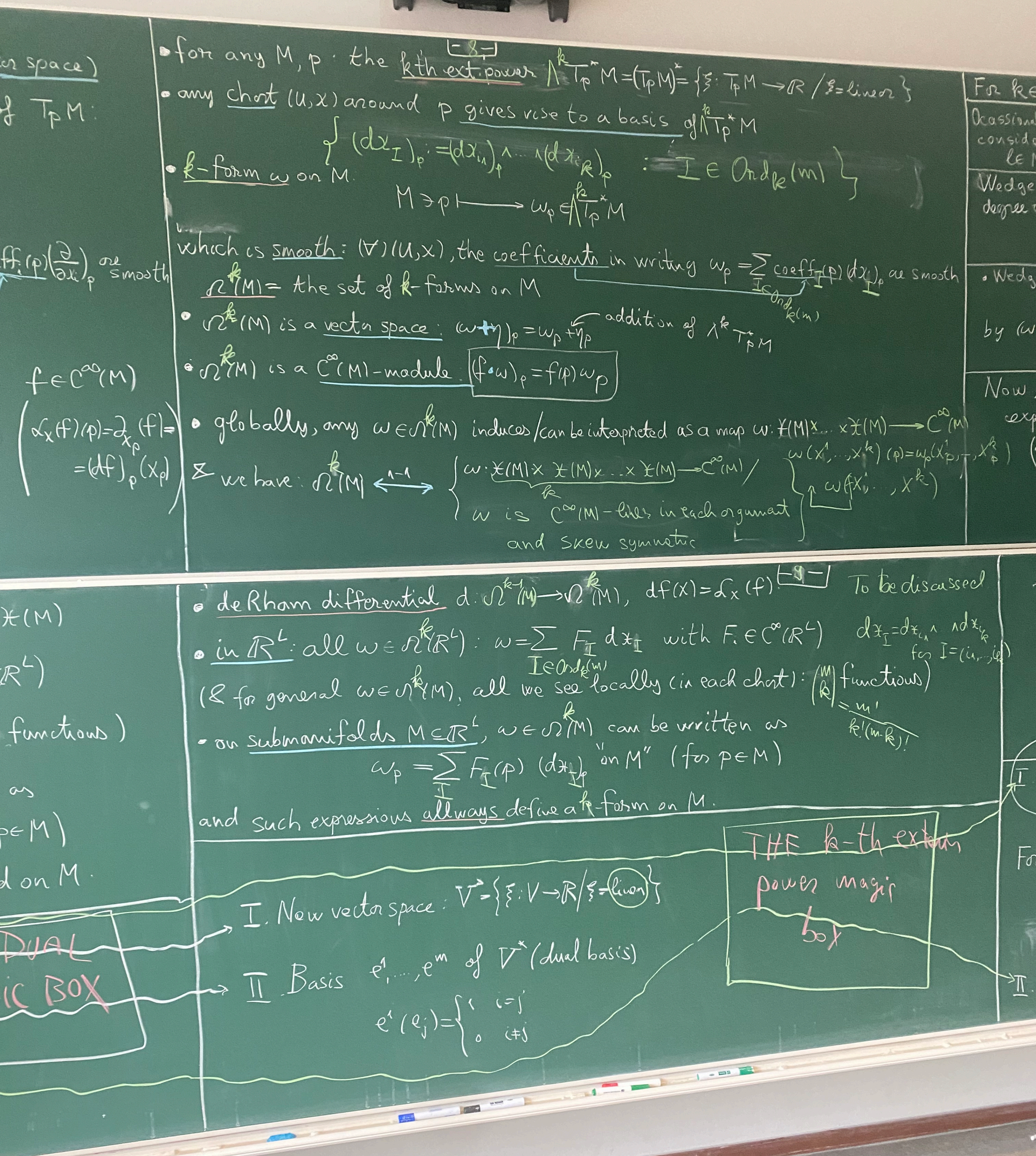
I. The k-th exterior power of V

"$\Lambda^k V^*$" $=\{\xi: \underbrace{V\times\dots\times V}_{k}\to\mathbb{R}$ / ξ = linear in each argument and skew symmetric $\xi(v_{\sigma(1)},\dots,v_{\sigma(k)})=\mathrm{sgn}(\sigma)\,\xi(v_1,\dots,v_k)$

For $\omega\in\Lambda^k V^*$, $\eta\in\Lambda^\ell V^*$ ⇒ we have $\omega\wedge\eta\in\Lambda^{k+\ell}V^*$ given by

$$(\omega\wedge\eta)(v_1,\dots,v_{k+\ell})=\sum_{\sigma\in S_{k,\ell}}\mathrm{sgn}(\sigma)\,\omega(v_{\sigma(1)},\dots,v_{\sigma(k)})\,\eta(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)})$$

II. The basis $\{e^I: I\in\mathrm{Ord}_k(m)\}$



r space)

f T_pM:

ff$_i(p)\left(\frac{\partial}{\partial x_i}\right)_p$ are smooth

$f\in C^\infty(M)$

$\left(\mathcal{L}_X(f)(p)=\partial_{X_p}(f)=\right.$
$\left.=(df)_p(X_p)\right)$

- for any M, p: the kth ext. power $\overset{\xi}{\Lambda^k T_p^*M}=(T_pM)^*=\{\xi: T_pM\to\mathbb{R}\,/\,\xi=\text{linear}\}$
- any chart (U,x) around p gives rise to a basis of $\Lambda^k T_p^*M$
  $$\left\{(dx_I)_p := (dx_{i_1})_p\wedge\dots\wedge(dx_{i_k})_p \;:\; I\in \mathrm{Ord}_k(m)\right\}$$
- k-form ω on M:
  $$M\ni p\longmapsto \omega_p\in\Lambda^k T_p^*M$$

which is smooth: $(\forall)(U,x)$, the coefficients in writing $\omega_p=\sum_{I\in\mathrm{Ord}_k(m)}\mathrm{coeff}_I(p)(dx_I)_p$ are smooth

$\Omega^k(M)=$ the set of k-forms on M

- $\Omega^k(M)$ is a vector space: $(\omega+\eta)_p=\omega_p+\eta_p$ ← addition of $\Lambda^k T_p^*M$
- $\Omega^k(M)$ is a $C^\infty(M)$-module: $\boxed{(f\cdot\omega)_p=f(p)\omega_p}$
- globally, any $\omega\in\Omega^k(M)$ induces/can be interpreted as a map $\omega:\mathfrak{X}(M)\times\dots\times\mathfrak{X}(M)\longrightarrow C^\infty(M)$

& we have: $\Omega^k(M)\xleftrightarrow{1-1}\left\{\omega:\underbrace{\mathfrak{X}(M)\times\mathfrak{X}(M)\times\dots\times\mathfrak{X}(M)}_{k}\to C^\infty(M)\,/\,\begin{matrix}\omega\text{ is }C^\infty(M)\text{-lines in each argument}\\ \text{and skew symmetric}\end{matrix}\right\}$

$\omega(X^1,\dots,X^k)(p)=\omega_p(X^1_p,\dots,X^k_p)$

$\omega(X^1,\dots,X^k)$

For k…

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$\ell\in$…

Wedge…
degree…

- Wedg…

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Now,…
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$\mathfrak{X}(M)$

$\mathbb{R}^L)$

functions)

as

$p\in M)$

d on M.

- deRham differential $d:\Omega^{k-1}(M)\longrightarrow\Omega^k(M)$, $df(X)=\mathcal{L}_X(f)$ $\boxed{-9-}$ To be discussed
- in $\mathbb{R}^L$: all $\omega\in\Omega^k(\mathbb{R}^L)$: $\omega=\sum_{I\in\mathrm{Ord}_k(m)}F_I\,dx_I$ with $F_i\in C^\infty(\mathbb{R}^L)$ $\quad dx_I=dx_{i_1}\wedge\dots\wedge dx_{i_k}$ for $I=(i_1,\dots,i_k)$

(& for general $\omega\in\Omega^k(M)$, all we see locally (in each chart): $\binom{m}{k}$ functions) $\quad \binom{m}{k}=\frac{m!}{k!(m-k)!}$

- on submanifolds $M\subseteq\mathbb{R}^L$, $\omega\in\Omega^k(M)$ can be written as
  $$\omega_p=\sum_I F_I(p)\,(dx_I)_p \quad \text{"on } M\text{"} \quad (\text{for } p\in M)$$

and such expressions allways define a k-form on M.

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THE k-th exterior
power magic
box

I. New vector space: $V^*=\{\xi:V\to\mathbb{R}\,/\,\xi=\text{linear}\}$

II. Basis $e^1,\dots,e^m$ of V^* (dual basis)

$$e^i(e_j)=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

I

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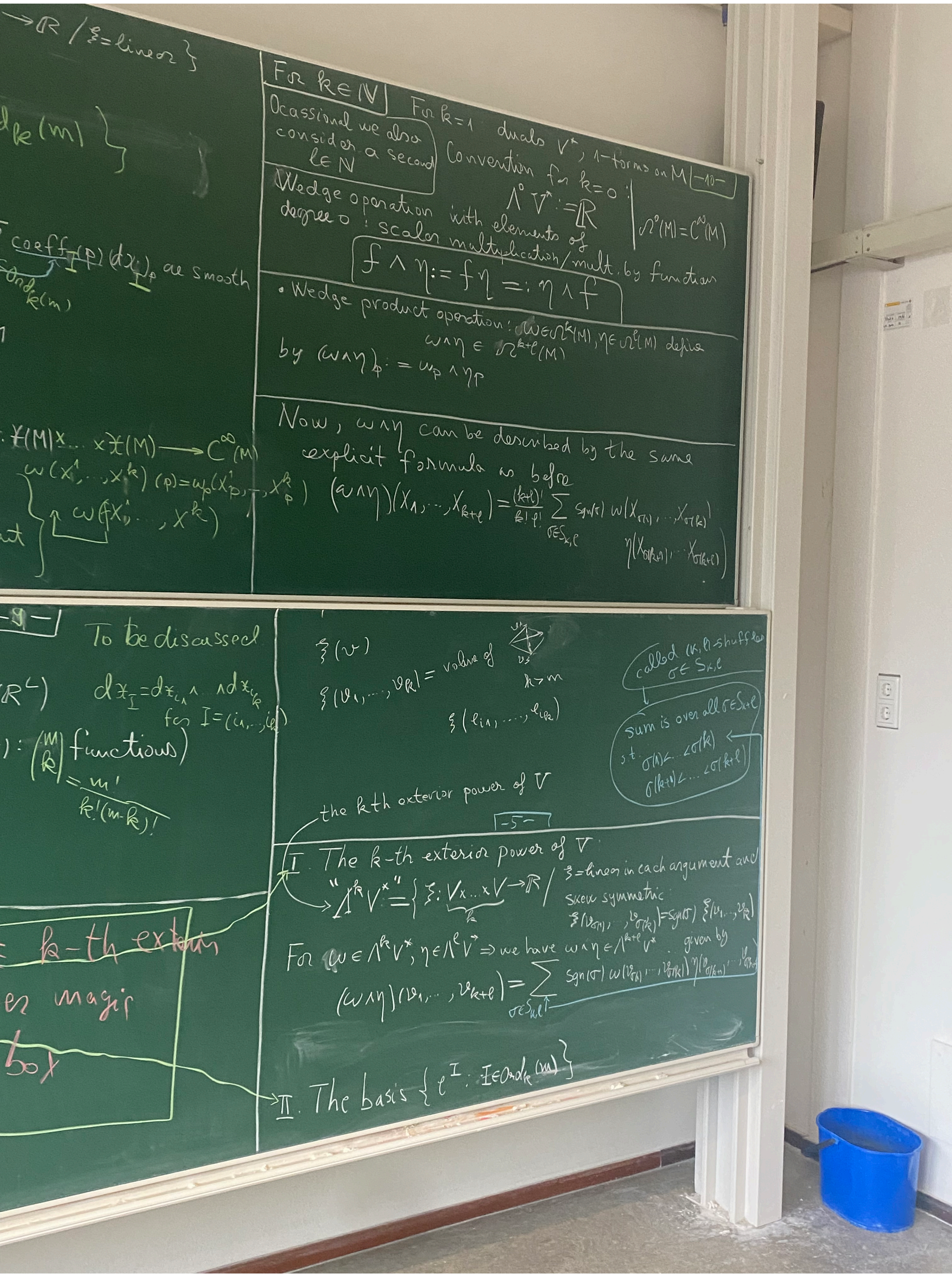
For



$\to \mathbb{R}$ / ξ = linear $\}$

$d_k(m)$ $\}$

coeff$_I(p)$ $(dx_I)_p$ are smooth

$\in \mathrm{Ord}_k(m)$

$\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$

$\omega(X^1, \ldots, X^k)(p) = \omega_p(X^1_p, \ldots, X^k_p)$

$\omega(fX^1, \ldots, X^k)$

nt

For $k \in \mathbb{N}$

Ocassional we also consider a second $\ell \in \mathbb{N}$

For $k=1$ duals V^*, 1-forms on M —10—

Convention for $k=0$: $\Lambda^0 V^* := \mathbb{R}$ | $\Omega^0(M) = C^\infty(M)$

Wedge operation with elements of degree 0: scalar multyplication / mult. by functions

$$\boxed{f \wedge \eta := f\eta =: \eta \wedge f}$$

- Wedge product operation: $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ define $\omega \wedge \eta \in \Omega^{k+\ell}(M)$ by $(\omega \wedge \eta)_p := \omega_p \wedge \eta_p$

Now, $\omega \wedge \eta$ can be described by the same explicit formula as before

$$(\omega \wedge \eta)(X_1, \ldots, X_{k+\ell}) = \frac{(k+\ell)!}{k!\,\ell!} \sum_{\sigma \in S_{k,\ell}} \mathrm{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)})$$

—9—

To be discussed

$\mathbb{R}^L)$

$dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ for $I = (i_1, \ldots, i_k)$

$): \binom{m}{k}$ functions$)$ $= \frac{m!}{k!(m-k)!}$

k-th exterior

er magic

box

$\xi(v)$

$\xi(v_1, \ldots, v_k)$ = volume of

$k > m$

$\xi(e_{i_1}, \ldots, e_{i_k})$

called (k,ℓ)-shuffles $\sigma \in S_{k,\ell}$

sum is over all $\sigma \in S_{k+\ell}$ s.t. $\sigma(1) < \ldots < \sigma(k)$, $\sigma(k+1) < \ldots < \sigma(k+\ell)$

the kth exterior power of V

—5—

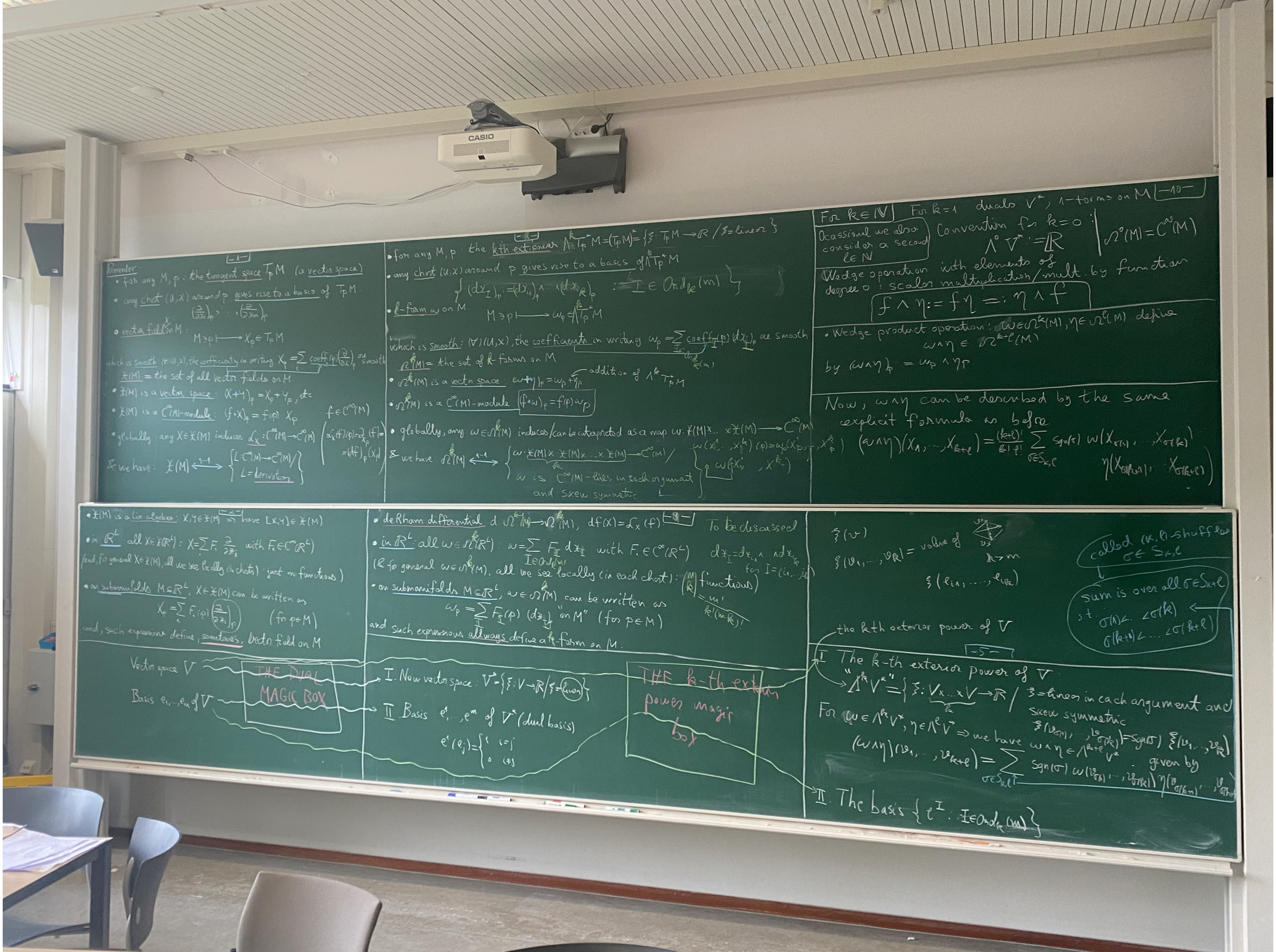
$\underline{I}$. The k-th exterior power of V:

$$\text{"}\Lambda^k V^*\text{"} = \{\xi : \underbrace{V \times \ldots \times V}_{k} \to \mathbb{R} \,/\, \xi = \text{linear in each argument and skew symmetric: } \xi(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma)\, \xi(v_1, \ldots, v_k)\}$$

For $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^\ell V^* \Rightarrow$ we have $\omega \wedge \eta \in \Lambda^{k+\ell} V^*$ given by

$$(\omega \wedge \eta)(v_1, \ldots, v_{k+\ell}) = \sum_{\sigma \in S_{k,\ell}} \mathrm{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\, \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

$\underline{II}$. The basis $\{e^I : I \in \mathrm{Ord}_k(m)\}$



—1—

Remember:

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which is smooth: $(\forall)(U,x)$, the coefficients in writing $X_p = \sum_i \text{coeff}_i(p)\left(\frac{\partial}{\partial x_i}\right)_p$ are smooth

$\mathfrak{X}(M)$ = the set of all vector fields on M.

- $\mathfrak{X}(M)$ is a vector space: $(X+Y)_p = X_p + Y_p$, etc
- $\mathfrak{X}(M)$ is a $C^\infty(M)$-module: $(f \cdot X)_p = f(p)\, X_p$ $\quad f \in C^\infty(M)$
- globally any $X \in \mathfrak{X}(M)$ induces $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$ $\left(\mathcal{L}_X(f)(p) = \partial_{X_p}(f) = (df)_p(X_p)\right)$

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—2—

- $\mathfrak{X}(M)$ is a Lie algebra: $X, Y \in \mathfrak{X}(M) \Rightarrow$ have $[X,Y] \in \mathfrak{X}(M)$
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(and, for general $X \in \mathfrak{X}(M)$, all we see locally (in charts): just m functions)

- on submanifolds $M \subseteq \mathbb{R}^L$, $X \in \mathfrak{X}(M)$ can be written as $X_p = \sum_i F_i(p) \boxed{\left(\frac{\partial}{\partial x_i}\right)_p}$ (for $p \in M$)

and, such expressions define, sometimes, vector field on M

Vector space V

Basis $e_1, \dots, e_m$ of V

THE DUAL MAGIC BOX

—8—

- for any M, p: the kth ext. power $\Lambda^k T_p^* M = (T_pM)^* = \{\xi : T_pM \to \mathbb{R} \,/\, \xi = \text{linear}\}$
- any chart (U,x) around p gives rise to a basis of $\Lambda^k T_p^* M$ $\left\{ (dx_I)_p := (dx_{i_1})_p \wedge \dots \wedge (dx_{i_k})_p : I \in \text{Ord}_k(m) \right\}$
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& we have $\Omega^k(M) \xleftrightarrow{1-1} \left\{ \omega : \underbrace{\mathfrak{X}(M) \times \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k} \to C^\infty(M) \,/\, \omega \text{ is } C^\infty(M)\text{-lin. in each argument and skew symmetric} \right\}$ $\quad \omega(fX^1, \dots, X^k)$

—9—

- deRham differential $d : \Omega^{k-1}(M) \to \Omega^k(M)$, $df(X) = \mathcal{L}_X(f)$ To be discussed
- in $\mathbb{R}^L$: all $\omega \in \Omega^k(\mathbb{R}^L)$: $\omega = \sum_{I \in \text{Ord}_k(m)} F_I\, dx_I$ with $F_I \in C^\infty(\mathbb{R}^L)$ $\quad dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$ for $I = (i_1, \dots, i_k)$

(& for general $\omega \in \Omega^k(M)$, all we see locally (in each chart): $\binom{m}{k}$ functions) $\quad = \frac{m!}{k!(m-k)!}$

- on submanifolds $M \subseteq \mathbb{R}^L$, $\omega \in \Omega^k(M)$ can be written as $\omega_p = \sum_I F_I(p)\,(dx_I)_p$ "on M" (for $p \in M$)

and such expressions allways define a k-form on M.

I. New vector space: $V^* = \{\xi : V \to \mathbb{R} \,/\, \xi = \boxed{\text{linear}}\}$

II. Basis $e^1, \dots, e^m$ of V^* (dual basis)

$$e^i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

THE k-th exterior power magic box

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For $k \in \mathbb{N}$ $\quad$ For $k=1$ duals V^*, 1-forms on M

Ocassional we also consider a second $\ell \in \mathbb{N}$

Convention for $k=0$: $\Lambda^0 V^* := \mathbb{R}$ $\quad \Omega^0(M) = C^\infty(M)$

Wedge operation with elements of degree 0: scalar multiplication/mult. by function

$$\boxed{f \wedge \eta := f\eta =: \eta \wedge f}$$

- Wedge product operation: $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ define $\omega \wedge \eta \in \Omega^{k+\ell}(M)$

by $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$

Now, $\omega \wedge \eta$ can be described by the same explicit formula as before

$$(\omega \wedge \eta)(X_1, \dots, X_{k+\ell}) = \frac{(k+\ell)!}{k!\,\ell!} \sum_{\sigma \in S_{k,\ell}} \text{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})$$

$\xi(v)$

$\xi(v_1, \dots, v_k)$ = volume of $v_1 \dots v_3$

$k > m$

$\xi(e_{i_1}, \dots, e_{i_k})$

called (k,ℓ)-shuffles $\sigma \in S_{k,\ell}$

sum is over all $\sigma \in S_{k+\ell}$ s.t. $\sigma(1) < \dots < \sigma(k)$, $\sigma(k+1) < \dots < \sigma(k+\ell)$

the kth exterior power of V

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I The k-th exterior power of V

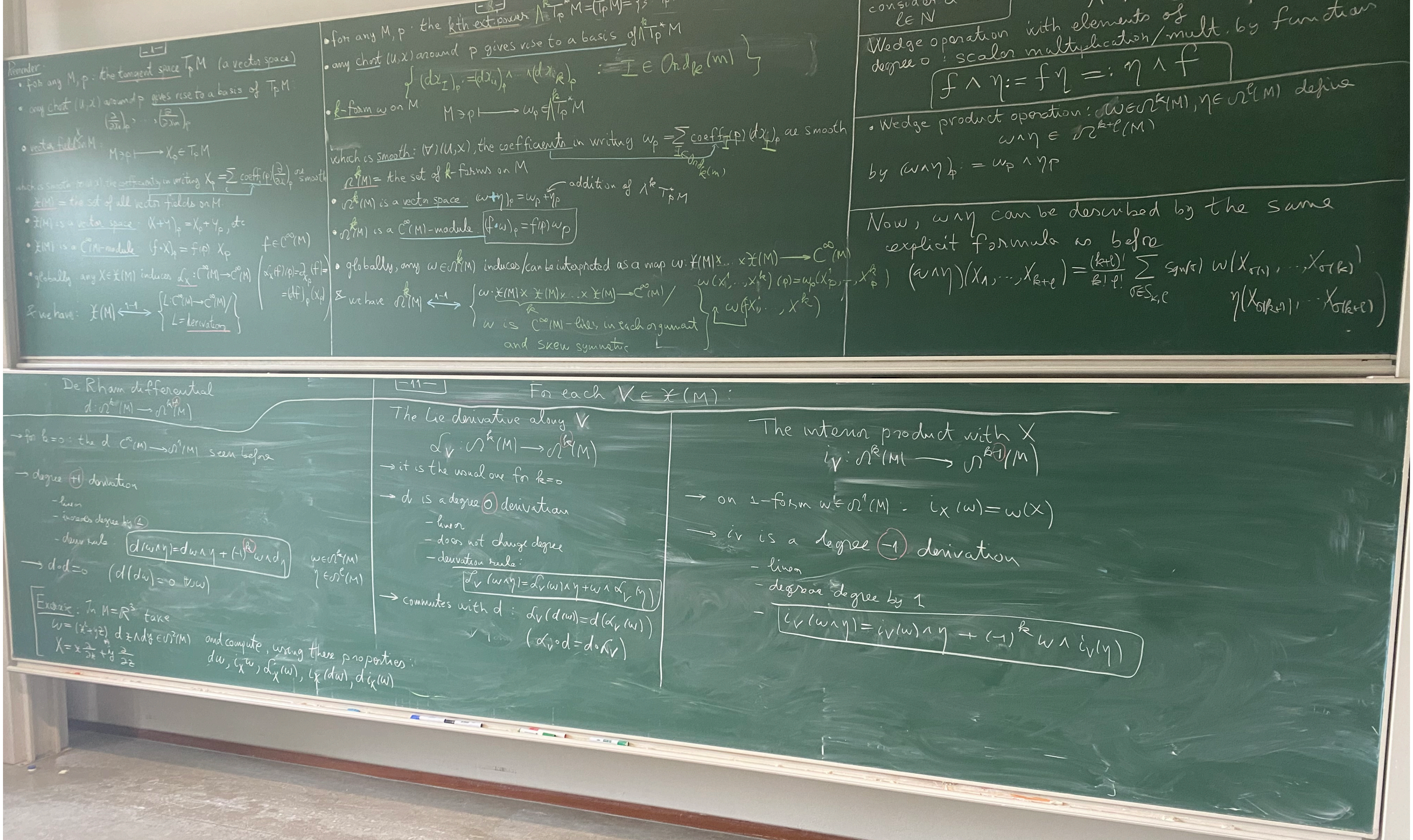
"$\Lambda^k V^*$" $= \{ \xi : \underbrace{V \times \dots \times V}_{k} \to \mathbb{R} \,/\, \xi = \text{linear in each argument and skew symmetric} \}$

$\xi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sgn}(\sigma)\, \xi(v_1, \dots, v_k)$

For $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^\ell V^* \Rightarrow$ we have $\omega \wedge \eta \in \Lambda^{k+\ell} V^*$ given by

$$(\omega \wedge \eta)(v_1, \dots, v_{k+\ell}) = \sum_{\sigma \in S_{k,\ell}} \text{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})$$

II. The basis $\{ e^I : I \in \text{Ord}_k(m) \}$



—1—

Reminder:

- for any M, p: the tangent space T_pM (a vector space)
- any chart (U, x) around p gives rise to a basis of T_pM: $\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$
- vector field X on M: $M \ni p \longmapsto X_p \in T_pM$

which is smooth: $(\forall)(U,x)$, the coefficients in writing $X_p = \sum_i \text{coeff}_i(p)\left(\frac{\partial}{\partial x_i}\right)_p$ are smooth

$\mathfrak{X}(M)$ = the set of all vector fields on M.

- $\mathfrak{X}(M)$ is a vector space: $(X+Y)_p = X_p + Y_p$, etc
- $\mathfrak{X}(M)$ is a $C^\infty(M)$-module: $(f \cdot X)_p = f(p)\, X_p$ $\quad f \in C^\infty(M)$
- globally any $X \in \mathfrak{X}(M)$ induces $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$ $\quad \left(\begin{aligned}\mathcal{L}_X(f)(p) &= \partial_{X_p}(f) = \\ &= (df)_p(X_p)\end{aligned}\right)$

& we have: $\mathfrak{X}(M) \xleftrightarrow{1-1} \left\{\begin{aligned} L &: C^\infty(M) \to C^\infty(M) / \\ L &= \text{derivation} \end{aligned}\right\}$

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- for any M, p: the kth ext. power $\Lambda^k T_p^*M = (T_pM) = \{$
- any chart (U, x) around p gives rise to a basis of $\Lambda^k T_p^* M$

$$\left\{ (dx_I)_p := (dx_{i_1})_p \wedge \dots \wedge (dx_{i_k})_p \;:\; I \in \text{Ord}_k(m) \right\}$$

- k-form ω on M: $M \ni p \longmapsto \omega_p \in \Lambda^k T_p^* M$

which is smooth: $(\forall)(U,x)$, the coefficients in writing $\omega_p = \sum_{I \in \text{Ord}_k(m)} \text{coeff}_I(p)\, (dx_I)_p$ are smooth

$\Omega^k(M)$ = the set of k-forms on M

- $\Omega^k(M)$ is a vector space: $(\omega + \eta)_p = \omega_p + \eta_p$ ← addition of $\Lambda^k T_p^* M$
- $\Omega^k(M)$ is a $C^\infty(M)$-module: $\boxed{(f \cdot \omega)_p = f(p)\, \omega_p}$
- globally, any $\omega \in \Omega^k(M)$ induces/can be interpreted as a map $\omega : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$

$\omega(X^1, \dots, X^k)(p) = \omega_p(X^1_p, \dots, X^k_p)$

& we have $\Omega^k(M) \xleftrightarrow{1-1} \left\{\begin{aligned} &\omega : \underbrace{\mathfrak{X}(M) \times \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k} \to C^\infty(M) / \\ &\omega \text{ is } C^\infty(M)\text{-lin. in each argument} \\ &\text{and skew symmetric} \end{aligned}\right\}$ $\quad \omega(fX^1, \dots, X^k)$

consider ...

$\ell \in \mathbb{N}$

Wedge operation with elements of degree 0: scalar multiplication/mult. by functions

$$\boxed{f \wedge \eta := f\eta =: \eta \wedge f}$$

- Wedge product operation: $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$ define

$$\omega \wedge \eta \in \Omega^{k+\ell}(M)$$

by $(\omega \wedge \eta)_p := \omega_p \wedge \eta_p$

Now, $\omega \wedge \eta$ can be described by the same explicit formula as before

$$(\omega \wedge \eta)(X_1, \dots, X_{k+\ell}) = \frac{(k+\ell)!}{k!\,\ell!} \sum_{\sigma \in S_{k,\ell}} \text{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})$$

De Rham differential

$$d : \Omega^k(M) \to \Omega^{k+①}(M)$$

→ for $k=0$: the d $C^\infty(M) \to \Omega^1(M)$ seen before

→ degree (+1) derivation

- linear
- increases degree by ①
- deriv. rule $\boxed{d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{Ⓚ}\omega \wedge d\eta}$ $\quad \omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$

→ $d \circ d = 0$ $\quad (d(d\omega) = 0 \;\; \forall \omega)$

Exercise: In $M = \mathbb{R}^3$ take

$\omega = (x^2 + yz)\, dz \wedge dy \in \Omega^2(M)$

$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$

and compute, using these properties:

$d\omega$, $i_X\omega$, $\mathcal{L}_X(\omega)$, $i_X(d\omega)$, $d\, i_X(\omega)$

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For each $V \in \mathfrak{X}(M)$:

The Lie derivative along V

$$\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega^{Ⓚ}(M)$$

→ it is the usual one for $k=0$

→ d is a degree ⓪ derivation

- linear
- does not change degree
- derivation rule:

$$\boxed{\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge \mathcal{L}_V(\eta)}$$

→ commutes with d: $\mathcal{L}_V(d(\omega)) = d(\mathcal{L}_V(\omega))$

$(\mathcal{L}_V \circ d = d \circ \mathcal{L}_V)$

The interior product with X

$$i_V : \Omega^k(M) \longrightarrow \Omega^{k-①}(M)$$

→ on 1-form $\omega \in \Omega^1(M)$: $i_X(\omega) = \omega(X)$

→ i_V is a degree (−1) derivation

- linear
- decrease degree by 1
- $\boxed{i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)}$

n $k=0$: the $d : C^\infty(M) \longrightarrow \Omega^1(M)$ seen before

→ degree (+1) derivation

- linear
- increases degree by ①
- deriv rule : $\boxed{d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^{k}\,\omega\wedge d\eta}$

$\omega\in\Omega^k(M)$

$\eta\in\Omega^\ell(M)$

→ $d\circ d = 0$ $\quad (d(d\omega) = 0 \ \forall\omega)$

Exercise : In $M=\mathbb{R}^3$ take

$\omega = (x^2+yz)\cdot dz\wedge dy \in \Omega^2(M)$ and compute, using these properties:

$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$

$d\omega$, $i_X\omega$, $\mathcal{L}_X(\omega)$, $i_X(d\omega)$, $d\,i_X(\omega)$

→ it

→ d

→ commu