-1-

Vector fields $M \ni p \longmapsto X_p \in T_pM$ $\mathfrak{X}(M)$

→ locally $X = \sum_{i=1}^{m} F_i \frac{\partial}{\partial x_i}$ on U ((U,x) = chart)

→ globally : $\mathcal{L}_X : C^\infty(M) \longrightarrow C^\infty(M)$ derivation

→ operations : $X+Y,\ f\cdot X,\ [X,Y]$ $\mathcal{L}_{[X,Y]}(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))$

→ flow of X : $\varphi_X = \varphi_X^t(p)$ $(t \in \mathbb{R}, p \in M)$ where

$\underbrace{(\forall)\, p \in M}$: $\underset{\mathbb{R}}{t} \longmapsto \underset{M}{\varphi_X^t(p)}$ (maximal) integral curve of X stating at p i.e. $\begin{cases} \frac{d}{dt} \varphi_X^t(p) = X_{\varphi_X^t(p)} \\ \varphi_X^0(p) = p \end{cases}$

If X = complete ⇒ $\underset{(\forall)t}{\varphi_X^t} : M \to M$ DIFFEOMORPHISMS

$\Omega^k(M)$

Convention: $\Omega^0(M) = C^\infty(M)$

-2-

Differential forms ω : $M \ni p \longmapsto \omega_p \in \Lambda^k T_p^*M$ (hence one can apply ω_p to k tangent vectors $\omega_p(v_p^1, \dots, v_p^k) \in \mathbb{R}$)

→ locally $\omega = \sum_{I \in \mathrm{Ind}_k(m)} F_I \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$ $(I = (i_1 < \dots < i_k))$

→ globally : $\omega : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{k} \longrightarrow C^\infty(M)$ $\begin{cases} \text{skew symmetric} \\ C^\infty(M)\text{-linear in each argument} \end{cases}$

→ operations : $\omega + \eta,\ f\cdot\omega, \dots$ $d : \Omega^0(M) \longrightarrow \Omega^1(M)$ $df(X) = \mathcal{L}_X(f)$

→ new : wedge product of $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ new form $\omega \wedge \eta \in \Omega^{k+l}(M)$:

$(\omega \wedge \eta)(X^1, \dots, X^{k+l}) = \sum_{\sigma \in S_{k,l}} \mathrm{sgn}(\sigma)\, \omega(X^{\sigma(1)}, \dots, X^{\sigma(k)})\, \eta(X^{\sigma(k+1)}, \dots, X^{\sigma(k+l)})$

& with basic properties $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$, $\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$

$\omega \wedge \omega = 0$ if $\omega \in \Omega^k(M)$, k = odd

→ forms are "$\boxed{\text{VERY}}$ NATURAL" : for ANY smooth map $F : M \longrightarrow N$ and $\omega \in \Omega^k(N)$

⇒ can talk about the pull-back of ω by F : $F^*(\omega)_p(X_p^1, \dots, X_p^k) = \omega_{F(p)}\big((dF)_p(X_p^1), \dots\big)$

$\boxed{F^*(\omega) \in \Omega^k(M)}$

-3-

"Natural" types of objects ξ: the type of objects [that can] be pulled-back by diffeomorphisms

Ex ① smooth function $\xi = f \in C^\infty(M)$

$F : M \longrightarrow N$: $F^*(g) = g \circ F$, $g \in C^\infty(M)$

② vector fields $X \in \mathfrak{X}(M)$

$\left.\begin{matrix} F : M \longrightarrow N \text{ diffeom} \\ X \in \mathfrak{X}(N) \end{matrix}\right\}$ $F^*(X)_p := (dF^{-1}) \dots$

③ diff. forms $\xi = \omega \in \Omega^k(M)$

For any natural type of o[bjects] one defines :

$\boxed{\mathcal{L}_V(\xi) = \text{the variation of } \dots \text{ along the flow of } \dots}$

-4-

Ex1. $\xi = f \in C^\infty(M)$

$\mathcal{L}_V(f) = \frac{d}{dt}\underbrace{(\varphi_V^t)^*(f)}_{f \circ \varphi_V^t} \Rightarrow \mathcal{L}_V(f)_p = \frac{d}{dt}\Big|_{t=0} f \dots$

$= (df)_p (\dots$

Ex2 : $\xi = X \in \mathfrak{X}(M)$

$\mathcal{L}_V(X) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(X) \Rightarrow \mathcal{L}_V(X)_p = \frac{d}{dt}\Big|_{t=0} (d\varphi_V^{-t}) \dots$

.... compute ...

$\Rightarrow \mathcal{L}_V(X) = [V, X]$

Ex3 :

-5-

De Rham differential

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ (De Rham)

Prop (Cor. 4.47 in notes) : On any manifold there is one and only one way to construct maps (DeRham) (for all k) such that :

(DeRham-0) d is $\mathbb{R}$-linear and, for $k=0$, $d : C^\infty(M) \longrightarrow \Omega^1(M)$ the usual

(DeRham-1) d satisfies derivation rule

$\boxed{d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta}$ $(\forall)\ \omega \in \Omega^k(M), \eta \in \Omega^l(M)$

(DeRham-2) $\boxed{d \circ d = 0}$.

Also useful :

Explicit formulas:

-6-

Lie derivatives w.r.t. $V \in \mathfrak{X}(M)$

$\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega^k(M)$ (*)

Prop (4.55 in notes) : For any V, ∃! way to define (*) such that :

(0) $\mathcal{L}_V$ is linear and, for $k=0$, $\mathcal{L}_V : C^\infty(M) \to C^\infty(M)$ the usual

(1) $\mathcal{L}_V$ satisfies derivation rule

$\boxed{\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge \mathcal{L}_V(\eta)}$

(2) $\boxed{\mathcal{L}_V \circ d = d \circ \mathcal{L}_V}$

Explicit formula :

-7-

Interior products w.r.t. $V \in \mathfrak{X}(M)$

$i_V : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$

Prop 4.57 : For any V, ∃! ... such that :

(0) i_V is linear and, for $k=1$, $i_V(\omega) = \omega(V)$

(1) i_V satisfies the derivation rule

$\boxed{i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)}$

Explicit formula :

$\boxed{d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k\,\omega\wedge d\eta}$ $(\forall)\,\omega\in\Omega^k(M),\ \eta\in\Omega^\ell(M)$

(De Rham-2) $\boxed{d\circ d = 0}$

Also useful:

Explicit formulas:

(2) $\boxed{\mathcal{L}_V\circ d = d\circ\mathcal{L}_V}$

Explicit formula:

Expl

-1-

Vector fields $M\ni p\longmapsto X_p\in T_pM$ $\quad\mathfrak{X}(M)$

$\rightarrow$ locally $X=\sum_{i=1}^{m} F_i\,\frac{\partial}{\partial x_i}$ on U $\quad((U,x)=\text{chart})$

$\rightarrow$ globally: $\mathcal{L}_X: C^\infty(M)\longrightarrow C^\infty(M)$ derivation

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If X = complete $\Rightarrow$ $\underset{(\forall)t}{\varphi_X^t}: M\to M$ DIFFEOMORPHISMS.

$\Omega^k(M)$

Convention: $\Omega^0(M)=C^\infty(M)$

-2-

Differential forms ω: $M\ni p\longmapsto \omega_p\in\Lambda^k T_p^*M$

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$\rightarrow$ globally: $\omega:\underbrace{\mathfrak{X}(M)\times\ldots\times\mathfrak{X}(M)}_{k}\longrightarrow C^\infty(M)$ $\begin{cases}\text{skew symmetric}\\ C^\infty(M)\text{-linear in e}\end{cases}$

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$(\omega\wedge\eta)(X^1,\ldots,X^{k+\ell})=\sum_{\sigma\in S_{k,\ell}}\mathrm{sgn}(\sigma)\,\omega(X^{\sigma(1)},\ldots,X^{\sigma(k)})\,\eta(X^{\sigma(k+1)},$

& with basic properties

$(\omega\wedge\eta)\wedge\zeta=\omega\wedge(\eta\wedge\zeta)$, $\quad\eta\wedge\omega=(-1)^{k\ell}\omega\wedge\eta$

$\rightarrow$ forms are "$\boxed{\text{VERY}}$ NATURAL": for ANY smooth map $F:M\rightarrow$

$\Rightarrow$ can talk about the pull-back of ω by F:

$\boxed{F^*(\omega)\in\Omega^k(M)}$: $\quad F^*(\omega)_p(X_p^1,\ldots,X_p^k)=\omega_{F(p)}((dF$

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Vector fields $M \ni p \longmapsto X_p \in T_pM$

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If X = complete ⟹ $\varphi_X^t : M \to M$ DIFFEOMORPHISMS $(\forall)\, t$

$\Omega^k(M)$

Convention: $\Omega^0(M) = C^\infty(M)$

Useful:

$F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$

$F^*(d\omega) = d(F^*\omega)$

Differe

→ loca

→ glob

→ ope

→ new

& with b

→ for

⟹

Lie derivatives w.r.t. $V \in \mathfrak{X}(M)$

$\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega^k(M) \quad (*)$

Prop (4.55 in notes): For any V, $\exists!$ way to

-4- Interior products w.r.t. $V \in \mathfrak{X}(M)$

$i_V : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$

Prop 4.57: For any V, $\exists!$ ... such that:

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Convention:

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$(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))$

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$F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$

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Differential forms ω -2- : $M \ni p \longmapsto \omega_p \in \Lambda^k T_p^* M$ (hence one can apply ω_p to k tangent vectors $\omega_p(v_p^1, \dots, v_p^k) \in \mathbb{R}$)

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$$F^*(\omega)_p (X_p^1, \dots, X_p^k) = \omega_{F(p)}\big((dF)_p(X_p^1), \dots\big).$$

$\boxed{F^*(\omega) \in \Omega^k(M)}$

Ex

$\mathcal{L}_V(f) =$

"Natural" type... be pu...

Ex: ① smoot... F

② vec...

③ dif...

For an... one de...

$\mathcal{L}_V(\xi$...

$\mathcal{L}_V(f) = \frac{d}{dt}(\varphi_V^t)^*(f) \Rightarrow \mathcal{L}_V(f)_{(p)} = \frac{d}{dt}\Big|_{t=0} f(\varphi_V^t(p))$

Remark:

-3-

"Natural types of objects ξ": the type of objects ξ that can be pulled-back by diffeomorphisms $F: M \to N$

$F^*(\xi) \qquad \xi$

Ex: ① smooth functions $\xi = f \in C^\infty(M)$

$F: M \longrightarrow N \xrightarrow{g} \mathbb{R}$: $F^*(g) = g \circ F$

$g \in C^\infty(M)$

② vector fields $X \in \mathfrak{X}(M)$:

$\left. \begin{array}{l} F: M \longrightarrow N \text{ diffeom} \\ X \in \mathfrak{X}(N) \end{array} \right\}$ $\underbrace{F^*(X)_p}_{\in M} := (dF^{-1})_{F(p)}\big(\underbrace{X_{F(p)}}_{\in T_{F(p)}N}\big)$

③ diff. forms $\xi = \omega \in \Omega^k(M)$

For any natural type of objects ξ and $V \in \mathfrak{X}(M)$ one defines:

$\boxed{\mathcal{L}_V(\xi) = }$ the variation of ξ along the flow of V $= \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\xi)$

$\mathcal{L}_V(\xi) = 0 \Leftrightarrow$
$\Leftrightarrow (\varphi_V^t)^*(\xi) = \xi$
$(\forall t)$

...ch that:

$i_V(\omega) = \omega(V)$

... rule

$\boxed{\ldots \wedge i_V(\eta)}$

$\ldots(\omega) \in \Omega^{k-1}(M)$

$\ldots(V, X^1, \ldots, X^{k-1})$

Ex1: $\xi = f \in C^\infty(M)$

$\mathcal{L}_V(f) = \frac{d}{dt} \underbrace{(\varphi_V^t)^*(f)}_{f \circ \varphi_V^t} \Rightarrow \mathcal{L}_V(f)_p = \frac{d}{dt}\Big|_{t=0} f(\varphi_V^t(p))$

$= (df)_p(V_p)$

Remark: usual $\mathcal{L}_V$

-4-

Ex2: $\xi = X \in \mathfrak{X}(M)$

$\mathcal{L}_V(X) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(X) \Rightarrow \mathcal{L}_V(X)_p = \frac{d}{dt}\Big|_{t=0} (d\varphi_V^{-t})_{\varphi_V^t(p)} (X_{\varphi_V^t(p)})$

.... compute ... $\Rightarrow$

Proposition

$\Rightarrow \mathcal{L}_V(X) = [V, X]$

Ex3: On form define $\mathcal{L}_V(\omega)$ using this principle; i.e.

$\mathcal{L}_V(\omega) := \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega)$

Applying def of pull-backs:

$\boxed{\mathcal{L}_V(\omega)_p(X_p^1, \ldots, X_p^k) = \frac{d}{dt}\Big|_{t=0} \omega_{\varphi_V^t(p)}\left((d\varphi_V^t)_p(X^1), \ldots\right)}$

"Natural types of objects ξ": the type of objects ξ that can be pulled-back by diffeomorphisms $F: M \to N$

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$\left.\begin{array}{l} F: M \longrightarrow N \text{ diffeom} \\ X \in \mathfrak{X}(N) \end{array}\right\}$ $\quad F^*(X)_p := (dF^{-1})_{F(p)}\left(X_{F(p)}\right)$, $\quad p \in M$, $\quad X_{F(p)} \in T_{F(p)}N$

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For any natural type of objects ξ and $V \in \mathfrak{X}(M)$ one defines:

$$\boxed{\mathcal{L}_V(\xi) = \text{the variation of } \xi \text{ along the flow of } V = \left.\frac{d}{dt}\right|_{t=0} (\varphi_V^t)^*(\xi)}$$

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-4-

<u>Ex1</u>: $\xi = f \in C^\infty(M)$

$$\mathcal{L}_V(f) = \frac{d}{dt}\underbrace{(\varphi_V^t)^*(f)}_{f\circ\varphi_V^t} \Rightarrow \mathcal{L}_V(f)_{p} = \frac{d}{dt}\Big|_{t=0} f(\varphi_V^t(p))$$

$$= (df)_p(V_p) \quad \text{usual } d_V$$

<u>Ex2</u>: $\xi = X \in \mathfrak{X}(M)$

$$\mathcal{L}_V(X) = \frac{d}{dt}\Big|_{t=0}(\varphi_V^t)^*(X) \Rightarrow \mathcal{L}_V(X)_p = \frac{d}{dt}\Big|_{t=0}(d\varphi_V^{-t})_{\varphi_V^t(p)}\left(X_{\varphi_V^t(p)}\right)$$

.... compute ... $\Rightarrow$

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<u>Ex3</u>:

-3-

...al types of objects "ξ": the type of objects ξ that can be pulled-back by diffeomorphisms $F: M \to N$, $\xi \mapsto F^*(\xi)$

① smooth functions $\xi = f \in C^\infty(M)$, $\;g: N \to \mathbb{R}$: $F^*(g) = g\circ F$

$F: M \longrightarrow N$, $g \in C^\infty(N)$

$X \in \mathfrak{X}(M)$: $F^*(X)_p := (dF^{-1})_{F(p)}\left(X_{F(p)}\right)$, $p \in M$, $X_{F(p)} \in T_{F(p)}N$

...nd $V \in \mathfrak{X}(M)$ ... $\mathcal{L}_V(\xi) = ...$

De Rham differential -5-

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ (De Rham)

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(DeRham-2) $\boxed{d \circ d = 0}$.

lso useful:

licit formulas:

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Explicit formula:

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Explicit formula:

Ex1

$\mathcal{L}_V(f)=$

Ex2

$\mathcal{L}_V(X)$

$\Rightarrow$

Ex

-1- $\mathfrak{X}(M)$

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$\omega \wedge \omega = 0$ if $\omega \in \Omega^k(M)$, k = odd

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$(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$, $\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta$

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"Natural" types ... be pul...

Ex: ① smooth ... F ...

② vect... F

③ diff...

For any ...

...ch that:

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$= (df)_p(V_p)$

Remark: usual $\mathcal{L}_V$

$\boxed{-4-}$

Ex2: $\xi = X \in \mathfrak{X}(M)$

$\mathcal{L}_V(X) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(X) \Rightarrow \mathcal{L}_V(X)_p = \frac{d}{dt}\Big|_{t=0} (d\varphi_V^{-t})_{\varphi_V^t(p)} \left(X_{\varphi_V^t(p)}\right)$

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$$\mathcal{L}_V(\omega)_p(X_p^1, \ldots, X_p^k) = \frac{d}{dt}\Big|_{t=0} \omega_{\varphi_V^t(p)}\left((d\varphi_V^t)_p(X^1), \ldots\right)$$

ξ ... t can

De Rham differential $\boxed{-5-}$

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M) \qquad \text{(De Rham)}$$

Prop (Cor. 4.47 in notes): On any manifold there is one and only one way to construct maps (DeRham) (for all k) such that:

(DeRham-0) d is $\mathbb{R}$-linear and, for $k=0$, $d : C^\infty(M) \to \Omega^1(M)$ the usual

(DeRham-1) d satisfies derivation rule

$$\boxed{d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta} \quad (\forall)\, \omega \in \Omega^k(M), \eta \in \Omega^\ell(M)$$

(DeRham-2) $\boxed{d \circ d = 0}$ $\qquad d(d(\omega)) = 0.$

Also useful: $F^* \circ d = d \circ F^*$

Explicit formulas:

$\boxed{-6-}$ Lie de...

$\mathcal{L}_V$

Prop (4.55 in notes): ...

define ($*$) such t...

(0) $\mathcal{L}_V$ is line...

(1) $\mathcal{L}_V$ satisfi...

(2) $\boxed{\mathcal{L}_V \circ d = \ldots}$

Explicit formul...

Prop.

$\mathcal{L}_V(\omega)(X_1, \ldots, X_k \ldots$

Pf of 1: $\mathcal{L}_V(\omega \ldots$

$\boxed{-1-}$ $\mathfrak{X}(M)$ | $\Omega^k(M)$ Convention: $C^\infty(M)$ | Differ... $\to$ ...

$X \in T_pM$

-6-

Lie derivatives w.r.t. $V \in \mathfrak{X}(M)$

$$\mathcal{L}_V : \Omega^k(M) \longrightarrow \Omega^k(M) \quad (*)$$

Prop (4.55 in notes): For any V, $\exists!$ way to define $(*)$ such that:

(0) $\mathcal{L}_V$ is linear and, for $k=0$, $\mathcal{L}_V : C^\infty(M) \to C^\infty(M)$ the usual

(1) $\mathcal{L}_V$ satisfies derivation rule

$$\boxed{\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge \mathcal{L}_V(\eta)}$$

(2) $\boxed{\mathcal{L}_V \circ d = d \circ \mathcal{L}_V}$ $\quad \mathcal{L}_V(d\omega) = d(\mathcal{L}_V(\omega))$

Explicit formula:

Prop.

$$\mathcal{L}_V(\omega)(X^1, \dots, X^k) = \mathcal{L}_V\Big(\overbrace{\omega(X^1, \dots, X^k)}^{\in C^\infty(M)}\Big) - \sum_{i=1}^{k} \omega(X^1, \dots, X^{i-1}, [V, X^i], \dots, X^k)$$

Pf of 1: $\mathcal{L}_V(\omega \wedge \eta) = \frac{d}{dt}\Big|_{t=0} (\varphi_V^t)^*(\omega \wedge \eta) = \frac{d}{dt}\Big|_{t=0}\Big((\varphi_V^t)^*(\omega) \wedge (\varphi_V^t)^*(\eta)\Big)$

-7-

Interior products w.r.t. $V \in \mathfrak{X}(M)$

$$i_V : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$$

Prop 4.57: For any V, $\exists!$ ... such that:

(0) i_V is linear and, for $k=1$, $i_V(\omega) = \omega(V)$

(1) i_V satisfies the derivation rule

$$\boxed{i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)}$$

Explicit formula: $\left.\begin{matrix} \omega \in \Omega^k(M) \\ V \in \mathfrak{X}(M) \end{matrix}\right\} i_V(\omega) \in \Omega^{k-1}(M)$

given by: $i_V(\omega)(X^1, \dots, X^{k-1}) = \omega(V, X^1, \dots, X^{k-1})$

-2-

$M \ni p \longmapsto \omega_p \in \Lambda^k T_p^* M$ (hence one can apply ω_p to k tangent vectors ... $)\in \mathbb{R}$)

"Natural ...

be ...

satisfies derivation rule

$\boxed{d(\omega\wedge\eta)=d\omega\wedge\eta+(-1)^k\omega\wedge d\eta}$ $(\forall)\,\omega\in\Omega^k(M),\ \eta\in\Omega^\ell(M)$

$\boxed{d\circ d=0}$ $\quad d(d(\omega))=0.$

$\boxed{F^*\circ d=d\circ F^*}$

$\frac{d}{dt}\Big|_{t=0}\omega_t=\lim\frac{\omega_t-\omega_0}{t}$

(1) $\mathcal{L}_V$ satisfies derivation rule the usual

$\boxed{\mathcal{L}_V(\omega\wedge\eta)=\mathcal{L}_V(\omega)\wedge\eta+\omega\wedge\mathcal{L}_V(\eta)}$

(2) $\boxed{\mathcal{L}_V\circ d=d\circ\mathcal{L}_V}$ $\quad\mathcal{L}_V(d\omega)=d(\mathcal{L}_V(\omega))$

Explicit formula: $\in C^\infty(M)$

(1) i_V satisfies the derivation rule

$\boxed{i_V(\omega\wedge\eta)=i_V(\omega)\wedge\eta+(-1)^k\omega\wedge i_V(\eta)}$

Explicit formula: $\left.\begin{matrix}\omega\in\Omega^k(M)\\ V\in\mathfrak{X}(M)\end{matrix}\right\}\ i_V(\omega)\in$

$$\frac{d}{dt}\Big|_{t=0}f(t)\cdot g(t)=f(0)\left(\frac{d}{dt}\Big|_{t=0}g(t)\right)+\left(\frac{d}{dt}\Big|_{t=0}f(t)\right)\cdot g(0)$$

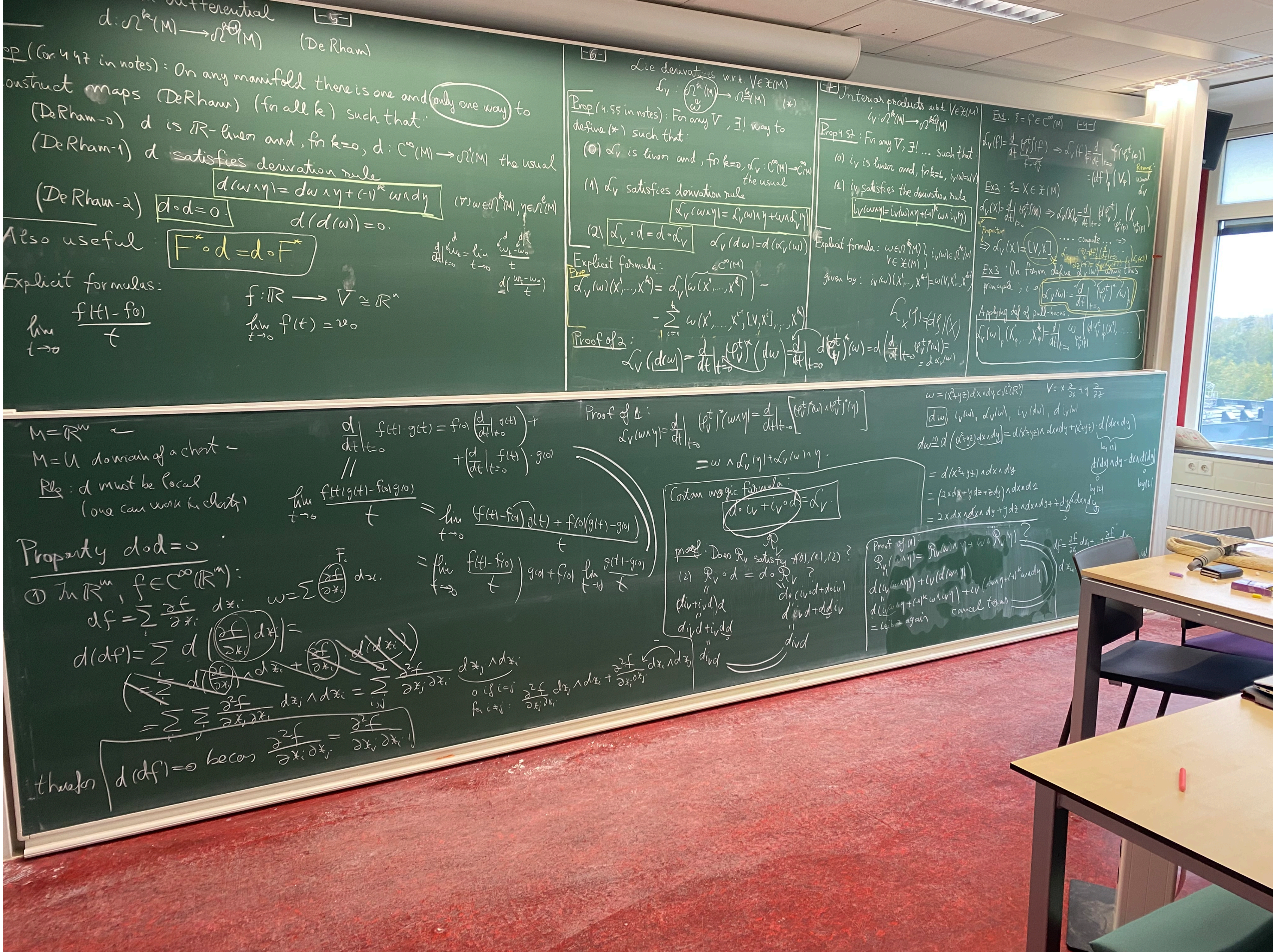
$$\lim_{t\to0}\frac{f(t)g(t)-f(0)g(0)}{t}=\lim_{t\to0}\frac{(f(t)-f(0))g(t)+f(0)(g(t)-g(0))}{t}$$

$$=\left(\lim_{t\to0}\frac{f(t)-f(0)}{t}\right)g(0)+f(0)\lim_{t\to0}\frac{g(t)-g(0)}{t}$$

Proof of 1:

$$\mathcal{L}_V(\omega\wedge\eta)=\frac{d}{dt}\Big|_{t=0}(\varphi_V^t)^*(\omega\wedge\eta)=\frac{d}{dt}\Big|_{t=0}\left[(\varphi_V^t)^*(\omega)\wedge(\varphi_V^t)^*(\eta)\right]$$

$$=\omega\wedge\mathcal{L}_V(\eta)+\mathcal{L}_V(\omega)\wedge\eta.$$



(De Rham)
On any manifold there is one and only one way to
construct maps (DeRham) (for all k) such that
(DeRham-0) d is ℝ-linear and, for k=0, d : C^∞(M) → Ω¹(M) the usual
(DeRham-1) d satisfies derivation rule
d(ω∧η) = dω∧η + (-1)^k ω∧dη
(DeRham-2) d∘d = 0
d(d(ω)) = 0
Also useful: F*∘d = d∘F*
Explicit formulas:
Lie derivatives w.r.t. V ∈ 𝔛(M)
Prop (4.55 in notes): For any V, ∃! way to
(0) ℒ_V is linear and, for k=0, ℒ_V : C^∞(M) → C^∞(M) the usual
(1) ℒ_V satisfies derivation rule
(2) ℒ_V∘d = d∘ℒ_V
Explicit formula:
Proof of 2:
Interior products w.r.t. V ∈ 𝔛(M)
(0) i_V is linear and, for k=1, i_V(ω) = ω(V)
(1) i_V satisfies the derivation rule
Explicit formula:
M = ℝ^n
M = U domain of a chart
Rk: d must be local
(one can work in charts)
Property d∘d = 0
① In ℝ^n, f ∈ C^∞(ℝ^n):
therefore d(df) = 0 because
Proof of 1:
Cartan magic formula
d∘i_V + i_V∘d = ℒ_V
proof: Does R_V satisfy (0), (1), (2)?
cancel terms