

De Rham differential -5-  
 $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  (De Rham)

**Prop** (Cor 4.47 in notes): On any manifold there is one and only one way to construct maps (De Rham) (for all  $k$ ) such that:  
 (De Rham-0)  $d$  is  $\mathbb{R}$ -linear and, for  $k=0$ ,  $d: C^0(M) \rightarrow \Omega^1(M)$  the usual (De Rham-1)  $d$  satisfies derivation rule  
 $d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$  ( $w \in \Omega^k(M), \eta \in \Omega^l(M)$ )  
 (De Rham-2)  $d \circ d = 0$ .

Also useful:  
 Explicit formulas:

-6- Lie derivatives w.r.t.  $V \in \mathfrak{X}(M)$   
 $L_V: \Omega^k(M) \rightarrow \Omega^k(M)$  (\*)

**Prop** (4.55 in notes): For any  $V, \exists!$  way to define (\*) such that:  
 (0)  $L_V$  is linear and, for  $k=0$ ,  $L_V: C^0(M) \rightarrow C^0(M)$  the usual  
 (1)  $L_V$  satisfies derivation rule  
 $L_V(w \wedge \eta) = L_V(w) \wedge \eta + w \wedge L_V(\eta)$   
 (2)  $L_V \circ d = d \circ L_V$   
 Explicit formula:

-7- Interior products w.r.t.  $V \in \mathfrak{X}(M)$   
 $i_V: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$

**Prop** 4.57: For any  $V, \exists!$  ... such that:  
 (0)  $i_V$  is linear and, for  $k=1$ ,  $i_V(\alpha(V)) = 0$   
 (1)  $i_V$  satisfies the derivation rule  
 $i_V(w \wedge \eta) = L_V(w) \wedge \eta + (-1)^k w \wedge i_V(\eta)$

Explicit formula:

**Ex 1:**  $\beta = f \in C^0(M)$  -4-  
 $i_V(f) = \frac{d}{dt} \left( \frac{d}{dt} f \right) \rightarrow d_V(f) = \frac{d}{dt} f$   
**Ex 2:**  $\beta = X \in \mathfrak{X}(M)$   
 $d_V(X) = \frac{d}{dt} (X^t) \rightarrow d_V(X) = \frac{d}{dt} (X^t)$   
 $\Rightarrow d_V(X) = [V, X]$  ... compute...  
**Ex 3:**

Vector fields  $M \ni p \mapsto X_p \in T_p M$  -1-  $\mathfrak{X}(M)$   
 $\rightarrow$  locally  $X = \sum_{i=1}^m F_i \frac{\partial}{\partial x_i}$  on  $U$  ( $(U, x) = \text{chart}$ )  
 $\rightarrow$  globally:  $L_X: C^\infty(M) \rightarrow C^\infty(M)$  derivation  
 $\rightarrow$  operations:  $X+Y, f \cdot X, [X, Y]$   
 $\rightarrow$  flow of  $X$ :  $\varphi_X^t = \varphi_X^t(p)$  ( $t \in \mathbb{R}, p \in M$ ) where  
 $(\forall) p \in M: \begin{cases} \frac{d}{dt} \varphi_X^t(p) = X_{\varphi_X^t(p)} \\ \varphi_X^0(p) = p \end{cases}$  i.e.  $\varphi_X^t$  (maximal) integral curve of  $X$  starting at  $p$   
 If  $X = \text{complete} \Rightarrow \varphi_X^t: M \rightarrow M$  DIFFEOMORPHISMS -2-

Differential forms  $\omega: M \ni p \mapsto \omega_p \in \Lambda^k T_p^* M$  (hence one can apply  $\omega_p$  to  $k$  tangent vectors  $v_1, \dots, v_k \in T_p M$ )  
 $\rightarrow$  locally  $\omega = \sum_{I \in \mathcal{I}_k} F_I dx_{i_1} \wedge \dots \wedge dx_{i_k}$  ( $I = (i_1, \dots, i_k)$ )  
 $\rightarrow$  globally:  $\omega: \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow C^\infty(M)$  -3-  
 $\rightarrow$  operations:  $\omega + \eta, f \cdot \omega, \dots; d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$   $d f(X) = d_X(f)$   
 $\rightarrow$  new wedge product of  $w \in \Omega^k(M)$  and  $\eta \in \Omega^l(M)$  new form  $w \wedge \eta \in \Omega^{k+l}(M)$ :  
 $(\text{sum}) (X^1, \dots, X^{k+l}) = \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \omega(X^{\sigma(1)}, \dots, X^{\sigma(k)}) \eta(X^{\sigma(k+1)}, \dots, X^{\sigma(k+l)})$   
 $\&$  with basic properties  $(w \wedge \eta) \wedge \zeta = w \wedge (\eta \wedge \zeta), \eta \wedge \omega = (-1)^k \omega \wedge \eta$   
 $\rightarrow$  forms are VERY NATURAL: for ANY smooth map  $F: M \rightarrow N$  and  $w \in \Omega^k(N)$   
 $\Rightarrow$  can talk about the pull-back of  $w$  by  $F$ :  
 $F^*(w) \in \Omega^k(M)$   $F^*(w)_p(X_p^1, \dots, X_p^k) = w(F(p)) (dF(X_p^1), \dots)$

Natural types of objects: the type of objects be pulled-back by diffeomorphisms  
**Ex 1** smooth functions  $f = f \in C^0(M)$   
 $F: M \rightarrow N$   $F^*(f) = f \circ F$   
**2** vector fields  $X \in \mathfrak{X}(M)$   
 $F: M \rightarrow N$  diff  $F^*(X) := (dF)^{-1}(X)$   
**3** diff. forms  $\omega \in \Omega^k(M)$   
 For any natural type of  $\omega$  one defines:  
 $d_V(\omega) = \frac{d}{dt} \omega(\varphi_X^t)$  the variation of  $\omega$  along the flow of  $V$



(DeRham-2)  $d \circ d = 0$

Also useful:

Explicit formulas:

$d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$  ( $\forall w \in \Omega^k(M), \eta \in \Omega^l(M)$ )

(2)  $\mathcal{L}_V \circ d = d \circ \mathcal{L}_V$

Explicit formula:

Vector fields  $M \ni p \mapsto X_p \in T_p M$   $\mathcal{X}(M)$

→ locally  $X = \sum_{i=1}^m F_i \frac{\partial}{\partial x_i}$  on  $U$  ( $(U, \chi) = \text{chart}$ )

→ globally  $\mathcal{L}_X : C^\infty(M) \rightarrow C^\infty(M)$  derivation

→ operations:  $X+Y, f \cdot X, [X, Y]$

→ flow of X:  $\varphi_X = \varphi_X^t(p)$  ( $t \in \mathbb{R}, p \in M$ ) where

$(\forall) p \in M: t \mapsto \varphi_X^t(p)$  (maximal) integral curve of X starting at p

If  $X = \text{complete} \Rightarrow \varphi_X^t : M \rightarrow M$  DIFFEOMORPHISMS

$\Omega^k(M)$   
Convention:  
 $\Omega^0(M) = C^\infty(M)$

Differential forms  $w: M \ni p \mapsto w_p \in \wedge^k T_p^* M$

→ locally  $w = \sum_{I \in \text{Ind}_k(M)} F_I \underline{dx_{i_1} \wedge \dots \wedge dx_{i_k}}$  ( $I = (i_1, \dots, i_k)$ )

→ globally  $w: \underbrace{\mathcal{X}(M) \times \dots \times \mathcal{X}(M)}_k \rightarrow C^\infty(M)$  skew symmetric  
 $C^\infty(M)$ -linear in  $e$

→ operations:  $w + \eta, f \cdot w, \dots, d: \Omega^q(M) \rightarrow \Omega^{q+1}(M)$

→ new wedge product of  $w \in \Omega^k(M)$  and  $\eta \in \Omega^l(M)$  new  
 $(w \wedge \eta)(X^1, \dots, X^{k+l}) = \sum_{\sigma \in S_{k+l}} \underbrace{\omega(X^{\sigma(1)}, \dots, X^{\sigma(k)})}_{\text{sign}(\sigma)} \eta(X^{\sigma(k+1)}, \dots, X^{\sigma(k+l)})$

& with basic properties  
 $(w \wedge \eta) \wedge \zeta = w \wedge (\eta \wedge \zeta), \eta \wedge w = (-1)^{kl} w \wedge \eta$

→ forms are VERY NATURAL: for ANY smooth map  $F: M \rightarrow N$

⇒ can talk about the pull-back of  $w$  by  $F$ :

$F^*(w) \in \Omega^k(M)$   $F_p^*(w)(X_p^1, \dots, X_p^k) = w(F_p(X_p^1), \dots, F_p(X_p^k))$



$$d: \Omega^k(M) \rightarrow \Omega^{k+1}(M) \quad (\text{De Rham})$$

Prop (Cor. 4.47 in notes): On any manifold there is one and only one way to

Prop (4.55 in

Vector fields  $M \ni p \mapsto X_p \in T_p M$  -1-  $\mathcal{X}(M)$

→ locally:  $X = \sum_{i=1}^m F_i \frac{\partial}{\partial x_i}$  on  $U$  ( $(U, x) = \text{chart}$ )

→ globally:  $\mathcal{L}_X: C^\infty(M) \rightarrow C^\infty(M)$  derivation

→ operations:  $X+Y, f \cdot X, [X, Y]$

$$\mathcal{L}_{[X, Y]}(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))$$

→ flow of X:  $\varphi_X = \varphi_X^t(p)$  ( $t \in \mathbb{R}, p \in M$ ) where

(A)  $p \in M$ :  $t \mapsto \varphi_X^t(p)$  (maximal) integral curve of  $X$  starting at  $p$

$$\text{i.e. } \begin{cases} \frac{d}{dt} \varphi_X^t(p) = X_{\varphi_X^t(p)} \\ \varphi_X^0(p) = p \end{cases}$$

If  $X = \text{complete} \Rightarrow \varphi_X^t: M \rightarrow M$  DIFFEOMORPHISMS.

(B)  $X$

Useful:

$$F^*(w \wedge \eta) = F^*(w) \wedge F^*(\eta)$$

$$F^*(dw) = d(F^*w)$$



Lie derivatives w.r.t.  $V \in \mathfrak{X}(M)$   
 $L_V : \Omega^k(M) \rightarrow \Omega^k(M) \quad (*)$

[-] Interior products w.r.t.  $V \in \mathfrak{X}(M)$   
 $i_V : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$   
 Prop 4.57: For any  $V$ ,  $\exists!$  ... such that

Prop (4.55 in notes): For any  $V$ ,  $\exists!$  way to

Differential forms  $w: M \ni p \mapsto \omega_p \in \wedge^k T_p^*M$  (hence one can apply  $\omega_p$  to  $k$  tangent vectors  $\omega_p(v_1^p, \dots, v_k^p) \in \mathbb{R}$ )  
 $(I = (i_1, \dots, i_k))$   
 $\rightarrow$  locally:  $w = \sum_{I \in \partial \Omega^k(M)} F_I dx_{i_1} \wedge \dots \wedge dx_{i_k}$

$\rightarrow$  globally:  $w: \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_k \rightarrow \mathbb{C}^\infty(M)$   
 $\left\{ \begin{array}{l} \text{skew symmetric} \\ \mathbb{C}^\infty(M) \text{ linear in each argument} \end{array} \right.$

$\rightarrow$  operations:  $w + \eta, f \cdot w, \dots, d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$   $df(X) = L_X(f)$

$\rightarrow$  new: wedge product of  $w \in \Omega^k(M)$  and  $\eta \in \Omega^l(M)$ . new form  $w \wedge \eta \in \Omega^{k+l}(M)$   
 $(w \wedge \eta)(X^1, \dots, X^{k+l}) = \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) \omega(X^{\sigma(1)}, \dots, X^{\sigma(k)}) \eta(X^{\sigma(k+1)}, \dots, X^{\sigma(k+l)})$

& with basic properties  $(w \wedge \eta) \wedge \zeta = w \wedge (\eta \wedge \zeta), \eta \wedge w = (-1)^{kl} w \wedge \eta$   
 $w \wedge w = 0$  if  $w \in \Omega^k(M)$   $k = \text{odd}$

$\rightarrow$  forms are "VERY NATURAL": for ANY smooth map  $F: M \rightarrow N$  and  $w \in \Omega^k(N)$   
 $\Rightarrow$  can talk about the pull-back of  $w$  by  $F$ :  
 $F^*(w)_p(X_p^1, \dots, X_p^k) = w(F(p))((dF)_p(X_p^1), \dots)$

$F^*(w) \in \Omega^k(M)$

$\Omega^k(M)$   
 Convention:  
 $\Omega^0(M) = \mathbb{C}^\infty(M)$

$(f) = L_X(L_Y(f)) - L_Y(L_X(f))$

i.e.  $\left\{ \begin{array}{l} \frac{d}{dt} \varphi_X^t(p) = X_{\varphi_X^t(p)} \\ \varphi_X^0(p) = p \end{array} \right.$

Useful:  
 $F^*(w \wedge \eta) = F^*(w) \wedge F^*(\eta)$   
 $F^*(dw) = d(F^*w)$

Natural type  
 be pu  
 Ex: ① smooth  
 F  
 ② vec  
 ③ dif  
 For an  
 one de  
 $d_V(\xi)$



such that:  $d_V(f) = \frac{d}{dt} \left( \frac{\varphi_t^*}{V} \right) (f) \rightarrow d_V(f) = \frac{d}{dt} \Big|_{t=0} f(\varphi_t^*(p))$

"Natural types of objects": the type of objects that can be pulled-back by diffeomorphisms  $F: M \rightarrow N$

Ex: ① smooth functions  $\xi = f \in C^\infty(M)$   $F^*(\xi) = f \circ F$

$F: M \rightarrow N$   $\xrightarrow{g \in C^\infty(N)}$   $F^*(g) = g \circ F$

② vector fields  $X \in \mathfrak{X}(M)$

$F: M \rightarrow N$  diffeom.  $F^*(X) := (dF^{-1})(X_{F(p)})$

$X \in \mathfrak{X}(N)$

③ diff. forms  $\xi = \omega \in \Omega^k(M)$

For any natural type of objects  $\xi$  and  $V \in \mathfrak{X}(M)$  one defines:

$d_V(\xi) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^*)^*(\xi)$  the variation of  $\xi$  along the flow of  $V$

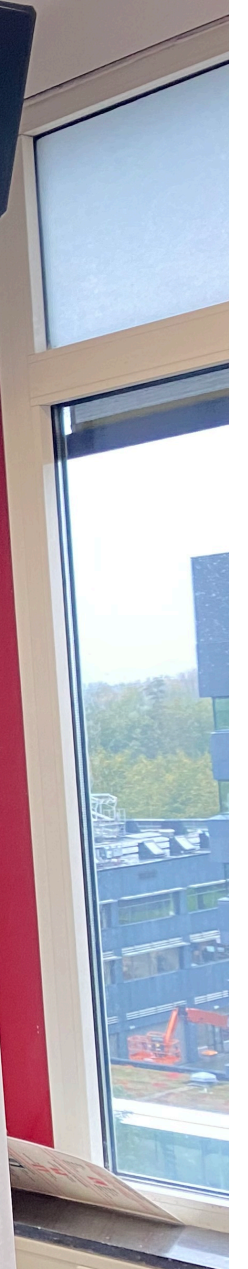
can to be done  $(v_p^k) \in \mathbb{R}$

ent

$L_X(f)$

$\Omega^{k+l}(M)$

$\Omega^k(N)$





ch that:

$$i_V(w) = w(V)$$

rule

$$i_V(\gamma)$$

$$w) \in \Omega^{k-1}(M)$$

$$(V, X^1, \dots, X^{k-1})$$

$$\xi = f \in C^\infty(M) \quad [-4-]$$
$$d_V(f) = \frac{d}{dt} \left( \underbrace{(\varphi_t^*)}_{f \circ \varphi_t} \right) (f) \Rightarrow d_V(f) = \frac{d}{dt} \Big|_{t=0} f(\varphi_t^*(p))$$

$$\text{Ex 2: } \xi = X \in \mathfrak{X}(M)$$

$$= (df)_p(V_p) \quad \text{Remark: usual } d_V$$

$$d_V(X) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^*)(X) \Rightarrow d_V(X)_p = \frac{d}{dt} \Big|_{t=0} (d\varphi_t^*)(X_{\varphi_t^*(p)})$$

Proposition

... compute ...  $\Rightarrow$

$$\Rightarrow d_V(X) = [V, X]$$

Ex 3: In form define  $d_V(w)$  using this principle: i.o.

$$d_V(w) := \frac{d}{dt} \Big|_{t=0} (\varphi_t^*)(w)$$

Applying def of pull-backs:

$$d_V(w)_p(X_p^1, \dots, X_p^{k-1}) = \frac{d}{dt} \Big|_{t=0} w \left( \underbrace{d\varphi_t^*}_{(d\varphi_t^*)_p} (X_p^1, \dots) \right)$$

$\xi$  can

g Meineszgeb



Natural types of objects  $\xi$ : the type of objects that can be pulled-back by diffeomorphisms  $F: M \rightarrow N$

Ex: ① smooth functions  $\xi = f \in C^\infty(M)$

$$F: M \rightarrow N \xrightarrow{g \in C^\infty(N)} \mathbb{R} : F^*(g) = g \circ F$$

② vector fields  $X \in \mathfrak{X}(M)$

$$F: M \rightarrow N \text{ diffeo} \left. \begin{array}{l} X \in \mathfrak{X}(M) \\ X \in \mathfrak{X}(N) \end{array} \right\} F^*(X) \Big|_M := (dF^{-1}) \Big|_M (X \Big|_{F(p)})$$

③ diff. forms  $\xi = \omega \in \Omega^k(M)$

For any natural type of objects  $\xi$  and  $V \in \mathfrak{X}(M)$  one defines:

$$\mathcal{L}_V(\xi) = \frac{d}{dt} \Big|_{t=0} (\varphi_t)^*(\xi)$$

the variation of  $\xi$  along the flow of  $V$

$$\begin{aligned} d_V(\xi) = 0 &\Leftrightarrow \\ \Leftrightarrow (\varphi_t)^*(\xi) = \xi & \\ \forall t \end{aligned}$$



Ex1.  $\xi = f \in C^\infty(M)$  [-4-]

$$d_V(f) = \frac{d}{dt} \underbrace{(\varphi_V^t)^*(f)}_{f \circ \varphi_V^t} \Rightarrow d_V(f)_p = \frac{d}{dt} \Big|_{t=0} f(\varphi_V^t(p))$$

$$= (df)_p (V_p) \quad \text{usual } d_V$$

Ex2:  $\xi = X \in \mathfrak{X}(M)$

$$d_V(X) = \frac{d}{dt} \Big|_{t=0} (\varphi_V^t)^*(X) \Rightarrow d_V(X)_p = \frac{d}{dt} \Big|_{t=0} (d\varphi_V^t)_{\varphi_V^t(p)} (X_{\varphi_V^t(p)})$$

... compute ...  $\Rightarrow$

$$\Rightarrow d_V(X) = [V, X]$$

Ex3:

[-3-]

al types of objects  $\xi$ : the type of objects that can be pulled-back by diffeomorphisms  $F: M \rightarrow N$

① smooth functions  $\xi = f \in C^\infty(M)$

$$F^*(g) = g \circ F$$

$$F^*(X)_{p \in M} = (dF^{-1})_{F(p)} (X_{F(p)})$$

and  $\forall X \in \mathfrak{X}(M)$



De Rham differential [-5-]  
 $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  (De Rham)

Prop (Cor 4.47 in notes): On any manifold there is one and only one way to construct maps (DeRham) (for all  $k$ ) such that:

(DeRham-0)  $d$  is  $\mathbb{R}$ -linear and, for  $k=0$ ,  $d: C^\infty(M) \rightarrow \Omega^1(M)$  the usual

(DeRham-1)  $d$  satisfies derivation rule

$$d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta \quad (w \in \Omega^k(M), \eta \in \Omega^l(M))$$

(DeRham-2)  $d \circ d = 0$ .

Also useful:

Explicit formulas:

Fields  $M \ni p \mapsto X_p \in T_p M$  [-1-]  $X(M)$   
 Locally:  $X = \sum_{i=1}^n F_i \frac{\partial}{\partial x_i}$  on  $U$  ( $(U, x) = \text{chart}$ )

Locally:  $L_X: C^\infty(M) \rightarrow C^\infty(M)$  derivation

Operations:  $X+Y, f \cdot X, [X, Y]$

Flow of  $X$ :  $\varphi_X^t = \varphi_X^t(p)$  ( $t \in \mathbb{R}, p \in M$ ) where  $\frac{d}{dt} \varphi_X^t(p) = X_{\varphi_X^t(p)}$  i.e.  $\varphi_X^t(p) = X_{\varphi_X^t(p)}$  (maximal) integral curve of  $X$

$\Omega^k(M)$   
 Convention:  $\Omega^0(M) = C^\infty(M)$

Lie derivatives w.r.t.  $V \in X(M)$  [-6-]  
 $L_V: \Omega^k(M) \rightarrow \Omega^k(M)$  (\*)

Prop (4.55 in notes): For any  $V, \exists!$  way to define (\*) such that:

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(1)  $L_V$  satisfies derivation rule

$$L_V(w \wedge \eta) = L_V(w) \wedge \eta + w \wedge L_V(\eta)$$

(2)  $L_V \circ d = d \circ L_V$

Explicit formula:

Interior products w.r.t.  $V \in X(M)$  [-7-]  
 $i_V: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$

Prop 4.57: For any  $V, \exists!$  ... such that:

(0)  $i_V$  is linear and, for  $k=1$ ,  $i_V(w) = w(V)$

(1)  $i_V$  satisfies the derivation rule

$$i_V(w \wedge \eta) = (i_V w) \wedge \eta + (-1)^k w \wedge i_V(\eta)$$

Explicit formula:

Differential forms  $w: M \ni p \mapsto \underline{w}_p \in \wedge^k T_p^* M$  [-2-] (hence one can apply up to  $k$  tangent vectors  $(v_1^p, \dots, v_k^p) \in \mathbb{R}^n$ )

$\rightarrow$  locally:  $w = \sum_{I \in \text{Ind}_k(n)} F_I dx_{i_1} \wedge \dots \wedge dx_{i_k}$  ( $I = (i_1, \dots, i_k)$ )

$\rightarrow$  globally:  $w: \underbrace{X(M) \times \dots \times X(M)}_k \rightarrow C^\infty(M)$   $\left\{ \begin{array}{l} \text{skew symmetric} \\ C^\infty(M)\text{-linear in each argument} \end{array} \right.$

$\rightarrow$  operations:  $w + \eta, f \cdot w, \dots, d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$   $df(X) = L_X(f)$

$\rightarrow$  new: wedge product of  $w \in \Omega^k(M)$  and  $\eta \in \Omega^l(M)$  new form  $w \wedge \eta \in \Omega^{k+l}(M)$ :

$$(w \wedge \eta)(X^1, \dots, X^{k+l}) = \sum_{\sigma \in S_{k+l}} \text{sgn}(\sigma) w(X^{\sigma(1)}, \dots, X^{\sigma(k)}) \eta(X^{\sigma(k+1)}, \dots, X^{\sigma(k+l)})$$

$w \wedge w = 0$  if  $w \in \Omega^k(M)$   $k = \text{odd}$

"Natural" types be pul

Ex: ① smooth F.

② vect F.

③ diff

For any  $v$



ch that:

$$i_V(\omega) = \omega(V)$$

rule

$$i_V(\gamma)$$

$$\omega \in \Omega^{k-1}(M)$$

$$(V, X^1, \dots, X^{k-1})$$

$$\xi = f \in C^\infty(M) \quad [4-1]$$
$$d_V(f) = \frac{d}{dt} \left( \underbrace{(\varphi_t^*)}_{f \circ \varphi_t^V} (f) \right) \Rightarrow d_V(f) = \frac{d}{dt} \Big|_{t=0} f(\varphi_t^V(p))$$

$$\text{Ex 2: } \xi = X \in \mathfrak{X}(M)$$

$$= (df)_p(V_p) \quad \text{Remark: usual } d_V$$

$$d_V(X) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^*)^*(X) \Rightarrow d_V(X)_p = \frac{d}{dt} \Big|_{t=0} (d\varphi_t^V)^*(X_{\varphi_t^V(p)})$$

Proposition

... compute ...  $\Rightarrow$

$$\Rightarrow d_V(X) = [V, X]$$

Ex 3: In form define  $d_V(\omega)$  using this principle: i.o.

$$d_V(\omega) := \frac{d}{dt} \Big|_{t=0} (\varphi_t^*)^*(\omega)$$

Applying def of pull-backs:

$$d_V(\omega)_p(X_p^1, \dots, X_p^k) = \frac{d}{dt} \Big|_{t=0} \omega_{\varphi_t^V(p)} \left( (d\varphi_t^V)_p(X^1), \dots \right)$$

$\xi$  can

g Meineszgeb



De Rham differential -5-  
 $d: \mathcal{O}^k(M) \rightarrow \mathcal{O}^{k+1}(M)$  (De Rham)

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$$d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta \quad (\forall w \in \mathcal{O}^k(M), \eta \in \mathcal{O}^l(M))$$

(DeRham-2)  $d \circ d = 0$   $d(d(w)) = 0$ .

Also useful:  $F^* \circ d = d \circ F^*$

Explicit formulas:

-6- Lie der  
 $\mathcal{L}_V$

Prop (4.55 in notes):  
 define  $(*)$  such that

(0)  $\mathcal{L}_V$  is linear

(1)  $\mathcal{L}_V$  satisfies

$$(2) \mathcal{L}_V \circ d =$$

Explicit formula  
Prop  
 $\mathcal{L}_V(w)(X^1, \dots, X^k)$

Prop 1:  $\mathcal{L}_V(w)$



-6-

Lie derivatives w.r.t.  $V \in \mathcal{X}(M)$

$$\mathcal{L}_V : \Omega^k(M) \rightarrow \Omega^k(M) \quad (*)$$

Prop (4.55 in notes): For any  $V$ ,  $\exists!$  way to define  $(*)$  such that:

(0)  $\mathcal{L}_V$  is linear and, for  $k=0$ ,  $\mathcal{L}_V : C^\infty(M) \rightarrow C^\infty(M)$  the usual

(1)  $\mathcal{L}_V$  satisfies derivation rule

$$\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge \mathcal{L}_V(\eta)$$

$$(2) \mathcal{L}_V \circ d = d \circ \mathcal{L}_V \quad \mathcal{L}_V(dw) = d(\mathcal{L}_V(w))$$

Explicit formula:

$$\begin{aligned} \text{Prop. } \mathcal{L}_V(w)(X^1, \dots, X^k) &= \mathcal{L}_V(\underbrace{w(X^1, \dots, X^k)}_{\in C^\infty(M)}) - \\ &\quad - \sum_{i=1}^k w(X^1, \dots, X^{i-1}, [V, X^i], \dots, X^k) \end{aligned}$$

$$\text{PF of 1: } \mathcal{L}_V(\omega \wedge \eta) = \frac{d}{dt} \Big|_{t=0} (\rho_V^t)^*(\omega \wedge \eta) = \frac{d}{dt} \Big|_{t=0} (\rho_V^t)^*(\omega) \wedge (\rho_V^t)^*(\eta)$$

-7- Interior products w.r.t.  $V \in \mathcal{X}(M)$

$$i_V : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$$

Prop 4.57: For any  $V$ ,  $\exists!$  ... such that:

(0)  $i_V$  is linear and, for  $k=1$ ,  $i_V(\omega) = \omega(V)$

(1)  $i_V$  satisfies the derivation rule

$$i_V(\omega \wedge \eta) = i_V(\omega) \wedge \eta + (-1)^k \omega \wedge i_V(\eta)$$

Explicit formula:  $\omega \in \Omega^k(M) \left\{ \begin{array}{l} i_V(\omega) \in \Omega^{k-1}(M) \\ V \in \mathcal{X}(M) \end{array} \right.$

$$\text{given by: } i_V(\omega)(X^1, \dots, X^{k-1}) = \omega(V, X^1, \dots, X^{k-1})$$

-2-

$$M \ni p \longmapsto \omega_p \in \wedge^k T_p^* M$$

(hence one can apply  $\omega_p$  to  $k$  tangent vectors  $(v^1, \dots, v^k) \in \mathbb{R}^k$ )

"Natural"  $\omega$  be  $p$



satisfies derivation rule  
 $d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$  (1)  $w \in \Omega^k(M), \eta \in \Omega^l(M)$   
 $d \circ d = 0$   
 $d(d(w)) = 0$   
 $F^* \circ d = d \circ F^*$

(1)  $\alpha_V$  satisfies derivation rule  
 $\alpha_V(w \wedge \eta) = \alpha_V(w) \wedge \eta + w \wedge \alpha_V(\eta)$   
(2)  $\alpha_V \circ d = d \circ \alpha_V$   
 $\alpha_V(dw) = d(\alpha_V(w))$   
Explicit formula:  $\in C^\infty(M)$

(1)  $i_V$  satisfies the derivation rule  
 $i_V(w \wedge \eta) = i_V(w) \wedge \eta + (-1)^k w \wedge i_V(\eta)$   
Explicit formula:  $w \in \Omega^k(M), v \in \mathfrak{X}(M)$

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} f(t) \cdot g(t) &= f(0) \left( \frac{d}{dt} \Big|_{t=0} g(t) \right) + \left( \frac{d}{dt} \Big|_{t=0} f(t) \right) \cdot g(0) \\ // \\ \lim_{t \rightarrow 0} \frac{f(t)g(t) - f(0)g(0)}{t} &= \lim_{t \rightarrow 0} \frac{(f(t) - f(0))g(t) + f(0)(g(t) - g(0))}{t} \\ &= \left( \lim_{t \rightarrow 0} \frac{f(t) - f(0)}{t} \right) g(0) + f(0) \lim_{t \rightarrow 0} \frac{g(t) - g(0)}{t} \end{aligned}$$

Proof of  $\Delta$ :

$$\alpha_V(w \wedge \eta) = \frac{d}{dt} \Big|_{t=0} (p_V^t)^*(w \wedge \eta) = \frac{d}{dt} \Big|_{t=0} \left[ (p_V^t)^*(w) \wedge (p_V^t)^*(\eta) \right]$$

$$= w \wedge \alpha_V(\eta) + \alpha_V(w) \wedge \eta$$



differential  $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$  (De Rham)

Prop (Cor 4.47 in notes): On any manifold there is one and only one way to construct maps (De Rham) (for all  $k$ ) such that:  
 (De Rham-0)  $d$  is  $\mathbb{R}$ -linear and, for  $k=0$ ,  $d: C^\infty(M) \rightarrow \Omega^1(M)$  the usual  
 (De Rham-1)  $d$  satisfies derivation rule  
 (De Rham-2)  $d \circ d = 0$

$d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$  ( $w \in \Omega^k(M), \eta \in \Omega^l(M)$ )  
 $d(d(w)) = 0$

Also useful:  
 Explicit formulas:

$F^* \circ d = d \circ F^*$

$f: \mathbb{R} \rightarrow V \cong \mathbb{R}^n$

$\lim_{t \rightarrow 0} \frac{f(t) - f_0}{t}$

$\lim_{t \rightarrow 0} f(t) = v_0$

Lie derivatives w.r.t  $V \in \mathfrak{X}(M)$   
 $\mathcal{L}_V: \Omega^k(M) \rightarrow \Omega^k(M)$  (\*)

Prop (4.55 in notes): For any  $V, \exists!$  way to define (\*) such that:

- (0)  $\mathcal{L}_V$  is linear and, for  $k=0$ ,  $\mathcal{L}_V: C^\infty(M) \rightarrow C^\infty(M)$  the usual
- (1)  $\mathcal{L}_V$  satisfies derivation rule
- (2)  $\mathcal{L}_V \circ d = d \circ \mathcal{L}_V$

Explicit formula:  
 $\mathcal{L}_V(w)(x^1, \dots, x^k) = d_V(w(x^1, \dots, x^k)) - \sum_{i=1}^k w(x^1, \dots, x^i, [V, x^i], \dots, x^k)$

Proof of 2:  
 $\mathcal{L}_V(dw) = \frac{d}{dt} \Big|_{t=0} (e^{tV})^*(dw) = \frac{d}{dt} \Big|_{t=0} d(e^{tV})^*(w) = d \left( \frac{d}{dt} \Big|_{t=0} (e^{tV})^*(w) \right) = d(\mathcal{L}_V w)$

Interior products w.r.t  $V \in \mathfrak{X}(M)$   
 $i_V: \Omega^k(M) \rightarrow \Omega^{k-1}(M)$

Prop 4.57: For any  $V, \exists!$  ... such that:

- (0)  $i_V$  is linear and, for  $k=1$ ,  $i_V(w) = w(V)$
- (1)  $i_V$  satisfies the derivation rule

Explicit formula:  
 $i_V(w) = w(V) - \sum_{i=1}^k w(x^1, \dots, x^i, [V, x^i], \dots, x^k)$

given by:  $i_V(x^1, \dots, x^k) = w(V, x^1, \dots, x^k)$

Ex 1:  $f = f \in C^\infty(M)$

$d_V(f) = \frac{d}{dt} \Big|_{t=0} (e^{tV})^*(f) \rightarrow d_V(f) = \frac{d}{dt} \Big|_{t=0} f(e^{tV}(p)) = (df)_p(V_p)$

Ex 2:  $\xi = X \in \mathfrak{X}(M)$   
 $d_V(X) = \frac{d}{dt} \Big|_{t=0} (e^{tV})^*(X) \Rightarrow d_V(X) = \frac{d}{dt} \Big|_{t=0} (X(e^{tV}(p)))$

Ex 3: On form  $\omega$  we have  $d_V(\omega) = \frac{d}{dt} \Big|_{t=0} (e^{tV})^*(\omega)$

Applying def of pull back:  
 $(e^{tV})^*(\omega) = \omega(e^{tV}(p))$

$M = \mathbb{R}^n$   
 $M = U$  domain of a chart -  
 Rk:  $d$  must be local (one can work in charts)

$\frac{d}{dt} \Big|_{t=0} f(t) \cdot g(t) = f_0 \left( \frac{d}{dt} \Big|_{t=0} g(t) \right) + \left( \frac{d}{dt} \Big|_{t=0} f(t) \right) \cdot g_0$

$\lim_{t \rightarrow 0} \frac{f(t)g(t) - f_0g_0}{t} = \lim_{t \rightarrow 0} \frac{(f(t)-f_0)g(t) + f_0(g(t)-g_0)}{t} = \lim_{t \rightarrow 0} \frac{f(t)-f_0}{t} g_0 + f_0 \lim_{t \rightarrow 0} \frac{g(t)-g_0}{t}$

Proof of 1:  
 $d_V(w \wedge \eta) = \frac{d}{dt} \Big|_{t=0} (e^{tV})^*(w \wedge \eta) = \frac{d}{dt} \Big|_{t=0} (e^{tV})^*(w) \wedge (e^{tV})^*(\eta) = d_V w \wedge \eta + w \wedge d_V \eta$

Cartan magic formula:  
 $d \circ i_V + i_V \circ d = \mathcal{L}_V$

proof: Does  $R_V$  satisfy (1), (2)?  
 (1)  $R_V \circ d = d \circ R_V$ ?  
 $d(i_V w + i_V d w) = d i_V w + d i_V d w$   
 $d i_V w + d d i_V w = d i_V w + d d i_V w$   
 $d i_V w + d d i_V w = d i_V w + d d i_V w$

Proof of (1):  
 $R_V(i_V w) = R_V(w(V)) = w(V)$   
 $d(i_V w) = d(w(V)) = (dw)(V) = \mathcal{L}_V w$   
 $d(i_V w) + i_V(d w) = \mathcal{L}_V w + i_V(d w) = d(i_V w)$   
 $d(i_V w) + i_V(d w) = d(i_V w) + i_V(d w)$   
 $=$  Leibniz again

Property  $d \circ d = 0$ :

(1) In  $\mathbb{R}^n, f \in C^\infty(\mathbb{R}^n)$ :  
 $df = \sum \frac{\partial f}{\partial x_i} dx_i, w = \sum \frac{\partial f}{\partial x_i} dx_i$   
 $d(df) = \sum d \left( \frac{\partial f}{\partial x_i} dx_i \right) = \sum \left( \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i + \frac{\partial f}{\partial x_i} d(dx_i) \right)$   
 $= \sum \sum \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i = \sum \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j = - \sum \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i = -d(df)$   
 therefore  $d(df) = 0$  beca  $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

$w = (x^2 y^2) dx \wedge dy \in \Omega^2(\mathbb{R}^2)$   $V = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$   
 $d(w) = d(x^2 y^2) dx \wedge dy = (2x y^2 dx + 2x^2 y dy) dx \wedge dy = 2x y^2 dx \wedge dx + 2x^2 y dy \wedge dx = -2x^2 y dx \wedge dy$

