—1—

Reminder: (U,x)-chart of $M \Rightarrow$ various basis at points $p \in M$:

$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ of T_pM

$(dx_1)_p, \dots, (dx_m)_p$ of T_p^*M — dual basis to — $(dx_i)_p\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$

wedges of k-such of $\Lambda^k T_p^*M$

$(dx_1)_p \wedge \dots \wedge (dx_m)_p$ of $\Lambda^m T_p^*M$ ($m = \dim M$!)

($\Omega^0(M) = C^\infty(M)$, $\Omega^k(M) = 0$ for $k > \dim M$)

& these are used to talk about coefficients (and smoothness) in charts of: vector fields, 1-forms, …, k-forms, …, m-forms ("TOP-FORMS")

Changing coordinates/charts: (U', x') another $\Rightarrow$

$$\Rightarrow \frac{\partial}{\partial x_i} = \sum_j \frac{\partial x_j'}{\partial x_i} \frac{\partial}{\partial x_j'}$$

$c = x' \circ x^{-1}$

(Hence, if $X \in \mathfrak{X}(M)$ has coefficients F_i in (U,x):

$$X = \sum_i F_i \frac{\partial}{\partial x_i} = \sum_j \underbrace{\left(\sum_i F_i \frac{\partial x_j'}{\partial x_i}\right)}_{F_j'} \frac{\partial}{\partial x_j'}$$

$$dx_i = \sum_j \frac{\partial x_i}{\partial x_j'} dx_j'$$

(Hence, for 1-forms ω:

$$\omega = \sum_i G_i\, dx_i = \sum_j \overbrace{\left(\sum_i G_i \frac{\partial x_i}{\partial x_j'}\right)}^{G_j'} dx_j'$$

—3—

Top-differential forms $\omega \in \Omega^m(M)$ ($m = \dim M$)

Locally: $\omega_p = F_x^\omega(x(p))\, (dx_1)_p \wedge \dots \wedge (dx_m)_p$

$U \xrightarrow{x} \Omega \subset \mathbb{R}^m$, $F_x^\omega: \Omega \to \mathbb{R}$ ← the coeff of ω in chart x

What if we change from x to x'? $\boxed{c = x' \circ x^{-1}}$

$$\omega = F_{x'}^\omega \circ x'\, dx_1' \wedge \dots \wedge dx_m' = F_x^\omega \circ x\, dx_1 \wedge \dots \wedge dx_m$$

$$F_{x'}^\omega \circ x' \left(\sum_{j_1} \frac{\partial x_1'}{\partial x_{j_1}} dx_{j_1}\right) \wedge \dots \wedge \left(\sum_{j_m} \frac{\partial x_m'}{\partial x_{j_m}} dx_{j_m}\right)$$

$$F_{x'}^\omega \circ x' \cdot \sum_{\sigma \in S_m} \frac{\partial x_1'}{\partial x_{\sigma(1)}} \cdots \frac{\partial x_m'}{\partial x_{\sigma(m)}}\, dx_{\sigma(1)} \wedge \dots \wedge dx_{\sigma(m)}$$

($dx_{\sigma(1)} \wedge \dots \wedge dx_{\sigma(m)} = \mathrm{sgn}(\sigma)\, dx_1 \wedge \dots \wedge dx_m$)

$$= F_{x'}^\omega \circ x' \underbrace{\left(\sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \frac{\partial x_1'}{\partial x_{\sigma(1)}} \cdots \frac{\partial x_m'}{\partial x_{\sigma(m)}}\right)}_{Jac(x' \circ x^{-1})} dx_1 \wedge \dots \wedge dx_m$$

Red $\Rightarrow$

$$\boxed{F_x^\omega = (F_{x'}^\omega \circ c) \cdot Jac(c)}$$

—4—

Problem: the absolute value …

ORIENTATIONS

Def:

- A volume form on M: any $\mu \in \Omega^m(M)$ s.t. $\mu_p \neq 0$ $(\forall)\, p \in M$ (μ_p is a basis of $\Lambda^m T_p^*M$ ⇓ any other $\omega \in \Omega^m(M)$ can be written as $\omega = f \cdot \mu$)
- Two volume forms μ, μ' are equivalent, $\mu \sim \mu'$ if $\mu' = f \cdot \mu$ with $f > 0$
- An orientation on M is an equivalence class w.r.t. $\sim$
- Given μ we denote by $\mathcal{O}_\mu$ the equiv class of μ, called the orientation induced by μ.

(Hence any orient on M is of type $\mathcal{O}_\mu$ & $\mathcal{O}_\mu = \mathcal{O}_{\mu'}$ iff $\mu' = f \cdot \mu$ with $f > 0$)

- ORIENTED MANIFOLD: manifold + a fixed orientation.
- Given $(M, \mathcal{O})$ oriented manifold, $\mathcal{O} = \mathcal{O}_\mu$, a chart (U, x) is called + oriented if $\mu_p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right) > 0$ $\forall\, p \in U$ ($\Longleftrightarrow$ $\mu = F\, dx_1 \wedge \dots \wedge dx_m$, F is > 0)

Main point: $(U, x), (U', x')$ + oriented charts $\Longrightarrow$ $c = x' \circ x^{-1}$ $\Rightarrow Jac(c) > 0$.

-1-

Reminder: (U,x)-chart of $M \Rightarrow$ various basis at points $p \in M$:

$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_m}\right)_p$ of T_pM

$(dx_1)_p, \dots, (dx_m)_p$ of T_p^*M dual basis to $\quad (dx_i)_p\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$

. . . . . .

wedges of k-such of $\Lambda^k T_p^*M$

. . . . . .

$(dx_1)_p \wedge \dots \wedge (dx_m)_p$ of $\Lambda^m T_p^*M$ $\quad$ ($m = \dim M$!)

$\left(\Omega^0(M) = C^\infty(M), \quad \Omega^k(M) = 0 \text{ for } k > \dim M \right)$

& these are used to talk about coefficients (and smoothness) in charts of: vector fields, 1-forms, ..., k-forms, ..., m-forms

"TOP-FORMS"

Changing coordinates/charts: (U', x') another $\Rightarrow$

$$\Rightarrow \frac{\partial}{\partial x_i} = \sum_j \frac{\partial x'_j}{\partial x_i} \frac{\partial}{\partial x'_j} \qquad (c = x' \circ x^{-1})$$

(Hence, if $X \in \mathfrak{X}(M)$ has coefficients F_i in (U,x):

$$X = \sum_i F_i \frac{\partial}{\partial x_i} = \sum_j \underbrace{\left(\sum_i F_i \frac{\partial x'_j}{\partial x_i} \right)}_{F'_j} \frac{\partial}{\partial x'_j}$$

$$dx_i = \sum_j \frac{\partial x_i}{\partial x'_j} dx'_j$$

(Hence, for 1-forms ω:

$$\omega = \sum_i G_i \, dx_i = \sum_j \overbrace{\left(\sum_i G_i \frac{\partial x_i}{\partial x'_j} \right)}^{G'_j} dx'_j$$

Problе...

Def...

(Hen...
&

Ma...

$c = x' \circ$ ...

Top-differential forms $\omega \in \Omega^m(M)$ -3- ($m = \dim M$)

Locally: $\omega_p = F^{\omega}_x(x(p)) \, (dx_1)_p \wedge \dots \wedge (dx_m)_p$ $\qquad U \subset \mathbb{R}^n$

Wh...

—2—

Standard integration in $\mathbb{R}^m$ (simple version): for each $\Omega \subseteq \mathbb{R}^m$ open

integration over Ω: $\int_\Omega : C_c^\infty(\Omega) \to \mathbb{R}$, $f \longmapsto \boxed{\int_\Omega f} = \int_\Omega f(x)\,dx = \int_\Omega f(x_1,\dots,x_m)\,dx_1 \dots dx_m$

$= \int_\Omega f(x_1,\dots,x_m)\,|dx_1 \dots dx$

with basic properties

(LIN) it is linear $\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$, $\int_\Omega \lambda\cdot f = \lambda \int_\Omega f$ for $\lambda \in \mathbb{R}$, $f, g \in C_c^\infty($

(CVF) change of variables formula: IF $c : \Omega \xrightarrow{\simeq} \Omega'$ DIFFEOMORFISM

$$\boxed{\int_{\Omega'} f = \int_\Omega (f\circ c)\cdot|\mathrm{Jac}(c)|} \quad (\forall)\, f \in C_c^\infty(\Omega').$$

$\Omega \xrightarrow{c} \Omega'$, $f\circ c : \Omega \to \mathbb{R}$, $f : \Omega' \to \mathbb{R}$

Reminder:

- $\mathrm{Jac}(c)$ is the smooth, real-valued function $x \longmapsto \det\left(\left(\frac{\partial c_i}{\partial x_j}(x)\right)_{i,j}\right)$
- $f \in C_c^\infty(\Omega) \iff f$ has compact support $\iff \exists\, K \subseteq \Omega$ compact s.t. $f = 0$ outside

$$\sum_j \ldots \partial x_j \qquad \omega = \sum_i G_i \, dx_i = \sum_j \left(\sum_i G_i \frac{\partial x_i}{\partial x'_j} \right) dx'_j$$

Top-differential forms $\omega \in \Omega^m(M)$ -3- $(m = \dim M)$

Locally: $\omega_p = F^{\omega}_{x}(x(p)) \, (dx_1)_p \wedge \ldots \wedge (dx_m)_p$

$U \xrightarrow{x} \Omega \subset \mathbb{R}^n$, $\quad F^{\omega}_{x} : \Omega \to \mathbb{R}$ ← the coeff of ω in chart x

What if we change from x to x'? $\quad \boxed{c = x' \circ x^{-1}}$

$$\omega = (F^{\omega}_{x'} \circ x') \, dx'_1 \wedge \ldots \wedge dx'_m = (F^{\omega}_{x} \circ x) \, dx_1 \wedge \ldots \wedge dx_m$$

$$F^{\omega}_{x'} \circ x' \cdot \left(\sum_{j_1} \frac{\partial x'_1}{\partial x_{j_1}} dx_{j_1} \right) \wedge \ldots \wedge \left(\sum_{j_m} \frac{\partial x'_m}{\partial x_{j_m}} dx_{j_m} \right)$$

$$= F^{\omega}_{x'} \circ x' \cdot \sum_{\sigma \in S_m} \frac{\partial x'_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial x'_m}{\partial x_{\sigma(m)}} \, dx_{\sigma(1)} \wedge \ldots \wedge dx_{\sigma(m)}$$

$dx_{\sigma(1)} \wedge \ldots \wedge dx_{\sigma(m)} \to \mathrm{sgn}(\sigma) \, dx_1 \wedge \ldots \wedge dx_m$

$$= F^{\omega}_{x'} \circ x' \underbrace{\left(\sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \frac{\partial x'_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial x'_m}{\partial x_{\sigma(m)}} \right)}_{\mathrm{Jac}(x' \circ x^{-1})} dx_1 \wedge \ldots \wedge dx_m$$

Red $\Rightarrow$

$$\boxed{F^{\omega}_{x} = (F^{\omega}_{x'} \circ c) \cdot \mathrm{Jac}(c)}$$

Problem: the absolute value ... [-4-] ORIENTATIONS

Def: • A volume form on M: any $\mu \in \Omega^m(M)$ s.t. $\mu_p \neq 0 \quad \forall p \in M$

• Two volume forms μ, μ' are equivalent, $\mu \sim \mu'$ if $\mu' = f \cdot \mu$ with $f > 0$

• An orientation on M is an equivalence class w.r.t. $\sim$

• Given μ we denote by $\mathcal{O}_\mu$ the equiv class of μ, called the orientation induced by μ.

(Hence any orient on M is of type $\mathcal{O}_\mu$ & $\mathcal{O}_\mu = \mathcal{O}_{\mu'}$ iff $\mu' = f \cdot \mu$ with $f > 0$)

• ORIENTED MANIFOLD: manifold + a fixed orientation.

• Given $(M, \mathcal{O})$ oriented manifold, $\mathcal{O} = \mathcal{O}_\mu$, a chart (U, χ) is called + oriented if $\mu_p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\right) > 0 \quad \forall p \in U$ $\left(\Longleftrightarrow \mu = F\, dx_1 \wedge \dots \wedge dx_m, \ F \text{ is } > 0\right)$

Main point: $(U, \chi), (U', \chi')$ + oriented charts $\Longrightarrow$

$c = \chi' \circ \chi^{-1} \Rightarrow \operatorname{Jac}(c) > 0$.

μ_p is a basis of $\Lambda^m T_p^* M$
$\Downarrow$
any other $\omega \in \Omega^m(M)$ can be written as $\omega = f \cdot \mu$ with $f \in$

- Given $(M,\mathcal{O})$ oriented ... is called + oriented if $\mu_p\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}\right)>0$

Main point: $(U,x),(U',x')$ + oriented charts $\Rightarrow$ $c=x'\circ x^{-1}$ $\Rightarrow$ $Jac(c)>0$.

$\omega\in\Omega_c^m(M)$ supported in U

—5—

Example 1: $M=\mathbb{R}^m$ standard volume form $\mu = dx_1\wedge\dots\wedge dx_m$.

Example 2: $M=S^1$: $\mu = y\,dx - x\,dy$ is a volume form because on S^1

$$\mu\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)=y^2+x^2=1\neq 0 \text{ at all points}$$

$x\,dx+y\,dy$ $\quad$ $x^2+y^2=1$ on S^1 $\quad$ $2x\,dx+2y\,dy=0$ on S^1

$\mu' = x\,dx+y\,dy = 0$

$\mu'' = y^2\,dx - xy\,dy$ problem at $p=(\pm1,0)$: $\mu''_p=0$

Example 3: $M=S^2$

$\mu = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$ a volume form. The standard volume form, and —"— orientation on S^2

At each $p\in S^2$: $\mu_p\in\Lambda^2T_p^*S^2$ i.e. $\mu_p: T_pS^2\times T_pS^2\to\mathbb{R}$.

I have to find $v_p\in T_pS^2$, $w_p\in T_pS^2$ s.t. $\mu_p(v_p,w_p)\neq 0$

$v=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$

$w=x\frac{\partial}{\partial z}-z\frac{\partial}{\partial x}$

Similarly S^m

Example 4: $M=\mathbb{P}^2$ is not orientable.

Rk: If M = orientable, connected $\Rightarrow$ M has precisely 2 orientations.

μ_0 $\quad$ $\mu = f\cdot\mu_0$, $f\neq 0$ $\quad$ $f>0\Rightarrow \mu\sim\mu_0$ $\quad$ $f<0\Rightarrow\mu\sim-\mu_0$

Sketch of proof:

Call $\omega\in\Omega_c^m(M)$ "small" ...

Step 1: Defining $\int_M\omega := \int_\Omega F_x ...$

: CVF

Step 2: Claim: any $\omega\in\Omega_c^m(M)$ (*) $\omega=\omega_1+\dots+\omega_k$

Step 3: Defining $\int_M\omega := \int_M\omega_1+\dots+\int_M ...$

Choose $K\subseteq M$ compact s.t. $\omega=0$ outside ...

Cover K by a finite number $\{U_1, ..., U_0\}$

Choose $\{\eta_0,\eta_1,\dots,\eta_k\}$ as before.

$\omega = 1\cdot\omega = (\eta_0+\eta_1+\dots+\eta_k)\cdot\omega = \eta_0 ...$

Step 3 uses twice partition ...

($m=\dim M$) $\quad$ F_x^ω the coeff of ω in chart x $\quad$ $c=x'\circ x^{-1}$ $\quad$ Red $\Rightarrow$ $F_x^\omega = (F_{x'}^\omega\circ c)\cdot Jac(c)$

Keep in mind: M = compact: $\Omega_c^m(M) = \Omega^m(M)$

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THEOREM: On any oriented manifold $(M, \mathcal{O})$, there exists and is unique

$$\int_M : \Omega_c^m(M) \longrightarrow \mathbb{R}$$

such that

(1) $\int_M$ is linear: $\int_M (\omega_1 + \omega_2) = \int_M \omega_1 + \int_M \omega_2$ etc.

(2) for any: → + ORIENTED CHART $\chi : U \to \Omega$ ($U \subseteq M$, $\Omega \subseteq \mathbb{R}^m$)

→ $\omega \in \Omega_c^m(M)$ supported in U

one has $\int_M \omega = \int_\Omega F_\chi^\omega$

HERE: ω = compactly supported ($\omega \in \Omega_c^m(M)$) in U $\iff$ K compact, $K \subseteq M$ s.t. $\omega_p = 0$ $(\forall) p \notin K$. ($K \subseteq U$)

-7-

Sketch of proof:

Call $\omega \in \Omega_c^m(M)$ "small" if it is supported in the domain U of some chart χ.

Step 1: Defining $\int_M \omega := \int_\Omega F_\chi^\omega$ where $\chi : U \to \Omega$ is as above (+ oriented) does not depend on the choice of χ

Ex: CVF

Step 2: Claim: any $\omega \in \Omega_c^m(M)$ can be written as

(*) $\omega = \omega_1 + \ldots + \omega_k$ with ω_i - small.

Step 3: Defining $\int_M \omega := \int_M \omega_1 + \ldots + \int_M \omega_k$ does not depend on the writing (*)

Choose $K \subseteq M$ compact s.t. $\omega = 0$ outside K

Cover K by a finite number $\{U_1, \ldots, U_k\}$ of domains of charts. $U_0 := M \setminus K$ — open cover of M

Choose $\{\eta_0, \eta_1, \ldots, \eta_k\}$ as before.

$$\omega = 1 \cdot \omega = (\eta_0 + \eta_1 + \ldots + \eta_k) \cdot \omega = \underbrace{\eta_0 \omega}_{0} + \underbrace{\eta_1 \omega}_{\omega_1 \text{ small}} + \ldots + \underbrace{\eta_k \omega}_{\omega_k \text{ small}}$$

Step 3 uses twice partitions

☒

ARTITIONS OF UNITY: M = manifold

For any open cover $\{U_0, \dots, U_k\}$ of M

$\exists$ a partition of unity $\{\eta_0, \dots, \eta_k\}$ subordinate to this cover, i.e., $\eta_i : M \to [0,1]$

smooth functions satisfying:

① $\sum_{i=0}^{k} \eta_i = 1$ (where 1 is the constant function $1 \in \mathbb{R}$)

② $\mathrm{supp}(\eta_i) \subseteq U_i$ (i.e.: $\exists A \subseteq M$ closed such that $A \subseteq U_i$, $\eta = 0$ outside A)

$(\forall f \in C^\infty(M)) : \sum_{i=0}^{k} \eta_i f = f$

"small"

is called + oriented if $\mu_p\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}\right) > 0$

Main point: $(U,x),(U',x')$ + oriented charts $\Rightarrow$ $c = x'\circ x^{-1} \Rightarrow Jac(c) > 0$.

-8-

Consequence: for any M compact & a volume form μ we can talk about:

$$vol_\mu(M) := \int_M \mu > 0 \text{ defined using } \mathcal{O} = \mathcal{O}_\mu.$$

Ex: $\mu = ydx - xdy$, $vol_\mu(S^1) = \int_{S^1}(xdy - ydx) = \dots = 2\pi$

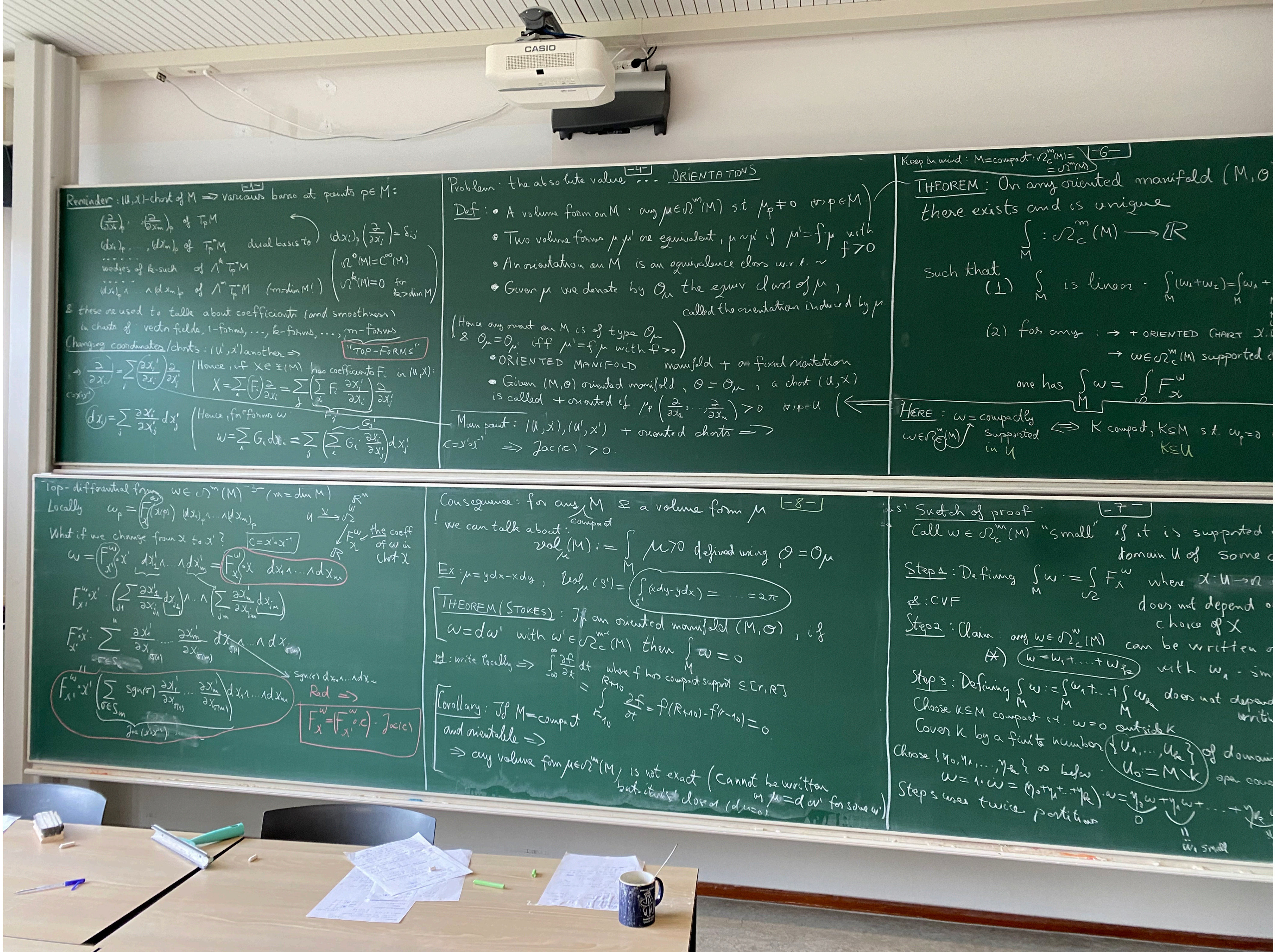
THEOREM (STOKES): If an oriented manifold $(M,\mathcal{O})$, if $\omega = d\omega'$ with $\omega' \in \Omega_c^{m-1}(M)$ then $\int_M \omega = 0$

Pf: write locally $\Rightarrow \int_{-\infty}^{\infty} \frac{\partial f}{\partial t} dt$ where f has compact support $\subseteq [r,R]$

$$= \int_{r-10}^{R+10} \frac{\partial f}{\partial t} = f(R+10) - f(r-10) = 0.$$

Corollary: If M = compact and orientable $\Rightarrow$

$\Rightarrow$ any volume form $\mu \in \Omega^m(M)$ is not exact (cannot be written as $\mu = d\omega'$ for some ω') but it is closed ($d\mu = 0$)


CASIO
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dual basis to
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& these are used to talk about coefficients (and smoothness)
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Two volume forms μ, μ' are equivalent, μ ~ μ' if μ' = f·μ with f > 0
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Such that (1) ∫_M is linear
(2) for any: → + ORIENTED CHART
→ ω ∈ Ω_c^m(M) supported
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HERE: ω = compactly supported in U ⇔ K compact, K ⊆ M s.t. ω_p = 0
Top-differential forms: ω ∈ Ω^m(M) (m = dim M)
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THEOREM (STOKES): If an oriented manifold (M, O), if ω = dω' with ω' ∈ Ω_c^{m−1}(M) then ∫_M ω = 0
Corollary: If M = compact and orientable ⇒
⇒ any volume form μ ∈ Ω^m(M) is not exact (cannot be written as μ = dω' for some ω') but it is closed (dμ = 0)
Sketch of proof
Call ω ∈ Ω_c^m(M) "small" if it is supported domain U of some
Step 1: Defining ∫_M ω := ∫ F_x^ω where x: U → does not depend on choice of x
Step 2: Claim any ω ∈ Ω_c^m(M) can be written
Step 3: Defining does not depend
Choose K ⊆ M compact s.t. ω = 0 outside K
Cover K by a finite number
Step 3 uses twice partition