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Reminder: differential forms $\omega \in \Omega^k(M)$ and operations:

- **wedges**: $\Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$, $(\omega, \eta) \longmapsto \omega \wedge \eta$
  (associative & graded commutative)
- **deRham** $d: \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, $\omega \longmapsto d(\omega)$
  (graded derivation rule, $\boxed{d \circ d = 0}$)
  $d(\omega \wedge \eta) = d\omega \wedge \eta \pm \pm \omega \wedge d\eta$
- **pull-backs**: $F: M \to N$ induces $F^*: \Omega^k(N) \to \Omega^k(M)$
  $k=0$: $F^*(f) = f \circ F$
  compatible with wedge: $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$
  compatible with DeRham: $F^*(d(\omega)) = d(F^*(\omega))$.
- **volume forms**: $\mu \in \Omega^m(M)$ s.t. $\mu_p \neq 0$ $(\forall)$ $p \in M$.

Ex: $\mu = dx_1 \wedge \ldots \wedge dx_m \in \Omega^m(\mathbb{R}^m)$

Ex: $\mu_{S^m} = \sum_{i=0}^{m} (-1)^i x_i \, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_m \in \Omega^m(S^m)$

Last time: $m=2$, $\mu_p \neq 0$ $\forall p$.

Rk: Given $\mu \Rightarrow$ any other $\omega \in \Omega^m(M)$ is $f \cdot \mu$ for some f

Rearrange the ...
(fix an m-d...
$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d}$ ...

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- **orientations** on M: main points:
  → each volume form μ gives rise to an orientation $\mathcal{O}_\mu$
  → for two volume forms μ, μ' one has: $\mathcal{O}_\mu = \mathcal{O}_{\mu'} \Leftrightarrow \mu' = f\mu$ with $f > 0$.

... and orientations allow us to talk about oriented charts.

- given $\boxed{(M, \mathcal{O}) \text{ oriented manifold}} \Rightarrow$ defined $\boxed{\int_M : \Omega^m_c(M) \to \mathbb{R}}$
  - linear
  - for ω = "small": usual $\int$ in $\mathbb{R}^m$
- given $\boxed{(M, \mu) \text{ manifold with a volume form}}$ (hence also an orientation)
  $\boxed{\mathrm{vol}_\mu(M) := \int_M \mu}$ (using $\mathcal{O}_\mu$) > 0

Thm (Stokes): $(M, \mathcal{O})$ = oriented $\Rightarrow \int_M \omega = 0$ for all ω of type $\omega = d\eta$

Corollary: When M = compact, any volume form $\mu \in \Omega^m(M)$ is closed (i.e. $d\mu = 0$) but not exact (i.e. not of type $d\eta$)

tions :

$\to \omega\wedge\eta$

$\omega)$

$M)$

$F^*(\eta)$

$(\omega))$.

M:

st time

$=2$

$p \neq 0 \ \forall p$.

some f

Rearrange the information
(fix an m-dimensional M).

... "De Rham complex of M":

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \to 0$$

$$H^k(M) := \Omega^k_{cl}(M) / \Omega^k_{ex}(M)$$

$\Omega^k_{cl}(M) = \{\omega \in \Omega^k(M) : d\omega = 0\}$ (closed k-forms)

$\cup$ $\quad d \circ d = 0$

$\Omega^k_{ex}(M) = \{\omega \in \Omega^k(M) : \omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)\}$ (exact k-forms)

i.e.: $H^k(M) = \{[\omega] \,/\, \omega \in \Omega^k(M), d\omega = 0\}$ where $[\omega] = [\omega'] \Leftrightarrow \omega - \omega' = d\eta$ for some η

The k-th De Rham cohom. space of M (a vector space: $[\omega] + [\omega'] = [\omega + \omega']$ etc)

Rk: M = compact $\Rightarrow$ all $H^k(M)$ are finite dimensional.

In this case the Euler characteristic of M is $\chi(M) = \sum_k (-1)^k \dim_{\mathbb{R}}(H^k(M))$

Rk: The wedge induces a similar operation

$$H^k(M) \times H^\ell(M) \longmapsto H^{k+\ell}(M), \quad [\omega] \wedge [\eta] := [\omega \wedge \eta]$$

$[\omega] = [\omega']$, $[\omega\wedge\eta] = [\omega' \wedge \eta]$

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Rearrange the information (fix an m-dimensional M) ... "De Rham complex of M":

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^m(M) \xrightarrow{d} 0 \to 0$$

$H^k(M) := \Omega^k_{cl}(M) / \Omega^k_{ex}(M)$

- $\Omega^k_{cl}(M) = \{\omega \in \Omega^k(M) : d\omega = 0\}$ (closed k-forms)
- $\cup$ $\quad d \circ d = 0$
- $\Omega^k_{ex}(M) = \{\omega \in \Omega^k(M) : \omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)\}$ (exact k-forms)

i.e.: $H^k(M) = \{[\omega] \,/\, \omega \in \Omega^k(M), d\omega = 0\}$ where $[\omega] = [\omega'] \Leftrightarrow \omega - \omega' = d\eta$ for some η

The k-th DeRham cohom space of M (a vector space: $[\omega] + [\omega'] = [\omega + \omega']$ etc.)

Thm (Poincaré Lemma): $H^k(\mathbb{R}^m) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & k \neq 0 \end{cases}$

Corollary: $(\forall)$ M, the maps $i_0, i_1 : M \to M \times \mathbb{R}$, $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$ induce the same map $i_0^* = i_1^* : H^k(M \times \mathbb{R}) \to H^k(M)$, $[\omega]$

-4- $\quad t \in \mathbb{R}$

Ex: $m = 1$: $0 \longrightarrow \Omega^0(\mathbb{R}) \xrightarrow{d} \Omega^1(\mathbb{R}) \longrightarrow 0$

$\Omega^0(\mathbb{R}) = C^\infty(\mathbb{R})$, $\Omega^1(\mathbb{R}) = \{g \cdot dt \,/\, g \in C^\infty(\mathbb{R})\}$

$f \longmapsto df = f'(t)dt$

Key:
- $f' = 0 \Rightarrow f = $ constant
- $\forall g \; \exists f$ s.t. $f' = g$

$m = 2$: exercise ... for general m, proceed inductively $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ ($\ni t$) do the same

Inspect the proof $\Rightarrow$ one can replace $\mathbb{R}^{m-1}$ by another manifold M

I.e. we look at products $M \times \mathbb{R}$ $\Rightarrow$ corollary

Look at arbitrary closed form on $M \times \mathbb{R}$: $\omega = \alpha_t + \beta_t \wedge dt$, $\quad \alpha_t \in \Omega^k(M)$, $\beta_t \in \Omega^{k-1}(M)$

$i_0 : M \to M \times \mathbb{R}$, $x \mapsto (x, 0)$

its diff. $d_{M \times \mathbb{R}}(\omega) = d_M \alpha_t - \dot{\alpha}_t \wedge dt + d\beta_t \wedge dt \Rightarrow \dot{\alpha}_t = d\beta_t$
$\quad = 0$

$\alpha_1 = \alpha_0 + \int_0^1 d\beta_t \, dt$

$\alpha_1 - \alpha_0 = d\gamma$, $\gamma = \int_0^1 \beta_t \, dt$

But $i_0^*(\omega) = \alpha_0$, $i_1^*(\omega) = \alpha_1$. $\quad [\alpha_0] = [\alpha_1]$

Functoriality of cohomology

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Any smooth $F: M \to N$ induces linear maps $F^*: H^k(N) \to H^k(M)$, $[\omega] \mapsto [F^*(\omega)]$

If $[\omega] = [\omega']$, $[F^*(\omega)] \overset{?}{=} [F^*(\omega')]$

Rk: Chain $\Rightarrow \begin{cases} G^* \circ F^* = (F \circ G)^* \\ Id_M^* = Id \end{cases} \Longrightarrow$ any diffeo F induces an isomorphism in cohomologie

Thm: (1) If $F_0, F_1: M \to N$ are homotopic ($F_0 \sim F_1$), then $F_0^*, F_1^*: H^k(N) \to H^k(M)$ coincide.

($F_0 \sim F_1$: $\exists\, H: M \times \mathbb{R} \to N$ smooth s.t. $\begin{cases} H(x,0) = F_0(x) \\ H(x,1) = F_1(x) \end{cases}$, $H(\cdot, t): M \to N$ for each t)

(2) If $F: M \to N$ is a homotopy equivalence (h.e.), then $F^*: H^k(N) \to H^k(M)$ is an isomorphism.

(h.e.: $\exists\, G: N \to M$ s.t. $F \circ G \sim Id_N$, $G \circ F \sim Id_M$)

Say: M and N are homotopic equiv if $\exists$ such an h.e. F

Ex3: If S^m has a nowhere vanishing vector field, we claim that $\begin{cases} id: S^m \to S^m \\ \tau: S^m \to S^m, \ \tau(x) = -x \end{cases}$ are homotopic.

$t=0$: x

$t=\frac{1}{2}$: V_x

$t=1$: $-x$

Pf: Let V be such a vector field $V: S^m \to \mathbb{R}^{m+1}$, $(V_p \perp p)$ $(\forall p \in S^m)$, $V_p \neq 0$

$$H(x,t) = \cos(\pi t) \cdot x + \sin(\pi t) \cdot \frac{V_x}{\|V_x\|} \in S^m$$

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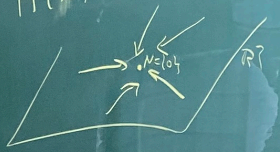
Ex1: $M = \mathbb{R}^m$, $N = \mathbb{R}^0 = \{0\}$ are h.e.

$\mathbb{R}^m \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{R}^0 = \{0\}$, $G(0) = p$

$F \circ G: \{0\} \to \{0\}$, $0 \mapsto 0$ Id

$G \circ F: \mathbb{R}^m \to \mathbb{R}^m$, $x \mapsto 0$. why homotopic to Id?

$H(x,t) = tx$ ($t=0$: 0; $t=1$: x)



Ex2: $M = \mathbb{R}^m \setminus \{0\}$, $N = S^{m-1}$ are h.e.

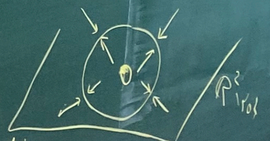
$\mathbb{R}^m_{\setminus\{0\}} \underset{G}{\overset{F}{\rightleftarrows}} S^{m-1}$

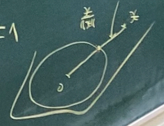
$G(x) = x$, $F(x) = \frac{x}{\|x\|}$

$F \circ G: S^{m-1} \to S^{m-1}$ is $Id_{S^{m-1}}$

$G \circ F: \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$, $x \mapsto \frac{x}{\|x\|}$ homotopic to Id by: $H(x,t) = t\frac{x}{\|x\|} + (1-t)x \in \mathbb{R}^m \setminus \{0\}$

($t=0$: $\frac{x}{\|x\|}$; $t=1$: x)





$H^k(M) \to H^k(M)$ coincide

$F_0 \sim F_1$ $\exists\, H: M \times \mathbb{R} \to N$ smooth s.t. $H(x,0) = F_0(x)$, $H(\cdot,t): M \to N$

$[F^*(\omega')]$

Pf of hairy ball theorem

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Assume $\exists$ a nowhere vanishing v.f. on S^m

$\Downarrow$

id and $\tau: S^m \to S^m$ are homotopic

$\Downarrow$

$\tau^* = id : H^m(S^m) \to H^m(S^m)$

Ψ

look at $[\mu] \overset{\tau^*}{\longmapsto} [\tau^*(\mu)] = (-1)^{m+1}[\mu]$

$\Bigg\} \Rightarrow [\mu] = (-1)^{m+1}[\mu]$

$[\mu]$ is non-zero!

$\Downarrow$

$(-1)^{m+1} = 1$

$\Downarrow$

$m = \text{odd}$. ▢

$\mu = \sum \pm x_i \, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_m$

$\tau^*(\mu) = \sum \pm (-x_i)\, d(-x_0) \wedge \ldots \wedge \hat{\ } \wedge \ldots \wedge d(-x_m)$ ($m+1$)

S^{2k+1}

$x_0 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2} + \ldots + x_{2k} \frac{\partial}{\partial x_{2k+1}} - x_{2k+1} \frac{\partial}{\partial x_{2k}}$

$F^*(\mu)$

$F(x_0, \ldots, x_m) = (-x_0, \ldots, -x_m)$

$\int_0 \beta_t \, dt$