$\mathbb{R}^n$

$$T_p\mathbb{R}^n \xrightarrow[\sim]{\text{standard}_p} \mathbb{R}^n$$

$\mathbb{R}^n$

$$\begin{array}{ccc} \left(\frac{\partial}{\partial x_i}\right)_p & \longmapsto & e_i \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ T_p\mathbb{R}^n & \xrightarrow[\sim]{\text{standard}_p} & \mathbb{R}^n \end{array}$$

$\mathbb{R}^n$

$$\begin{array}{ccc} \left(\frac{\partial}{\partial x_i}\right)_p & \longmapsto & e_i \\ \cap & & \cap \\ \frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n & \xrightarrow[\sim]{\text{standard}_p} & \mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \ldots\right) \end{array}$$

$$= \sum_i \frac{d\gamma_i}{dt}(0) \cdot e_i$$

$\mathbb{R}^n$

$$\sum_i \underbrace{\frac{d\gamma_i}{dt}(0)} \left(\frac{\partial}{\partial x_i}\right)_p \qquad \left(\frac{\partial}{\partial x_i}\right)_p \longmapsto e_i$$

$$\frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n \xrightarrow[\sim]{\text{standard}_p} \mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \ldots\right)$$

$$= \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)} \cdot e_i$$

$$\begin{array}{ccccc}
\sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\left(\frac{\partial}{\partial x_i}\right)_p & \left(\frac{\partial}{\partial x_i}\right)_p & \longmapsto & e_i & \\
\shortparallel & \cap & & \cap & \\
\frac{d\gamma}{dt}(0) \in & T_p\mathbb{R}^n & \xrightarrow[\sim]{\text{standard}_p} & \mathbb{R}^n & \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \dots\right) = \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\, e_i
\end{array}$$

$$\begin{tikzcd}
\mathbb{R}^n \\
M \arrow[u, hook]
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathbb{R}^n \\
M \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
\left(\frac{\partial}{\partial x_i}\right)_p \arrow[r, mapsto] \arrow[d, phantom, "\cap" rotate=-90] & e_i \arrow[d, phantom, "\cap" rotate=-90] \\
T_p\mathbb{R}^n \arrow[r, "\mathrm{standard}_p", "\sim"'] & \mathbb{R}^n \\
T_pM \arrow[u, hook] \arrow[r, "\sim"] & T_pM \arrow[u, hook]
\end{tikzcd}
$$

$$\sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\left(\frac{\partial}{\partial x_i}\right)_p = \frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n$$

$$\mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \dots\right) = \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)} \cdot e_i$$

$$\begin{tikzcd} \mathbb{R}^n \\ M \arrow[u, hook] \end{tikzcd}$$

$$\sum_i \underbrace{\frac{d\gamma^e_i}{dt}(0)} \left(\frac{\partial}{\partial x_i}\right)_p \qquad \left(\frac{\partial}{\partial x_i}\right)_p \longmapsto e_i$$

$$\begin{tikzcd} \frac{d\gamma^e}{dt}(0) \in T_p\mathbb{R}^n \arrow[r, "\text{standard}_p", "\sim"'] & \mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \dots\right) = \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)} \cdot e_i \\ T_pM \arrow[u, hook, "\text{if } \gamma \in \text{Curves}_p(M)"] \arrow[r, "\sim"] & T_pM \arrow[u, hook] \end{tikzcd}$$

$$\sum_i \underbrace{\frac{d\gamma^i}{dt}(0)}\left(\frac{\partial}{\partial x_i}\right)_p = \frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n$$

$$\begin{tikzcd}
\left(\frac{\partial}{\partial x_i}\right)_p \arrow[rr, mapsto] \arrow[d, phantom, "\cap"] & & e_i \arrow[d, phantom, "\cap"] \\
T_p\mathbb{R}^n \arrow[rr, "\mathrm{Standard}_p", "\sim"'] & & \mathbb{R}^n \\
T_pM \arrow[u, hook] \arrow[rr, "\sim"] & & T_pM \arrow[u, hook]
\end{tikzcd}$$

$$\mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \dots\right) = \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\, e_i$$

if $\gamma \in \mathrm{Curves}_p(M)$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[dr, "h"] & \\
M \arrow[u, hook] & \mathbb{R}^k
\end{tikzcd}$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[ddr, "h"] & \\ M \arrow[u, hook] & \\ & \mathbb{R}^k \end{tikzcd}$$

$$\sum_i \underbrace{\frac{d\gamma^i}{dt}(0)}\left(\frac{\partial}{\partial x_i}\right)_p = \frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n \ni \left(\frac{\partial}{\partial x_i}\right)_p \longmapsto e_i \in \mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \dots\right) = \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)} \cdot e_i$$

if $\gamma \in \mathrm{Curves}_p(M)$

$$\begin{tikzcd} T_p\mathbb{R}^n \arrow[rr, "\mathrm{standard}_p", "\sim"'] \arrow[ddr, "(dh)_p"] & & \mathbb{R}^n \arrow[ddr, "(dh)_p"] & \\ T_pM \arrow[u, hook] \arrow[rr, "\sim"] & & T_pM \arrow[u, hook] & \\ & T_{h(p)}\mathbb{R}^k \arrow[rr, "\sim"] & & \mathbb{R}^k \end{tikzcd}$$

$$\sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\left(\frac{\partial}{\partial x_i}\right)_p \qquad \left(\frac{\partial}{\partial x_i}\right)_p \longmapsto e_i$$

$$\frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n \xrightarrow[\sim]{\text{standard}_p} \mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \dots\right) = \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)} \cdot e_i$$

if $\gamma \in \mathrm{Curves}_p(M)$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[ddr, "h"] & & T_p\mathbb{R}^n \arrow[rr, "\sim"] \arrow[ddr, "(dh)_p"] & & \mathbb{R}^n \arrow[ddr, "(dh)_p"] & \\
M \arrow[u, hook] & & T_pM \arrow[u, hook] \arrow[rr, "\sim"] & & T_pM \arrow[u, hook] & \\
& \mathbb{R}^k & & T_{h(p)}\mathbb{R}^k \arrow[rr, "\sim"] & & \mathbb{R}^k
\end{tikzcd}$$

$$(dh)_p(e_i) = \frac{\partial h}{\partial x_i}(p)$$

$$= \left(\frac{\partial h_1}{\partial x_i}(p), \dots, \frac{\partial h_k}{\partial x_i}(p)\right)$$

$$= \sum_j \frac{\partial h_j}{\partial x_i}(p) \cdot e_j$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[ddr, "h"] & \\
M \arrow[u, hook] & \\
& \mathbb{R}^k
\end{tikzcd}$$

$$\sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\left(\frac{\partial}{\partial x_i}\right)_p = \frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n, \qquad \left(\frac{\partial}{\partial x_i}\right)_p \longmapsto e_i, \qquad e_i \in \mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \dots\right) = \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\cdot e_i$$

$$\begin{tikzcd}
T_p\mathbb{R}^n \arrow[rr, "\text{standard}_p", "\sim"'] \arrow[dd, "(dh)_p"'] & & \mathbb{R}^n \arrow[dd, "(dh)_p"] \\
T_pM \arrow[u, hook, "\text{if } \gamma \in \mathrm{Curves}_p(M)"] \arrow[rr, "\sim"] & & T_pM \arrow[u, hook] \\
T_{h(p)}\mathbb{R}^k \arrow[rr, "\sim"] & & \mathbb{R}^k
\end{tikzcd}$$

$$\begin{aligned}
(dh)_p(e_i) &= \frac{\partial h}{\partial x_i}(p) \\
&= \left(\frac{\partial h_1}{\partial x_i}(p), \dots, \frac{\partial h_k}{\partial x_i}(p)\right) \\
&= \sum_j \frac{\partial h_j}{\partial x_i}(p) \cdot e_j
\end{aligned}$$

$$\boxed{(dh)_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right) = \sum_j \frac{\partial h_j}{\partial x_i}(p)\left(\frac{\partial}{\partial x_j}\right)_p}$$

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "h"] & & \mathbb{R}^k \\
M \arrow[u, hook] \arrow[rr, "h' = h|_M"] & & N \arrow[u, hook]
\end{tikzcd}
$$

$$\sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\left(\frac{\partial}{\partial x_i}\right)_p \qquad \left(\frac{\partial}{\partial x_i}\right)_p \longmapsto e_i$$

$$\frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n \qquad \mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \dots\right) = \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\, e_i$$

if $\gamma \in Curves_p(M)$

$$
\begin{tikzcd}
T_p\mathbb{R}^n \arrow[rr, "\text{standard}_p", "\sim"'] \arrow[ddrr, "(dh)_p"] & & \mathbb{R}^n \arrow[ddrr, "(dh)_p"] & & \\
T_pM \arrow[u, hook] \arrow[rr, "\sim"] & & T_pM \arrow[u, hook] & & \\
& & T_{h(p)}\mathbb{R}^k \arrow[rr, "\sim"] & & \mathbb{R}^k
\end{tikzcd}
$$

$$(dh)_p(e_i) = \frac{\partial h}{\partial x_i}(p)$$
$$= \left(\frac{\partial h_1}{\partial x_i}(p), \dots, \frac{\partial h_k}{\partial x_i}(p)\right)$$
$$= \sum_j \frac{\partial h_j}{\partial x_i}(p)\, e_j$$

$$\boxed{(dh)_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right) = \sum_j \frac{\partial h_j}{\partial x_i}(0)\left(\frac{\partial}{\partial x_j}\right)_p}$$

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[ddr, "h"] & \\
M \arrow[u, hook] \arrow[dr, "h' = h|_M"'] & \\
& \mathbb{R}^k \\
& N \arrow[u, hook]
\end{tikzcd}
$$

$\sum_i \underbrace{\frac{d\gamma_i}{dt}(0)}\left(\frac{\partial}{\partial x_i}\right)_p$ " $\left(\frac{\partial}{\partial x_i}\right)_p \longmapsto e_i$

$\frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n \xrightarrow[\sim]{\text{standard}_p} \mathbb{R}^n \ni \frac{d\gamma}{dt}(0) = \left(\frac{d\gamma_1}{dt}(0), \dots\right) = \sum_i \underbrace{\frac{d\gamma_i}{dt}(0)} \cdot e_i$

if $\gamma \in \mathrm{Curves}_p(M)$

$$
\begin{tikzcd}
T_p\mathbb{R}^n \arrow[rr, "\text{standard}_p", "\sim"'] \arrow[ddr, "(dh)_p"] & & \mathbb{R}^n \arrow[dd, "(dh)_p"] \\
T_pM \arrow[u, hook] \arrow[r, "\sim"] \arrow[ddr, "(dh')_p"'] & T_pM \arrow[ur, hook] \arrow[ddr, "(dh')_p"] & \\
& T_{h(p)}\mathbb{R}^k \arrow[r, "\sim"] & \mathbb{R}^k \\
& T'_{h'(p)}N \arrow[u, hook] \arrow[r, "\sim"] & T_{h'(p)}N \arrow[u, hook]
\end{tikzcd}
$$

$$(dh)_p(e_i) = \frac{\partial h}{\partial x_i}(p) = \left(\frac{\partial h_1}{\partial x_i}(p), \dots, \frac{\partial h_k}{\partial x_i}(p)\right) = \sum_j \frac{\partial h_j}{\partial x_i}(p) \cdot e_j$$

$$\boxed{(dh)_p\left(\left(\frac{\partial}{\partial x_i}\right)_p\right) = \sum_j \frac{\partial h_j}{\partial x_i}(0)\left(\frac{\partial}{\partial x_j}\right)_p}$$