MANIFOLDS(WISB342) EXAM, NOVEMBER 7, 2023

Exercise 1 (1pt)**.** Show that the composition of two immersions is again an immersion.

Exercise 2 (1pt)**.** If $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, show that $\mathcal{L}_X(f) = 0$ if and only if f is constant along the integral curves of X, i.e., for any integral curve $\gamma : I \to M$ of X, $f \circ \gamma$ is constant.

Exercise 3. Let M be an arbitrary manifold. Recall that a differential form ω on M is called closed if $d\omega = 0$, and it is called exact if there exists a form η such that $\omega = d\eta$.

(a) (0.5pt) Show that the wedge product of two closed forms is again closed.
(b) (0.5pt) Show that for any form ω of even degree, $\omega \wedge d\omega$ is closed.
(c) (0.5pt) In (b), is it true that $\omega \wedge d\omega$ is actually exact?
(d) (0.5pt) Is the conclusion from (b) true also for forms of odd degrees?

Exercise 4 (6.5pt)**.** Consider

$$M := \{(x, y, z, u) \in \mathbb{R}^4 : (x^2 + y^2 + z^2 - 1)^2 + (xy - u)^2 = 0\}$$

(a) (0.5pt) Show that M is a 2-dimensional embedded submanifold of $\mathbb{R}^4$
(b) (0.5pt) Show that the following define vector fields on M:

$$V_1 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + yz\frac{\partial}{\partial u}$$
$$V_2 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + xz\frac{\partial}{\partial u}$$

(c) (0.5pt) Compute $[V_1, V_2]$
(d) (1pt) Compute the flow of the vector field on M given by

$$V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (y^2 - x^2)\frac{\partial}{\partial u}.$$

(e) (1pt) Consider the form

$$\omega = 2xdy \wedge dz + zdx \wedge dy - du \wedge dz \in \Omega^2(M)$$

and compute $\omega(V_1, V_2)$.
(f) (0.5pt) Show that ω is closed.
(g) (1pt) compute $i_V(\omega)$ and $L_V(\omega)$.
(h) (0.5p) Show that ω is nowhere zero.
(i) (1pt) Find a diffeomorphism $F : S^2 \to M$ and compute the pull-backs by F of ω, V_1, V_2 and V.

Exercise 5 (2pt)**.** Let M be a manifold. Recall that, when X is a vector field, ϕ_X^t denotes the flow of X. Show that for any vector field X one has

(a) $\phi_{cX}^t = \phi_X^{ct}$ for any constant $c \in \mathbb{R}$ and all t for which one of the sides is defined.
(b) if M is compact then $\phi_{5X}^t\left(\phi_X^{-t}(p)\right) = \phi_{2X}^t\left(\phi_X^{2t}(p)\right)$ for all $p \in M$ and all $t \in \mathbb{R}$.

NOTES:

- Please write down your name and the student no
- The order of the exercises is completely unrelated to their difficulty.
- The mark for the exam is the minimum between 10 and the total number of points you collect.
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please:
  - include all your computations that support your claims. E.g. in items (c), (d), (e), (g) of Exercise 4 (and similar items where you have to compute something) please do not just write down the final result (that will not count!) but explain how you got it/provide the details.
  - in items (c) and (d) of Exercise 3 please do not just answer with "yes" or "no" (if it "yes" provide a proof, if it is "no" provide a proof as well (e.g. find a counterexample)). Also, in item (i) of Exercise 4, please explain (i.e. prove) why the F that you describe is a diffeomorphism.