

HOMEWORK 2 (SEPTEMBER 20, 2023)

Exercise 1. Consider the function

$$f : \mathbb{P}^1 \rightarrow S^1, \quad f([x : y]) = \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{2xy}{x^2 + y^2} \right)$$

and do the following:

- (1) show that f is well-defined;
- (2) using the atlas on \mathbb{P}^1 that we discussed (two charts) and the atlas on S^1 given by the stereographic projection with respect to the north and south pole (again two charts), please write down explicitly the resulting representations of f , as in Definition 2.28 (in total $2 \times 2 = 4$ possibilities).
- (3) in each case, please make sure that you also write down explicitly the domain of those representations. Deduce that f is smooth;
- (4) show that f is actually a diffeomorphism;
- (5) in general, for any projective space \mathbb{P}^m , there is the so-called Hopf map, simply given by

$$H : S^m \rightarrow \mathbb{P}^m, \quad H(x_0, x_1, \dots, x_m) = [x_0 : x_1 : \dots : x_m]$$

or, more geometrically: it sends $P \in S^m$ to the line \overline{OP} through the origin and P . You are not asked to prove anything about this map (yet). For $m = 1$ compute $f \circ H$ and, using that, describe f on a picture (think of how you would describe the map to a colleague, on the blackboard, without being allowed to use formulas).

- (6) not part of the homework, but nevertheless instructive: for general m , what can you say about H ? Is it smooth? Is it a diffeomorphism? Is it a submersion or immersion?
(Just to make sure: for the last item, if you decide to do it, you will get “only feedback” and not points!)