# HOMEWORK 3 (SEPTEMBER 27, 2023)

The projective spaces $\mathbb{P}^n$ were defined "abstractly" and the natural question is: can they be seen inside some Euclidean space. Or, more precisely, can one find a smooth embedding of $\mathbb{P}^n$ into some Euclidean space (of some dimension, possibly very large)? The homework is related to this question for $n = 2$.

We start with some remarks that you may find useful, but I would like to make it clear: *they are not really necessary for the homework in the technical sense; instead, they are meant as insightful comments that may improve understanding and make you be more at ease working with the projective spaces.*

*Remark* 1. The question of embedding $\mathbb{P}^n$ in some Euclidean space should be clear now for $n = 1$: indeed, the previous homework can be re-interpreted as providing a smooth embedding of $\mathbb{P}^1$ into $\mathbb{R}^2$, namely

$$\mathbb{P}^1 \to \mathbb{R}^2, \quad [x : y] \mapsto \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{2xy}{x^2 + y^2}\right)$$

(with image the circle $S^1$). Of course, you could (and should) have wondered first of whether there is such an embedding into $\mathbb{R}$. The answer is no, and the course Inleiding Topologie provides you with the tools to prove it in 2-3 lines.

*Remark* 2. Also $\mathbb{P}^2$ is something that you have seen in the course Inleiding Topologie, when producing spaces by starting with a (filled) square and gluing some its sides- depending on how you did the gluing you got spaces like the torus, the Klein bottle and $\mathbb{P}^2$. While the torus sits nicely in the 3-dimensional space, you were easily convinced that the other two need more dimensions. Since the actual proof of that fact is not so easy, it is still instructive to search for explicit embeddings into $\mathbb{R}^3$ and see what actually goes wrong (and, if you search hard enough ... you may actually discover an immersion of $\mathbb{P}^2$ in $\mathbb{R}^3$ ... and maybe even Boy's surface).

*Remark* 3. In general, when looking for an embedding of a manifold $M$ into some Euclidean space $\mathbb{R}^n$

(1) it is wise to check whether $M$ is compact ... as the life may get a lot easier.
(2) in any case, when you start looking/guessing your embedding $f : M \to \mathbb{R}^n$, the first thing to take care of is that it really "allows you to put $M$ inside of $\mathbb{R}^n$ as a set" or, more precisely: that $f$ is injective.
(3) then you have to take care that $f$ is "injective in a smooth way" or, more precisely: that $f$ is an immersion.
(4) finally, by Theorem 2.64, you are now left with checking that $f$ is also a topological embedding (or maybe you are not left with anything ... if you did the first check).

*Remark* 4. This is a remark about trying to find explicit functions defined on $\mathbb{P}^n$. For instance real-valued functions $P : \mathbb{P}^n \to \mathbb{R}$. Since the points of $\mathbb{P}^n$ are parametrised by coordinates $x_0, \ldots, x_n$ from $\mathbb{R}^{n+1}$ it is natural to look for formulas depending on those coordinates:

$$P([x_0 : x_1 : \ldots : x_n]) = f(x_0, x_1, \ldots, x_n).$$

However, since multiplying such coordinates by a constant $\lambda$ describes the same point of $\mathbb{P}^n$, for $P$ to be well-defined $f$ should satisfy

$$f(\lambda x_0, \lambda x_1, \dots, \lambda x_n) = f(x_0, x_1, \dots, x_n)$$

for all $\lambda$ and $x_i$s. For instance, when $n = 1$, looking at simple formulas like

$$x^2 + y^3,\ x^2 + y^2, \frac{2xy}{x^2 + y^2}, \frac{x}{x^2 + y^2}, \frac{xy^3}{2x^4 + 2x^3y + x^2y^2 + y^4}$$

only the third and the fifth ones work. More generally, when looking for functions on $\mathbb{P}^n$ that are based on polynomial expressions, we end up looking at quotients of homogeneous polynomials of the same degree. One should also pay attention to the fact that the denominator should be non-zero whenever $x = (x_0, x_1, \dots, x_n) \neq 0$; with that in mind, the most natural choice for the denominator would be the norm of $x$ (or powers of it).

*Remark* 5. When working with points in $\mathbb{P}^n$, to avoid larger expressions (e.g. coming from the denominators mentioned above), one may use instead of arbitrary $x \in \mathbb{R}^{n+1} \setminus \{0\}$ points only on the sphere. Geometrically that is due to the fact that any line through the origin intersects the sphere in at least one point, while algebraically that is based on the fact that we can renormalise $x$ to $\overline{x} = \frac{1}{||x||}x \in S^n$:

$$[x_0 : x_1 : \dots : x_n] = [\overline{x}_0 : \overline{x}_1 : \dots : \overline{x}_n], \quad \text{where } \overline{x}_i = \frac{1}{||x||}x_i.$$

**Exercise 1.** Consider first the map

$$f : \mathbb{P}^2 \to \mathbb{R}^3, \quad f([x : y : z]) = \left( \frac{xy}{x^2 + y^2 + z^2}, \frac{yz}{x^2 + y^2 + z^2}, \frac{zx}{x^2 + y^2 + z^2} \right).$$

1. is the map $f$ injective?
2. is the map $f$ smooth?
3. is the map $f$ an immersion?

Now, let's fix the problems that you detected (but keep on reading even if you didn't detect any problem), by adding an extra-coordinate: consider

$$g_\lambda : \mathbb{P}^2 \to \mathbb{R}^3, \quad g_\lambda([x : y : z]) = \left( f([x : y : z]), \frac{x^2 + \lambda y^2}{x^2 + y^2 + z^2} \right),$$

where $\lambda \in \mathbb{R}$ is a constant. By the same arguments as above, this is a smooth map (you can use that without having to prove it). Show that:

4. for $\lambda = 0$: $g_0$ is still not a smooth embedding
5. for $\lambda = 1$: $g_1$ is still not a smooth embedding
6. for $\lambda = -1$: $g_{-1}$ IS a smooth embedding.

Please explain your answers.

Note: admittedly, this is an exercise that requires doing some computations. But it that kind of computations that you should do yourself once, and know how to do (they are often useful to make precise, and correct, things that are based on intuition). On the other hand, please try to find arguments that reduce the amount of computations ...