HOMEWORK 4 (OCTOBER 4, 2023)

This exercise is to help you work with tangent vectors, as (abstractly) defined in the lectures, when looking inside $\mathbb{R}^n$ and embedded submanifolds. Therefore, we will use the tangent vectors introduced (abstractly) in the lectures, such as:

- The canonical basis

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p \in T_p\mathbb{R}^n. \tag{1}$$

  Note that this allows us to write/represent an arbitrary tangent vector vector $v \in T_p\mathbb{R}^n$ uniquely as

$$v = \lambda_1 \cdot \left(\frac{\partial}{\partial x_1}\right)_p + \ldots + \lambda_n \left(\frac{\partial}{\partial x_n}\right)_p, \tag{2}$$

  with $\lambda_i \in \mathbb{R}$.
- For $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$, its "speed" as an element

$$\frac{d\gamma}{dt}(0) \in T_p\mathbb{R}^n \quad (p = \gamma(0)) \tag{3}$$

Via the standard identification

$$\text{standard}_p : T_p\mathbb{R}^n \to \mathbb{R}^n, \quad v \mapsto v^{\text{Id}}$$

sending v to its value at the identity chart, the canonical basis (1) corresponds to the standard basis $e_1, \ldots, e_n$ of the vector space $\mathbb{R}^n$, and the abstract speed (3) corresponds to the usual derivative. But we work with the more abstract objects. Please also keep in mind that, given two tangent vectors $v, w \in T_pM$, at some point $p \in M$, to check that

$$v = w$$

it suffices to check that, for some chart χ around p, $v^\chi = w^\chi$.

Next, for an embedded submanifold $M \subset N$, we have discussed how to interpret the resulting inclusion $T_pM \subset T_pN$, for $p \in M$. In particular, for embedded submanifolds $M \subset \mathbb{R}^n$, one has

$$T_pM \subset T_p\mathbb{R}^n \quad (p \in M).$$

While arbitrary tangent vectors to $\mathbb{R}^n$ look like (2), the question is: when does such a vector (2) actually belong to T_pM? For instance, when M is just a point (a 0-dimensional submanifold!), the answer is: only when all λ_i vanish!

The main tool you have at hand is to use "speeds of curves": to show that $v \in T_p\mathbb{R}^n$ belongs to T_pM you have to find a curve γ in M such that $v = \frac{d\gamma}{dt}(0)$ (... well, there is also the regular value theorem, that you can use in extreme cases ...).

Exercise: Please do the following:

(1) For arbitrary curves $\gamma : I \to M$ in a manifold M, defined on some interval $I \subset \mathbb{R}$, make sense of (define) the speeds at arbitrary times:

$$\frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M.$$

(2) When $M \subset \mathbb{R}^n$, using what we asked you to keep mind, show that

$$\frac{d\gamma}{dt}(t) = \gamma_1'(t)\left(\frac{\partial}{\partial x_1}\right)_{\gamma(t)} + \ldots + \gamma_n'(t)\left(\frac{\partial}{\partial x_n}\right)_{\gamma(t)},$$

where $\gamma_i : I \to \mathbb{R}$ are the components of γ and $\gamma_i'(t) \in \mathbb{R}$ the usual derivatives.

(3) Similarly, show that, for any $h : \mathbb{R}^n \to \mathbb{R}^k$ smooth and $p \in \mathbb{R}^n$, its differential between the abstract tangent spaces (white or green, your preference), $(dh)_p : T_p\mathbb{R}^n \to T_{h(p)}\mathbb{R}^k$, is still given by "the usual formula's", i.e. it sends

$$\left(\frac{\partial}{\partial x_i}\right)_p \mapsto \sum_j \frac{\partial h_j}{\partial x_i}(p) \cdot \left(\frac{\partial}{\partial x_j}\right)_p,$$

where h_j is the $j-th$ component of h.

(4) And now something handy when working on embedded submanifolds $M \subset \mathbb{R}^n$ and you have to work with the differential of smooth functions $f : M \to \mathbb{R}^k$ which may happen to be restrictions to M of smooth functions $h : \mathbb{R}^n \to \mathbb{R}^k$. I.e., assume that $f = h|_M$ with h as before. Show that the differential

$$(df)_p : T_pM \to T_{f(p)}\mathbb{R}^k$$

is the restriction of the differential

$$(dh)_p : T_p\mathbb{R}^n \to T_{h(p)}\mathbb{R}^k$$

to the subspace $T_pM \subset T_p\mathbb{R}^n$. (Hint: use the inclusion. And don't forget the chain rule!).

(5) For the sphere $S^2 \subset \mathbb{R}^3$, and $p = (1, 0, 0) \in S^2$, show that

$$\left(\frac{\partial}{\partial y}\right)_p, \left(\frac{\partial}{\partial z}\right)_p \in T_p\mathbb{R}^3$$

both belong to T_pS^2 (Hint: use (2) for the appropriate curves γ in S^2).

(6) The two tangent vectors will be send by the differential $(df)_p$ of the map

$$f : S^2 \to \mathbb{R}^2, \quad f(x, y, z) = (xy, xz)$$

to two tangent vectors to $\mathbb{R}^2$ (at what point?). Compute them!

(7) Using (6), deduce that f is a submersion at $p = (1, 0, 0)$.