HOMEWORK 7 (OCTOBER 25, 2023)

In the exercise item (d), (e) and (i) are worth a point, item (j) three points, and the rest each one a point. Hence, with this exercise, you can earn more than 10 points (and they will all count in the formula for the average of the homeworks). You should think of (j) as being a bonus exercise, in the sense that it is more difficult than the rest.

Exercise 1. On the 2-sphere S^2 we consider the vector field given by

$$V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

and the 1-form and the 2-form given by the following formulas (restricted to S^2):

$$\omega = z^2(ydx - xdy), \quad \sigma = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

Please do the following:

(a) compute $d\sigma$ and $d\omega$;

(b) compute $i_V(\omega)$ and show that $i_V(\sigma) = -dz$;

(c) show that, for any $p = (x, y, z) \in S^2$, σ_p is non-zero (as an element in $\Lambda^2 T_p^* S^2$, i.e., as a skew-symmetric bilinear map $T_pS^2 \times T_pS^2 \to \mathbb{R}$!);

(d) find the points $p = (x, y, z) \in S^2$ at which $i_V(\sigma)_p = 0$;

(e) compute $L_V(\omega)$ and $L_V(\sigma)$ using the main properties of the Lie derivatives (as discussed in the class);

(f) compute $i_V(d(\omega))$ and $d(i_V(\omega))$ and check that their sum precisely $L_V(\omega)$;

(g) show that $\omega = F^*(ydx - xdy)$ where $F : S^2 \to \mathbb{R}^2$, $F(x, y, z) = (yz, xz)$;

(h) show that if a form ζ on S^2 can be written as the pull-back by F of a form on $\mathbb{R}^2$, then $i_V(\zeta)$ must vanish at the point $q = (1, 0, 0)$. Deduce that σ cannot be written as such a pull-back.

(i) find a function $f \in C^\infty(S^2)$ such that $d\omega = f \cdot \sigma$.

(j) show that σ cannot be written as the pull-back of a 2-form on $\mathbb{R}^2$ by any smooth function $G : S^2 \to \mathbb{R}^2$.