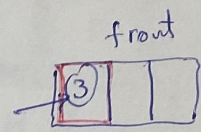
Theorem 1.38 & 1.37: Given $M \subseteq \mathbb{R}^L$, $p \in M$, the following conditions around p are equivalent:



(1) m-dimensional manifold condition ~~atop~~:

$\exists$ smooth chart of M

$$x : \underset{\substack{\cap \text{ open} \\ M}}{U} \longrightarrow \underset{\substack{\cap \text{ open} \\ \mathbb{R}^m}}{\Omega} \qquad \text{with } p \in U.$$

(2) m-dimensional **parametrization:**

$\exists$ homeomorphism

$$\text{par} : \underset{\substack{\cap \text{ open} \\ \mathbb{R}^m}}{\Omega} \longrightarrow \underset{\substack{\cap \text{ open} \\ M}}{U} \qquad \text{with } p \in U$$

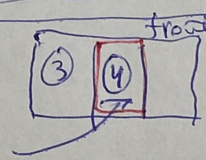
such that, as a map $\text{par} : \Omega \to \mathbb{R}^L$, is a smooth immersion.

(3) m-dimensional **implicit equation:**

$\exists$ smooth submersion

$$\text{eq} : \underset{\substack{\cap \text{ open} \\ \mathbb{R}^L}}{\tilde{U}} \longrightarrow \mathbb{R}^{L-m}$$

such that $M \cap \tilde{U} = \{ q \in \tilde{U} : \text{eq}(q) = 0 \}$.



(4) $\exists$ smooth chart of $\mathbb{R}^L$ around p

$$\tilde{x} : \underset{\substack{\cap \text{ open} \\ \mathbb{R}^L}}{\tilde{U}} \longrightarrow \underset{\substack{\cap \text{ open} \\ \mathbb{R}^L}}{\tilde{\Omega}}$$

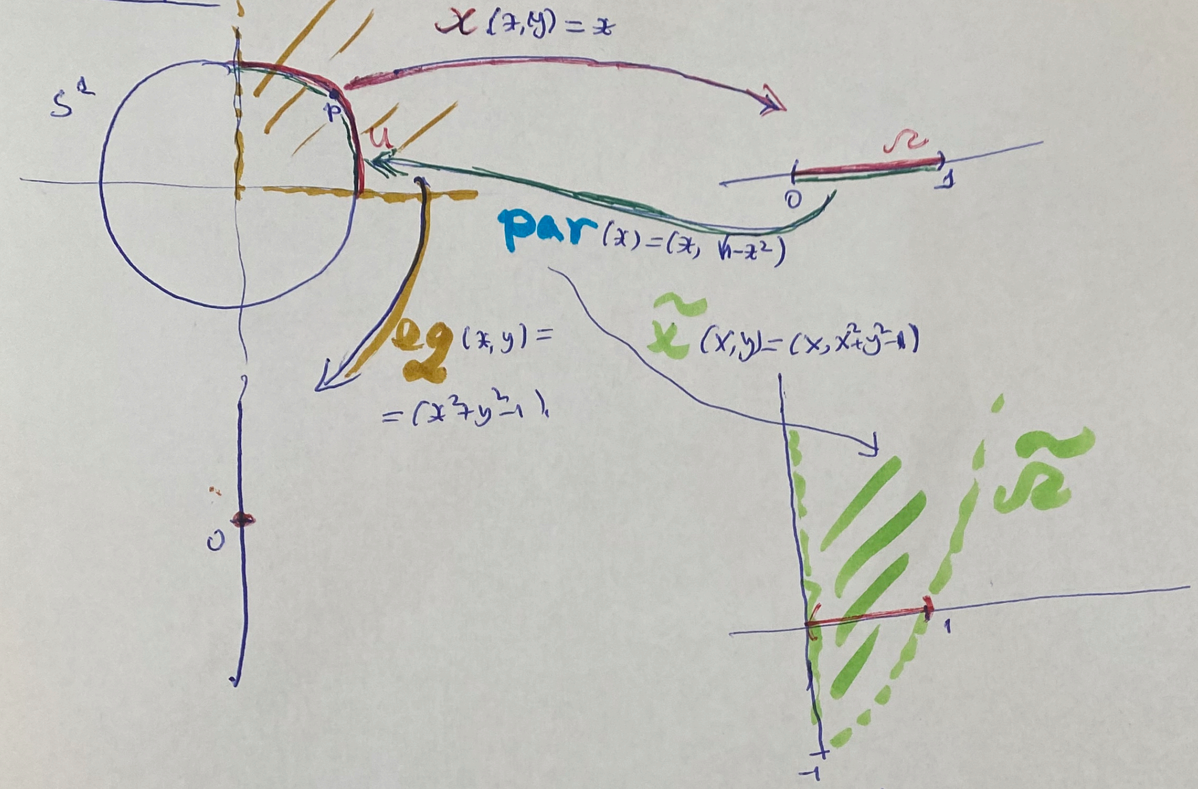
that is adapted to M in the sense that, denoting

$U = \tilde{U} \cap M$ (open in M!)

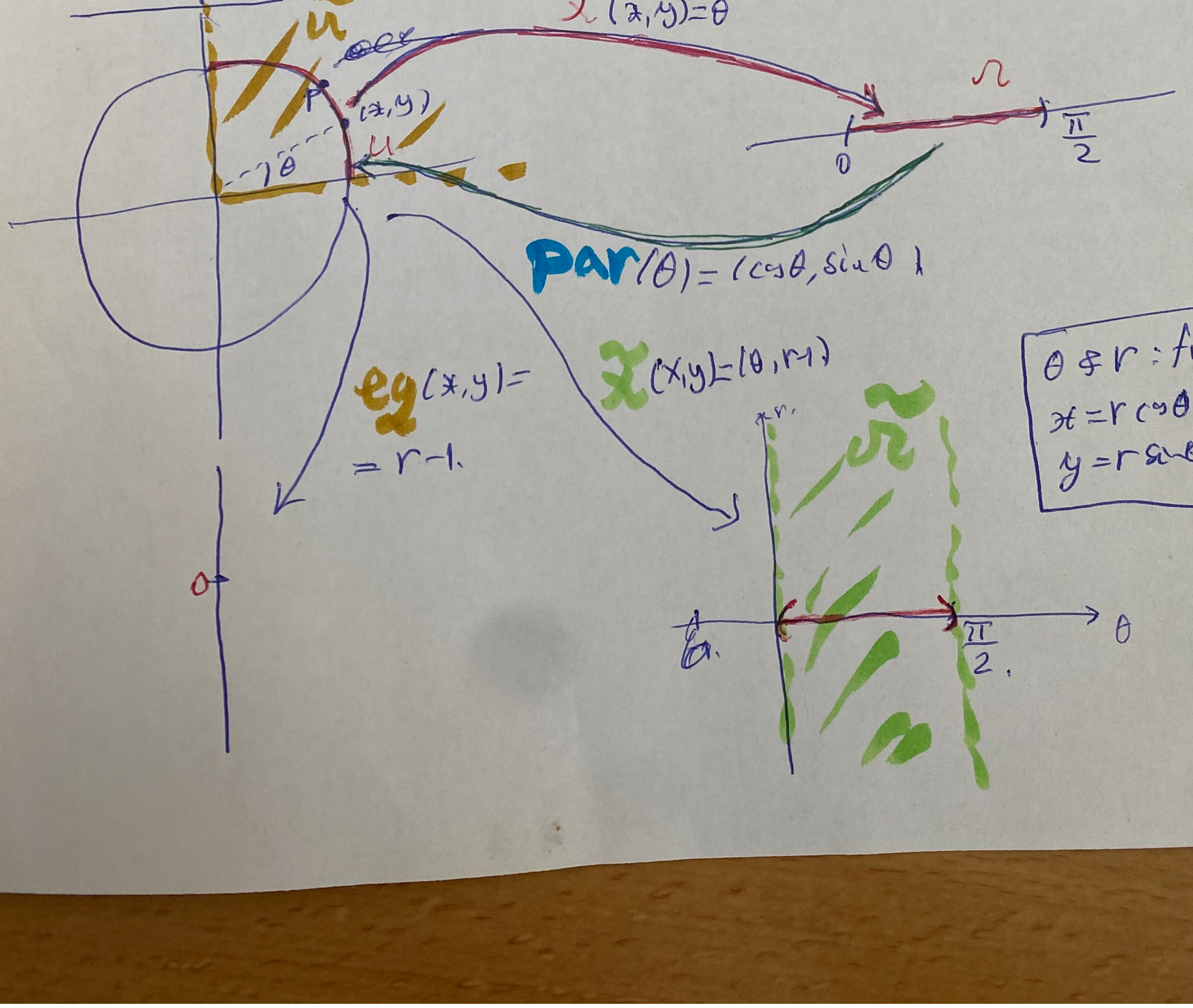
$\Omega = \tilde{U} \cap (\mathbb{R}^m \times \{0\})$ (open in $\mathbb{R}^m$!)

one has $\quad \tilde{x}(U) = \Omega$.

Example:



Example: same $M = S^1$ but other charts:



θ & r: from
$x = r\cos\theta$
$y = r\sin\theta$.