Chapter 6
Previous exams

Here are some of the last exams- but you may have seen already quite a few of the questions already in the previous exercises. Anyway, here are the complete exams.

6.1 Exam 2022/2023 (November 8-th, 2022)

Exercise 6.1 (1pt). You know already, from one of the homeworks, that $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$. Show now that $\mathbb{P}^2$ can be embedded in S^4. (Hint: do not look for complicated formulas).

Exercise 6.2 (2pt). Let $f : M \to N$ be a smooth map between two manifolds of dimensions m and n, respectively, let $N_0 \subset N$ be a (smooth, embedded) submanifold and we are interested in the pre-image

$$M_0 = f^{-1}(N_0) := \{x \in M : f(x) \in N_0\}.$$

The RVT (regular value theorem) tells us that if N_0 consists of a single point and we require f to be a submersion at all points in M_0, then M_0 is a submanifold of M of dimension $n-m$, whose tangent spaces are given by the kernels of the differentials of f:

$$T_xM_0 = \{v \in T_xM : (df)_x(v) = 0\} \quad (\text{for } x \in M_0).$$

Here we want to generalise the RVT to more general submanifolds N_0. To that end, we replace the submersion condition by the condition that "f is transverse to N_0" by which we mean: for each $x \in M_0$ one has $(df)_x(T_xM) + T_{f(x)}N_0 = T_{f(x)}N$ or, more explicitly: any element $w \in T_{f(x)}N$ can be written as

$$w = (df)_x(v) + w_0, \quad \text{with } v \in T_xM, w_0 \in T_{f(x)}N_0.$$

a) Further assuming that $N_0 = g^{-1}(z)$ for some submersion $g : N \to P$ and $z \in P$ into yet another manifold P, show that, indeed, M_0 is an submanifold of M.
b) Describe the dimension of M_0 in terms of the dimensions of M, N and N_0.
c) Describe the tangent spaces of M_0 in terms of the ones of M, N, N_0 and the differential of f.
d) Finally, show that the conclusions above hold in general (without assuming g).

Exercise 6.3 (7pts). Consider the following curve in $\mathbb{R}^3$:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = (t^2, t^3, t).$$

a) Show that the following is a submanifold of $\mathbb{R}^3$ containing γ:

$$M = \{(x, y, z) \in \mathbb{R}^3 : y = xz\}.$$

b) Show that the following defines a vector field on M

$$V := 2z\frac{\partial}{\partial x} + (x+2z^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

c) Show that γ is an integral curve of V.
d) Here are three more subspaces of $\mathbb{R}^3$ that contain γ:

$$\{(x,y,z) \in \mathbb{R}^3 : x = z^2\}, \quad \{(x,y,z) \in \mathbb{R}^3 : x^3 = y^2\}, \quad \{(x,y,z) \in \mathbb{R}^3 : y = z^3\}.$$

Among them, only one is not a submanifold of $\mathbb{R}^3$. Which one, and why is it not?
e) Find vector fields $X, Y \in \mathfrak{X}(M)$ which, at each point in M, give a basis of the tangent space of M.
f) Compute $[X,Y]$.
g) [1pt] Compute the flow of V, describing explicitly all the diffeomorphisms ϕ_V^t induced by V.
h) Find a non-zero 1-form $\theta \in \Omega^1(\mathbb{R}^3)$ such that $\theta|_M = 0$.
i) Find a 1-form on M which is not exact, i.e. cannot be written as df for some $f \in C^\infty(M)$.
j) Show that $\mu = (dx \wedge dz)|_M$ is a volume form.
k) For $\omega = e^y(x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy)|_M \in \Omega^2(M)$ find $f \in C^\infty(M)$ such that $\omega = f \cdot \mu$.
l) Compute $i_V(\omega)$.
m) Also compute $L_V(\omega)$, writing the result in the form $g \cdot \mu$ (with g a function explicitly computed).

Exercise 6.4 (1pt). Let's wonder whether a manifold M can admit a volume form μ of type $\mu = \theta \wedge \theta$, for some other differential form θ on M. Show that if this happens then the dimension of M is divisible by 4. Then show that this can happen already on $M = \mathbb{R}^4$ (provide an explicit example!).

NOTES:

- Please write down your name CLEARLY (this is important for me to be able to give you feed-back via Microsoft Teams), and do not forget to mention also the student no!

- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please include all your computations that support your claims.

- Each subquestion (labelled by a letter) is worth 0.5 points, except for g) of Exercise 3 (1pt, if you provide all details, as mentioned above).

- The mark for the exam is the minimum between 10 and the total number of points you collect.

- The order of the exercises is completely unrelated to their difficulty (in particular, there is absolutely no reason to be scared about the first or the last exercise!).

6.2 Retake 2021/2022 (December, 2021)

Note: Please give all details. All the questions below are worth 0.5 points except for the ones in blue which are worth 1 pt (with a grand total of 11.5pt, of which only 10 are needed to receive the maximum mark).

Exercise 6.5. On the 2-sphere S^2, on which we use the coordinate functions $(x,y,z) \in \mathbb{R}^3$:

(a) Show that the function
$$f : S^2 \to S^2, \quad f(x,y,z) = (x^2+y^2-z^2, 2yz, 2xz)$$
is not a submersion.
(b) Show that $(xz \cdot dy - yz \cdot dx) \wedge dz = (1-z^2) \cdot dx \wedge dy$ (an equality of 2-forms on S^2).
(c) Show that the pull-back via f of the volume form $\sigma = x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy$ satisfies
$$f^*\sigma = 4dx \wedge dy = 4z \cdot \sigma.$$
(d) Find the points $p \in S^2$ at which f fails to be a submersion.
(e) Compute $\int_{S^2} f^*\sigma$ (where you can use any orientation on S^2 that you want).

Exercise 6.6. Let $F : M \to N$ be a smooth map between two manifolds. Recall that, for vector fields $X \in \mathfrak{X}(M)$, $V \in \mathfrak{X}(N)$, one says that X is F-projectable to V if
$$(dF)_p(X_p) = V_{F(p)}$$
for all $p \in M$. Show that:

(a) If $X \in \mathfrak{X}(M)$ if F-projectable to $V \in \mathfrak{X}(N)$, then $L_X(F^*(f)) = F^*(L_V(f))$ for all $f \in C^\infty(N)$, where $F^* : C^\infty(N) \to C^\infty(M)$, $F^*(f) = f \circ F$.
(b) Then show that the converse of (a) holds as well.
(c) Deduce that, if $X \in \mathfrak{X}(M)$ is F-projectable to $V \in \mathfrak{X}(N)$ and $Y \in \mathfrak{X}(M)$ is F-projectable to $W \in \mathfrak{X}(N)$ then $[X,Y]$ is F-projectable to $[V,W]$.

Exercise 6.7. Assume that M is a compact manifold, $f : M \to S^1$ is a smooth map and that there exists a vector field $V \in \mathfrak{X}(M)$ that is f-projectable to $\frac{\partial}{\partial \varphi} = -y \cdot \frac{\partial}{\partial x} + x \cdot \frac{\partial}{\partial y} \in \mathfrak{X}(S^1)$. We also consider the pull-back via f of the canonical 1-form $d\varphi = -y \cdot dx + x \cdot dy \in \Omega^1(S^1)$,
$$\omega := f^*(d\varphi) \in \Omega^1(M).$$
Show that:

(a) For each $\alpha \in \mathbb{R}$, the fiber
$$M_\alpha := \{p \in M : f(p) = e^{i\alpha}\}$$
is an embedded submanifold of M.
(b) ω is closed.
(c) $\omega(V) = 1$.
(d) $\mathscr{L}_V(\omega) = 0$.
(e) $\phi_V^t(p) \in M_{\alpha+t}$ for any $p \in M_\alpha$, and that the flow of V gives rise to diffeomorphisms
$$\phi_V^t|_{M_\alpha} : M_\alpha \xrightarrow{\sim} M_{\alpha+t} \quad \text{(for all } t, \alpha \in \mathbb{R}).$$

Exercise 4. On $\mathbb{R}^3$ we define the following operation ("product"):

$$(x_1, y_1, z_1) \star (x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and we consider the associated left translations, one for each $p \in \mathbb{R}^3$,

$$L_p : \mathbb{R}^3 \to \mathbb{R}^3, \quad L_p(q) := p \star q.$$

We define the vector fields $X^1, X^2, X^3 \in \mathfrak{X}(\mathbb{R}^3)$ which, at $p \in \mathbb{R}^3$, are obtained by applying the differential of L_p at the origin, $(dL_p)_0 : T_0\mathbb{R}^3 \to T_p\mathbb{R}^3$, to

$$\left(\frac{\partial}{\partial x}\right)_0, \quad \left(\frac{\partial}{\partial y}\right)_0, \quad \left(\frac{\partial}{\partial z}\right)_0.$$

(a) Compute X^1, X^2, X^3 explicitly and show that they are dual to

$$\theta_1 = dx, \quad \theta_2 = dy, \quad \theta_3 = -xdy + dz,$$

i.e. $\theta_i(X^j) = \delta_i^j$ (1 if $i = j$ and 0 otherwise).

(b) Show that $[X^1, X^2] = X^3, [X^2, X^3] = 0, [X^3, X^1] = 0$.

(c) Find a vector field V such that $\theta^1(V) = 1$, $\theta^2(V) = 0$ and which commutes with X^1, X^2, X^3 (i.e. $[V, X^1] = 0$, $[V, X^2] = 0$, $[V, X^3] = 0$).

(d) For V that you found in (c), compute the flow ϕ_V^t of V explicitly.

(e) Show that ϕ_V^t preserves θ_3, i.e. $(\phi_V^t)^* \theta_3 = \theta_3$.

(f) Deduce that $\mathscr{L}_V(\theta_3) = 0$.

(g) Prove directly (by an algebraic computation) that $\mathscr{L}_V(\theta_3) = 0$.

6.3 Exam 2021/2022 (November 8-th, 2021)

Note: please do not forget to write down your name and student number. Also, please motivate your answers. The first exercise is worth 1.5 point, while in the second exercise all the questions (a)-(p) are worth 0.5p except for the ones in blue which are worth 1p. The mark for the exam is the minimum between 10 and the total number of points you collect.

Exercise 6.8. (*1.5 pt*) Show that, on any manifold M,

$$L_{f\cdot X}(\omega) = f\cdot L_X(\omega) + df \wedge i_X(\omega)$$

for any smooth function $f \in C^\infty(M)$, any vector field $X \in \mathfrak{X}(M)$ and any differential form $\omega \in \Omega^k(M)$

Exercise 6.9. We consider the space of two by two matrices with real coefficients, identified to the Euclidean space $\mathbb{R}^4$ with coordinate functions denoted by x, y, z, t:

$$N := \left\{ A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : x, y, z, t \in \mathbb{R} \right\}$$

and, inside it, the space of matrices of determinant 1:

$$M := \left\{ A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} : xt - yz = 1 \right\}.$$

First show that:

(a) M is an embedded submanifold of N.
(b) The following define vector fields tangent to M:

$$V^1 := z\frac{\partial}{\partial x} + t\cdot\frac{\partial}{\partial y}$$

$$V^2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t}.$$

$$V^3 = -y\cdot\frac{\partial}{\partial x} + x\cdot\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}$$

(c) $f : M \to \mathbb{R}^2$, $\quad f\begin{pmatrix} x & y \\ z & t \end{pmatrix} = (z, t)$ is a submersion.
(d) Because t is used to denote the last coordinate in M, the time-parameter of curves will be denoted by s. Compute the flows of the vector fields V_1, V_2 and V_3 and show they are of type

$$\phi^s_{V_1}(A) = A_1(s)\cdot A, \quad \phi^s_{V_2}(A) = A_2(s)\cdot A, \quad \phi^s_{V_3}(A) = A\cdot A_3(s)$$

where $A_i(s)$ are matrices that depend on s. For instance, you should get

$$\phi^s_{V_1} = \begin{pmatrix} x + s\cdot z & y + s\cdot t \\ z & t \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x & y \\ z & t \end{pmatrix}, \quad \text{hence } A_1(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

(e) Show that $[V_1, V_3] = 0$ and $[V_2, V_3] = 0$.
(f) Show that $[V_1, V_2] = 2V_1$.
(g) Find another vector field V on M which admits the following integral curve:

$$\gamma(s) = A_1(s)\cdot A_3(s) = \begin{pmatrix} \cos s - s\cdot\sin s & \sin s + s\cdot\cos s \\ -\sin s & \cos s \end{pmatrix}$$

(h) Given an example of a 1-form θ on N which is nonzero but such that $\theta|_M = 0$.

(i) Using the 1-forms on M:

$$\theta_1 = -y \cdot dx + x \cdot dy + \frac{x^2+y^2}{z^2+t^2}\,(t \cdot dz - z \cdot dt)\,,$$

$$\theta_2 = -\frac{1}{z^2+t^2}(z \cdot dz + t \cdot dt), \quad \theta_3 = \frac{1}{z^2+t^2}(-t \cdot dz + z \cdot dt),$$

show that V^1, V^2, V^3 induce a basis of T_pM at each $p \in M$.

(j) Show that $d\theta_2 = 0$ and $d\theta_3 = 0$.

(k) Consider now $\omega := dx \wedge dy \wedge dz$ as a form on M (the restriction to M). Compute $i_{V^1}(\omega)$, and then $\omega(V^1, V^2, V^3)$.

(l) Show that ω is not a volume form on M and find all $p \in M$ at which $\omega_p = 0$.

(m) Show that $L_{V^1}(\omega) = -dx \wedge dz \wedge dt$ in two ways: one using Cartan's formula, and one using flows.

(n) Consider the following parametrization of M:

$$\begin{cases} x = r \cdot \cos\alpha - \frac{u}{r}\sin\alpha & y = r \cdot \sin\alpha + \frac{u}{r}\cos\alpha \\ z = -\frac{1}{r}\sin\alpha & t = \frac{1}{r}\cos\alpha \end{cases} \qquad u \in \mathbb{R},\ r \in (0,\infty),\ \alpha \in (0, 2\pi)$$

which we interpret as the inverse χ^{-1} of a chart of M. What is the domain U of χ?

(o) Hence the u, r, α determined by the equations above (all functions of (x,y,z)) are precisely the components $\chi_1(x,y,z)$, $\chi_2(x,y,z)$ and $\chi_3(x,y,z)$ of $\chi(x,y,z)$. Show that the resulting vector field $\frac{\partial}{\partial \chi_1}$ coincides with V_1 at all points $p \in U$.

(p) Let $\mu := \theta_1 \wedge \theta_1 \wedge \theta_2$. Compute the coefficient of μ w.r.t. the chart χ,

$$f^{\mu}_{\chi} \in C^{\infty}(\mathbb{R} \times (0,\infty) \times (0,2\pi)).$$

6.4 Retake 2020/2021 (January 8-th, 2021)

Note: The points below add up together to 12 points, so you do not have to solve everything to get the maximum mark of 10! Please motivate all your answers: do not just answer with "yes" or "no" but also provide arguments; do not just write down the final result, but also explain how you found it.

Exercise 6.10. (*1 pt*) If $\omega \in \Omega^k(M)$ is a closed differential form on a manifold M show that the operation

$$\Omega^l(M) \to \Omega^{k+l}(M), \quad \theta \mapsto \omega \wedge \theta$$

takes closed forms to closed forms, and exact forms to exact forms.

Exercise 6.11. (*1.5 pt*) Does there exists a vector field X on $\mathbb{R}^3$ with flow given by

$$\phi_X^t(x,y,z) = (x+t, y+tz, z)?$$

But a vector field Y with

$$\phi_Y^t(x,y,z) = (x+t, y+tz, z+t)?$$

Exercise 6.12. Consider the following differential 1-forms on S^2:

$$\theta_1 = x \cdot dy - y \cdot dx + z \cdot dz, \quad \theta_2 = x \cdot dy + y \cdot dx + z \cdot dz \quad \text{(restricted to } S^2!\text{)}$$

and the vector field $V = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \in \mathfrak{X}(S^2)$.

(a) (*0.5 pt*) Which one of the 1-forms above is closed and which not?
(b) (*0.5 pt*) Which one of the 1-forms above is exact and which not?
(c) (*0.5 pt*) Is $\theta_1 \wedge \theta_2$ a volume form?
(d) (*0.5 pt*) Is $\theta_1 \wedge \theta_2$ exact?
(e) (*0.5 pt*) Compute $i_V(\theta_1 \wedge \theta_2)$.
(f) (*0.5 pt*) Show that $L_V(\theta_1) = 0$.
(g) (*1 pt*) Compute $(\phi_V^t)^* \theta_1$, where ϕ_V^t is the flow of V.

Exercise 6.13. Let $E : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, we denote by E_x, E_y and E_z the partial derivatives of E with respect to the variables x, y and z (each a smooth function on $\mathbb{R}^3$), and we construct the following three vector fields on $\mathbb{R}^3$:

$$\widetilde{U} = E_z \cdot \frac{\partial}{\partial y} - E_y \cdot \frac{\partial}{\partial z}, \quad \widetilde{V} = E_x \cdot \frac{\partial}{\partial z} - E_z \cdot \frac{\partial}{\partial x}, \quad \widetilde{W} = E_y \cdot \frac{\partial}{\partial x} - E_x \cdot \frac{\partial}{\partial y}.$$

We also consider

$$M_E := \{(x,y,z) \in \mathbb{R}^3 : E(x,y,z) = 0\}$$

and we assume that, at every point $p \in M_E$, at least one of the three vector fields above does not vanish. Show that:

(a) (*0.5 pt*) M_E is a 2-dimensional submanifold of $\mathbb{R}^3$.
(b) (*0.5 pt*) $\widetilde{U}, \widetilde{V}$ and $\widetilde{W}$ are tangent to M_E at each point $p \in M_E$. Hence they define three vector fields on M_E- and those will be denoted

$$U, V, W \in \mathfrak{X}(M_E).$$

(c) (*0.5 pt*) U, V and W span the tangent space T_pM_E at each $p \in M_E$.

(d) (*1 pt*) as a consequence of (c), at every point $p \in M$, the Lie bracket $[V,W]_p$ and the similar ones are linear combinations of U_p, V_p and W_p. Show that there are smooth functions f,g and h on M_E such that

$$[V,W] = f \cdot U + g \cdot V + h \cdot W,$$

and similarly for the other two Lie brackets.

(e) (*0.5 pt*) Consider the projection on the first two factors

$$\mathrm{pr}_{1,2} : M_E \to \mathbb{R}^2, \quad \mathrm{pr}_{1,2}(x,y,z) = (x,y).$$

Describe the points at which $\mathrm{pr}_{1,2}$ fails to be a submersion. And the same for the similar projections $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,3}$.

(f) (*0.5 pt*) Show that, around each point $p \in M_E$, at least one of the projections $\mathrm{pr}_{1,2}$, $\mathrm{pr}_{1,3}$ and $\mathrm{pr}_{2,3}$ can be used as a chart, i.e. there exists an open neighborhood U of p and a chart of type of type (U,χ), with χ being the restriction to U of one of those three projections.

Exercise 6.14. Consider the 3-dimensional projective space $\mathbb{P}^3$.

(a) (*0.5 pt*) Show that

$$p : (0,\pi) \times (0,\pi) \times (0,\pi) \to \mathbb{P}^3,$$

$$(\phi_1,\phi_2,\phi_3) \mapsto [\cos\phi_1 \ : \ \sin\phi_1 \cdot \cos\phi_2 \ : \ \sin\phi_1 \cdot \sin\phi_2 \cdot \cos\phi_3 \ : \ \sin\phi_1 \cdot \sin\phi_2 \cdot \sin\phi_3]$$

is a parametrization of $\mathbb{P}^3$, i.e. a diffeomorphism into an open $U \subset \mathbb{P}^3$.

(b) (*1.5 pt*) Assuming that you know already that $\mathbb{P}^3$ is orientable, and using any orientation that you prefer, compute

$$\int_{\mathbb{P}^3} f \cdot df_0 \wedge df_1 \wedge df_2,$$

where $f, f_0, f_1, f_2 : \mathbb{P}^3 \to \mathbb{R}$ are the functions which, for $(x_0,x_1,x_2,x_3) \in S^2$, send:

$$[x_0 : x_1 : x_2 : x_3] \xrightarrow{f_i} (x_i)^2, \quad [x_0 : x_1 : x_2 : x_3] \xrightarrow{f} x_0 \cdot x_1 \cdot x_2 \cdot x_3.$$

You can express the result in terms of $I_k = \int_0^\pi \sin^k \phi$ without further numerical computations (those would be obtained by further remarking that $I_k = \frac{k-1}{k} I_{k-2}$ and $I_0 = \pi$, $I_1 = 2$.)

Note: The points below add up together to 13 points, so you do not have to solve everything to get the maximum mark of 10! Please motivate all your answers: do not just answer with "yes" or "no" but also provide arguments; do not just write down the final result, but also explain how you found it.

Exercise 6.15. Define
$$M = \{(x,y,z,t) \in \mathbb{R}^4 : xt = yz + 12\}.$$

(a) (*0.5 pt*) Show that M is an embedded submanifold of $\mathbb{R}^4$.
(b) (*0.5 pt*) Compute the tangent space of M at the point $p_0 = (4,0,0,3)$.
(c) (*0.5 pt*) Find a vector field $V \in \mathfrak{X}(M)$ with the property that its flow satisfies

$$\phi_V^{2s}(p_0) = (4\cos s, 4\sin s, -3\sin s, 3\cos s).$$

(d) (*1 pt*) Show that also
$$X^1 := \frac{1}{2}\left(y\cdot\frac{\partial}{\partial x} + x\cdot\frac{\partial}{\partial y} + t\cdot\frac{\partial}{\partial z} + z\cdot\frac{\partial}{\partial t}\right)$$
defines a vector field tangent to M and compute $X^2 := [X^1, V], X^3 := [X^1, X^2]$.
(e) (*1 pt*) Consider the spheres centred at the origin
$$S_r^3 = \{(x,y,z,t) \in \mathbb{R}^3 : x^2 + y^2 + z^2 + t^2 = r^2\}$$
(one for each $r > 0$), and show that their intersections with M
$$M_r := M \cap S_r^3,$$
are embedded submanifolds of $\mathbb{R}^4$ (be aware that, depending on how you approach this, you may end up doing some very time consuming computations ...).
(f) (*0.5 pt*) When $r = 5$ (so that $p_0 \in M_r$) compute the tangent space of M_5 at p_0.
(g) (*0.5 pt*) Show that the following defines a submersion from M_5 to S^1:
$$f : M_5 \to S^1, \quad f(x,y,z,t) = (x - t, y + z).$$
(h) (*0.5 pt*) Show that M_5 is diffeomorphic to a torus.

Exercise 6.16. Return to the last exercise and, in (d) above, just assume that
$$X^1 := \frac{1}{2}\left(y\cdot\frac{\partial}{\partial x} + x\cdot\frac{\partial}{\partial y} + t\cdot\frac{\partial}{\partial z} + z\cdot\frac{\partial}{\partial t}\right),$$
$$X^2 := \frac{1}{2}\left(-x\cdot\frac{\partial}{\partial x} + y\cdot\frac{\partial}{\partial y} - z\cdot\frac{\partial}{\partial z} + t\cdot\frac{\partial}{\partial t}\right),$$
$$X^3 := \frac{1}{2}\left(-y\cdot\frac{\partial}{\partial x} + x\cdot\frac{\partial}{\partial y} - t\cdot\frac{\partial}{\partial z} + z\cdot\frac{\partial}{\partial t}\right)$$
and they satisfy
$$[X^1, X^2] = X^3, \quad [X^2, X^3] = -X^1, \quad [X^3, X^1] = -X^2$$
(you do not have to prove these identities). Show that:

(a) (*0.5 pt*) The 1-forms
$$\theta_1 = (-z\cdot dx + t\cdot dy + x\cdot dz - y\cdot dt)/12,$$
$$\theta_2 = (-t\cdot dx - z\cdot dy + y\cdot dz + x\cdot dt)/12,$$
$$\theta_3 = (\ z\cdot dx + t\cdot dy - x\cdot dz - y\cdot dt)/12$$
satisfy $\theta_i(X^j) = \delta_i^j$ (1 if $i = j$ and 0 otherwise.

(b) (*0.5 pt*) Deduce that X^1_p, X^2_p, X^3_p form a basis of T_pM for all $p \in M$.
(c) (*0.5 pt*) Deduce that two 2-forms $\eta, \xi \in \Omega^2(M)$ coincide if and only if

$$\eta(X^1,X^2) = \xi(X^1,X^2), \quad \eta(X^2,X^3) = \xi'(X^2,X^3), \quad \eta(X^3,X^1) = \xi(X^3,X^1).$$

(d) (*0.5 pt*) Show that $d\theta_1 = \theta_2 \wedge \theta_3, \quad d\theta_2 = \theta_3 \wedge \theta_1, \quad d\theta_3 = -\theta_1 \wedge \theta_2$.

Exercise 6.17. (*1.5 pt*) Show that there exists a vector field $X \in \mathscr{X}(S^2)$ with flow given by

$$\phi^t_X(x,y,z) = \left(\frac{(1+x)e^t - (1-x)e^{-t}}{(1+x)e^t + (1-x)e^{-t}}, \frac{2y}{(1+x)e^t + (1-x)e^{-t}}, \frac{2z}{(1+x)e^t + (1-x)e^{-t}} \right).$$

Make sure you give all the details. Is the vector field that you found complete?

Exercise 6.18. Assume that M is a compact manifold, $f : M \to S^1$ is a smooth map and that there exists a vector field $V \in \mathfrak{X}(M)$ that is projectable to $\frac{\partial}{\partial\theta}$, i.e.

$$(df)_p(V_p) = \left(\frac{\partial}{\partial\theta}\right)_{f(p)} \qquad \text{for all } p \in M.$$

We also consider the pull-back via f of the canonical 1-form $d\theta \in \Omega^1(S^1)$,

$$\omega := f^*(d\theta) \in \Omega^1(M).$$

Show that:

(a) (*0.5 pt*) for each $\alpha \in \mathbb{R}$, the fiber

$$M_\alpha := \{p \in M : f(p) = e^{i\alpha}\}$$

is an embedded submanifold of M.
(b) (*1.5 pt*) ω is closed, $\omega(V) = 1$ and $\mathscr{L}_V(\omega) = 0$.
(c) (*1 pt*) for any $p \in M_\alpha$, $\phi^t_V(p) \in M_{\alpha+t}$ and the flow of V gives rise to diffeomorphisms

$$\phi^t_V|_{M_\alpha} : M_\alpha \xrightarrow{\sim} M_{\alpha+t} \quad (\text{for all } t, \alpha \in \mathbb{R}).$$

Exercise 6.19. (*1.5 pt*) Consider the following copy of the torus in $\mathbb{R}^3$:

$$\mathbb{T}^2 := \{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 = 1\}.$$

Choose an orientation on $\mathbb{T}^2$ and compute

$$\int_{\mathbb{T}^2} \omega, \quad \text{where } \omega = (x\,dy \wedge dz)|_{\mathbb{T}^2}.$$

Notes:

1. please add you name and student number (as usual);
2. of course, if you prefer, you can write your solution/explanations (or just part of them) in Dutch;
3. ... but please write all details! do not just "communicate" to us a formula, but show how you obtained it (in that way we see that you understand what you do and, in particular, even if you make mistakes in the computation, you may still get quite a few points if the details show a good path to the solution);
4. please be aware that the points above add up to a total of 13 (i.e. you do not have to do all of them correctly in order to obtain a 10);
5. in a sequence of items, if you are not able to do one of them, then move to the next one (and you are allowed to use the item that you just skipped, even if you did not do it). But, hopefully, that will not be necessary;
6. overall advice: do not hurry up too much- i.e. think a little bit before any question and before jumping to do a computation (just think of what you know, what should be done, etc, so that you do not waste your time because of an unfortunate choice of strategy).

6.5 Retake 2019/2020 (January 6-th, 2020)

Consider the 2-dimensional torus

$$\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$$

(equipped with the product manifold structure) and the vector field $\frac{\partial}{\partial \theta_1}$ given by

$$\left(\frac{\partial}{\partial \theta_1}\right)_{(z_1,z_2)} := \left.\frac{d}{dt}\right|_{t=0} (e^{i\cdot t} z_1, z_2) \in T_{(z_1,z_2)}\mathbb{T}^2,$$

and similarly we define $\frac{\partial}{\partial \theta_2}$. For any smooth function $f : \mathbb{T}^2 \to \mathbb{R}$, we use the notations $\frac{\partial f}{\partial \theta_i}$ for the resulting Lie derivatives (again functions on the torus):

$$\frac{\partial f}{\partial \theta_1}(z_1,z_2) = \left.\frac{d}{dt}\right|_{t=0} f(e^{i\cdot t} z_1, z_2), \quad \text{etc.}$$

We will also consider the more general vector fields on the torus:

$$X^f := \frac{\partial}{\partial \theta_1} + f \cdot \frac{\partial}{\partial \theta_2} \in \mathfrak{X}(\mathbb{T}^2). \qquad (*)$$

defined for any smooth function f on T^2. Recall also that for any $R > r > 0$, one has a ”concrete model” of $\mathbb{T}^2$ inside $\mathbb{R}^3$, namely

$$T^2_{R,r} := \{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\},$$

which is the image of the map

$$F : \mathbb{T}^2 \to \mathbb{R}^3, \quad (e^{i\theta_1}, e^{i\theta_2}) \mapsto ((R + r\cdot\cos\theta_1)\cdot\cos\theta_2,\ (R + r\cdot\cos\theta_1)\cdot\sin\theta_2,\ r\cdot\sin\theta_1).$$

You may assume (e.g. from Topology) that you know already that F is a homeomorphism between $\mathbb{T}^2$ and $T^2_{R,r}$.

Below is a list of questions for you. Some explanations:

- the list is rather long, but that should make the questions easier to answer.
- each question is worth 0.5 pt, except for those from numbers 1, 7, 15, 16 and 19 which are 1 pt. In particular, you do not have to answer all the questions in order to obtain the maximum of points (10 pt).
- IMPORTANT: if you do a computation, please do not just give the final result- but include the actual computation. If you just through in the final result, there will be no points given. On the other hand, keep in mind that if you provide the details but you make a mistake on the way (and the final result is wrong), you may still get quite a few points.

And here are the questions:

1. For $i \in \{1,2\}$ compute $(dF)_p(\frac{\partial}{\partial \theta_i})$ at arbitrary points $p = (e^{i\cdot\theta_1}, e^{i\cdot\theta_2}) \in \mathbb{T}^2$ and show that F is an immersion at each point.
2. Deduce that F is a diffeomorphism between $\mathbb{T}^2$ and $T^2_{R,r}$.
3. We want to compute the vector fields $\frac{\partial}{\partial \theta_1}$ and $\frac{\partial}{\partial \theta_2}$ by moving them to $T^2_{R,r}$ (via F) and decomposing them w.r.t. the basis $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ of the tangent spaces of $\mathbb{R}^3$. In other words, we want to compute $F_*\left(\frac{\partial}{\partial \theta_1}\right)$ and $F_*\left(\frac{\partial}{\partial \theta_2}\right)$. Show that

$$F_*\left(\frac{\partial}{\partial \theta_2}\right)_q = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad \text{for all } q = (x,y,z) \in T^2_{R,r},$$

   and find a similar formula for $F_*\left(\frac{\partial}{\partial \theta_1}\right)$.
4. Compute the flows of $\frac{\partial}{\partial \theta_1}$ and $\frac{\partial}{\partial \theta_2}$ and then deduce that $[\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}] = 0$.

5. Looking at the vector fields of type $(*)$, show that for any two smooth functions f, g on $\mathbb{T}^2$, there exists another smooth function h such that $\frac{\partial}{\partial\theta_1} + [X^f, X^g] = X^h$.
6. For each real number $\lambda \in \mathbb{R}$ consider the vector field X^λ (i.e. $(*)$ obtained when f is the constant function λ). Compute the maximal integral curve γ of X^λ starting at $(1,1) \in \mathbb{T}^2$.
7. With γ as above (and λ arbitrary constant) show that:
   - the image of γ is always an immersed submanifold of $\mathbb{T}^2$
   - but, if λ is irrational, then that image is not an embedded submanifold of $\mathbb{T}^2$.
8. Show that there exist, and they are unique, two 1-forms $d\theta_1$ and $d\theta_2$ on $\mathbb{T}^2$ with the property that

$$d\theta_i\left(\frac{\partial}{\partial\theta_j}\right) = \delta_{i,j} \text{ (1 if } i = j \text{ and 0 otherwise)}.$$

9. Show that for any smooth function f on $\mathbb{T}^2$ one has

$$df = \frac{\partial f}{\partial\theta_1} \cdot d\theta_1 + \frac{\partial f}{\partial\theta_2} \cdot d\theta_2.$$

10. In contrast with what the notation may suggest, $d\theta_1$ and $d\theta_2$ are not exact forms. To see this, you are asked to prove something a bit more general: on any compact manifold M, any 1-form that is exact must vanish at at least one point in M.
11. However, show that $d\theta_1$ and $d\theta_2$ are both closed.
12. Show that $d\theta_2 \wedge d\theta_1$ is a volume form on $\mathbb{T}^2$ and compute the corresponding volume.
13. Show that there do not exist smooth functions f, g on $\mathbb{T}^2$ such that

$$\frac{\partial g}{\partial\theta_1} - \frac{\partial f}{\partial\theta_2} = 1.$$

(hint: volumes are strictly positive, hence non-zero).

14. Consider again the vector field $(*)$ (with f again an arbitrary smooth function). Since $L_{X^f}(d\theta_2 \wedge d\theta_1)$ is again a 2-form on $\mathbb{T}^2$, it is a function times the volume form $d\theta_2 \wedge d\theta_1$. Compute that function.
15. Deduce that the flow of X^f preserves our volume form, i.e.

$$\left(\phi^t_{X^f}\right)^* (d\theta_2 \wedge d\theta_1) = d\theta_2 \wedge d\theta_1 \quad \text{for all } t,$$

if and only if $f = f(z_1, z_2)$ depends only on the first coordinate z_1.

16. Show that there is no closed 2-form $\tilde{\eta}$ on $\mathbb{R}^3$ such that $F^*\tilde{\eta} = d\theta_2 \wedge d\theta_1$. (hint: think of a more general statement, for any volume form on a compact embedded submanifold of a Euclidean space).
17. Show that the restrictions of the standard 1-forms dx, dy, dz from $\mathbb{R}^3$ to $T^2_{R,r}$ satisfy:

$$x \cdot dx + y \cdot dy + z \cdot dz = \frac{R}{\sqrt{x^2+y^2}}(x \cdot dx + y \cdot dy) \quad (\text{on } T^2_{R,r}).$$

18. Compute the pull-backs of dx, dy and dz to $\mathbb{T}^2$ (in terms of θ_1 and θ_2).
19. For $\sigma = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(\mathbb{R}^3)$, restrict it to $T_{R,r}$ and compute

$$\int_{T_{R,r}} \sigma|_{T_{R,r}} \quad (\text{use any orientation you prefer}).$$

(hint: you may want to compute $F^*\sigma$, and hopefully you will obtain the expression $r(R + r \cdot \cos\theta_1)(r + R\cos\theta_1) \cdot d\theta_2 \wedge d\theta_1$).

20. We are now trying to write the volume form $d\theta_2 \wedge d\theta_1$, moved to $T^2_{R,r}$ (via the diffeomorphism F), in terms of the standard coordinates x, y, z on $\mathbb{R}^3$ Show that

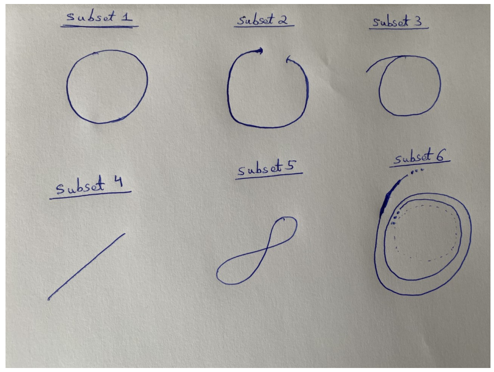
$$\eta := \frac{1}{r^2(x^2+y^2)}\left[\left(\sqrt{x^2+y^2}-R\right)\cdot\sigma + Rz\cdot dx\wedge dy\right]$$

works. In other words, interpreting η as a 2-form on $(\mathbb{R}^2\setminus\{0\})\times\mathbb{R}$, show that

$$F^*\eta = d\theta_2\wedge d\theta_1.$$

6.6 Exam 2019/2020 (November 6th, 2019)

Exercise 6.20. (*1.5 pt*) Which of the following subsets of $\mathbb{R}^2$ can be the image of an integral curve of a vector field $X \in \mathfrak{X}(\mathbb{R}^2)$. (Explanation: subset 1 is a circle, subset 2 is an open part of a circle, subset 4 is an infinite line, subset 6 describes a spiral approaching a circle. Please explain you answer; but when you describe a vector field, you do not have to write down a formula- just draw it on the picture)



Exercise 6.21. (*1.5 pt*) Which of the subsets of $\mathbb{R}^2$ from the previous picture describes an embedded submanifold of $\mathbb{R}^2$. But an immersed submanifold? (please motivate your answer, but you can use the picture instead of explicit formulas)

Exercise 6.22. (*1.5 pt*) Let $\theta \in \Omega^1(M)$ be a closed 1-form and define

$$d_\theta : \Omega^k(M) \to \Omega^{k+1}(M), \quad d_\theta(\omega) := d\omega + \theta \wedge \omega.$$

Show that $d_\theta \circ d_\theta = 0$ if and only if θ is closed.

Exercise 6.23. ($4 \times 0.5 = 2$ *pt*) Consider

$$f : S^3 \to \mathbb{R}, \quad f(x,y,z,t) = x^2 + y^2 - z^2 - t^2.$$

(a) show that the zero-set $M_0 := f^{-1}(0)$ is an embedded submanifold of S^3.
(b) show that M_0 is diffeomorphic to the 2-torus.
(c) find all the points at which f is a submersion.
(d) give an example of a 1-form θ on S^3, different from the zero form, such that $\theta|_{M_0} = 0$.

Exercise 6.24. ($1+1+1 = 3$ *pt*) On the following subspace of the 2-sphere

$$N := \{(x,y,z) \in S^2 : z \neq 0\}$$

consider the restriction of $\frac{1}{z} \cdot dx \wedge dy$ to N:

$$\sigma_0 := \frac{1}{z} \cdot dx \wedge dy \Big|_N .$$

(a) show that σ_0 can be extended to the entire S^2, i.e. there exists a (smooth) 2-form $\sigma \in \Omega^2(S^2)$ such that $\sigma|_N = \sigma_0$.
(b) compute $i_X(\sigma)$ and $L_X(\sigma)$ for $X = y \cdot \frac{\partial}{\partial x} - x \cdot \frac{\partial}{\partial y}$.

(c) compute $\int_{S^2}(x+y+z)\cdot\sigma$ (use your favourite orientation of S^2).

Exercise 6.25. ($0.5+1.5+0.5=2.5$ *pt*) On the 2-sphere we consider the following curve:

$$\gamma:\mathbb{R}\to S^2,\quad \gamma(t)=\left(\frac{t}{\sqrt{1+t^2}},\frac{\cos t}{\sqrt{1+t^2}},\frac{\sin t}{\sqrt{1+t^2}}\right).$$

(a) draw a picture of γ.
(b) find a vector field X on S^2 (explicit formulas!) for which γ is an integral curve.
(c) did you check how smooth your vector field is? please do!

6.7 Retake 2018/2019 (January 2019)

Exercise 6.26. Consider the unit circle
$$\S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$
Identifying $\mathbb{C}$ with $\mathbb{R}^2$ via $x + iy \mapsto (x, y)$ we give $\mathbb{C}$ its standard manifold structure.

a) Give a smooth map $f : \mathbb{C} \to \mathbb{R}$ which has $1 \in \mathbb{R}$ as a regular value. Conclude that $\S^1$ is an embedded submanifold of $\mathbb{C}$.
b) By inspecting its differential, show that the map
$$p : \mathbb{R} \to \S^1, \quad t \mapsto e^{it}$$
is a local diffeomorphism.
c) Show that the vector field $\frac{\partial}{\partial t} \in \mathfrak{X}(\mathbb{R})$ is p-projectable to a unique smooth vector field $\frac{\partial}{\partial \theta}$ on $\S^1$.
d) Show that there is a unique smooth 1-form on $\S^1$, denoted by $d\theta$, such that $p^* d\theta = dt \in \Omega^1(\mathbb{R})$.
e) Is $d\theta$ closed? Is it a volume form?
f) Compute the integral
$$\int_{\S^1} d\theta.$$
g) Is $d\theta$ exact?

In the coming exercise we consider the 2-dimensional torus:
$$T^2 := S^1 \times S^1$$
equipped with the product manifold structure. Write (t_1, t_2) for the standard coordinates in $\mathbb{R}^2$. By part (a) of the previous exercise, the map
$$q : \mathbb{R}^2 \to T^2, \quad (t_1, t_2) \mapsto (e^{it_1}, e^{it_2})$$
is a local diffeomorphism and by part (c) of that same exercise, the vector fields $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_2}$ on $\mathbb{R}^2$ are q-projectable to unique vector fields on T^2, denoted by $\frac{\partial}{\partial \theta_1}$ and $\frac{\partial}{\partial \theta_2}$.

Exercise 6.27. Let c be an irrational real number and consider the vector field X on T^2 defined as:
$$X = \frac{\partial}{\partial \theta_1} + c\frac{\partial}{\partial \theta_2}.$$

a) Compute the maximal integral curve γ of X starting at $(1,1) \in T^2$.
b) Prove that the image of γ is an immersed submanifold of T^2.
c) Prove that the image of γ is not an embedded submanifold of T^2.

Exercise 6.28. This exercise is about orientability. Recall that a manifold M is orientable if it admits an atlas $\mathscr{A}$ such that:
$$\mathrm{Jac}(\chi_1 \circ \chi_2^{-1})_x > 0, \quad \forall x \in \chi_2(U_1 \cap U_2)$$
and all charts $(U_1, \chi_1), (U_2, \chi_2) \in \mathscr{A}$.

a) Show that if M is an orientable n-dimensional manifold, then there is a volume form $\omega \in \Omega^n(M)$.
b) Show that if ω_1 and ω_2 are two volume forms, then there is a unique no-where vanishing smooth function $f \in C^\infty(M)$ such that
$$f \cdot \omega_1 = \omega_2.$$

In the rest of this exercise we aim to prove that $\mathbb{R}P^2$ is not orientable. Suppose by contradiction that it is orientable. Then it admits a volume-form $\omega \in \Omega^2(\mathbb{R}P^2)$.

c) Consider the map
$$\pi : S^2 \to \mathbb{R}P^2, \quad (x, y, z) \mapsto [x : y : z].$$

Show that π is a local diffeomorphism and conclude that $\pi^*\omega$ is a volume-form on S^2.

d) Show that there is a no-where vanishing function $f \in C^\infty(S^2)$ such that:

$$f \cdot (\pi^*\omega) = z dx \wedge dy + y dz \wedge dx + x dy \wedge dz,$$

where $x, y, z : S^2 \to \mathbb{R}$ are the projections onto the three components in $\mathbb{R}^3$.

e) Compute the pull-back of both sides with the anti-podal map

$$\iota : S^2 \to S^2, \quad p \mapsto -p$$

and derive a contradiction.

6.8 Exam 2018/2019 (November 6th, 2018)

Exercise 6.29. (*1 pt*) Show that, on any manifold, for any $\theta \in \Omega^k(M)$ with k odd, one has $\theta \wedge \theta = 0$. Then find an example of a manifold M and a 2-form M such that $\theta \wedge \theta \neq 0$.

Exercise 6.30. (*3× 0.5= 1.5 pt*) This is about vector fields on $\mathbb{R}^3$. Do the following:

(a) find a vector field X for which $\gamma(t) = (e^{2t}, e^t \sin t, e^t \cos t)$ is an integral curve.
(b) find one more such vector field, but different from X.
(c) for one the vector fields you found, compute its flow.

Exercise 6.31. (*4× 1= 4 pt*) Consider

$$M_4 := \{(x,y,z) \in \mathbb{R}^3 : x^4 + y^4 + z^4 = 1\} \subset \mathbb{R}^3,$$

and the map from M_4 to the 2-sphere S^2 given by

$$f : M_4 \to S^2, \quad f(x,y,z) = (x^2, y^2, z^2)$$

(a) Show that M_4 is a submanifold of $\mathbb{R}^3$ and f is a smooth map.
(b) Compute the tangent space of M_4 at the point $p = (\frac{1}{\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, 0)$; more precisely, show that it is spanned by

$$\left(\frac{\partial}{\partial x}\right)_p - \left(\frac{\partial}{\partial y}\right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial z}\right)_p \in T_pM_4.$$

Similarly at the point $q = (\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}})$.
(c) Show that f is not an immersion at p, but it is a local diffeomorphism around q.
(d) Show that M_4 is not diffeomorphic to

$$M_3 := \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 1\} \subset \mathbb{R}^3.$$

but it is diffeomorphic to S^2.

Exercise 6.32. Let $\theta_1, \ldots, \theta_n \in \Omega^1(M)$ be $n = \dim M$ one-forms on M which are dual to a set $\{X^1, \ldots, X^n\}$ of vector fields on M, in the sense that

$$\theta^i(X_j) = \delta_i^j \quad (1 \text{ if } i = j \text{ and } 0 \text{ otherwise}).$$

(a) (*1 pt*) show that all the 1-forms $\theta_1, \ldots, \theta_n$ are closed if and only if the vector fields $X^1, \ldots, X^n$ are pairwise commuting, i.e. $[X_i, X_j] = 0$ for all i and j.
(b) (*1.5 pt*; the most difficult- you may want to leave it for later) show that for any family of vector fields $X^1, \ldots, X^n$ that are pairwise commuting and which are linearly independent at some point $x_0 \in M$, there exists a chart (U, χ) near x_0 such that

$$X^1_x = \left(\frac{\partial}{\partial \chi_1}\right)_x, \quad \ldots \quad , X^n_x = \left(\frac{\partial}{\partial \chi_n}\right)_x \quad \text{for all } x \in U.$$

(c) (*0.5 pt*) Deduce that if $\{\theta_1, \ldots, \theta_n\}$ is a set of closed 1-forms on the n-dimensional manifold M, that are linearly independent at some point $x_0 \in M$, then there exists a chart (U, χ) of M around x_0 such that $\theta_i|_U = d\chi_i$ on U, for all $i \in \{1, \ldots, n\}$.

Exercise 6.33. Let M be a manifold, and $X,Y \in \mathfrak{X}(M)$ two vector fields. We look at

$$L_X \circ i_Y - i_Y \circ L_X : \Omega^k(M) \to \Omega^{k-1}(M) \quad \text{(for all } k \geq 0).$$

Show that:

(a) (*0.5 pt*) on 1-forms $\theta \in \Omega^1(M)$, it coincides with $i_{[X,Y]}$.
(b) (*1 pt*) it satisfies a Leibniz-type identity that you have to write down yourself.
(c) (*1 pt*) $L_X \circ i_Y - i_Y \circ L_X = i_{[X,Y]}$.

Exercise 6.34. (*2.5 pt*) Compute the following integral

$$\int_{S^2} \big((x+y^{2017})dy \wedge dz + (y+z^{2018})dz \wedge dx + (z+x^{2019})dx \wedge dy\big)$$

(you can use any orientation on S^2 that you want).

6.9 Retake 2017/2018 (January 3rd, 2018)

Note: The questions in blue are worth 1 point, and the rest 0.5 points. The points below add up together to 13.5, so you do not have to solve everything to get the maximum mark of 10! Also, please be aware: blue really means that ”it is worth more points”, and not that ”it is more difficult”.

Exercise 6.35. Compute the flow of $X = -x^2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$.

Exercise 6.36. Show that $M := \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 4\}$ is an embedded submanifold of $\mathbb{R}^3$ and compute the tangent space $T_pM \subset \mathbb{R}^3$ at the point $p = (1,1,-1)$.

Exercise 6.37. Assume that M is an S^1-manifold, i.e. a manifold together with an action of the circle S^1- i.e. a smooth map

$$M \times S^1 \to M, \quad (\lambda, p) \mapsto \lambda \cdot p$$

such that $p \cdot 1 = p$ and $(p\cdot\lambda)\cdot\lambda' = p\cdot(\lambda\lambda')$ for all $\lambda, \lambda' \in S^1$, $p \in M$. We define the following vector field on M:

$$V_p = \frac{d}{dt}\bigg|_{t=0} p \cdot e^{2\pi i t}. \tag{6.9.1}$$

(a) Write down the flow of V using the action of S^1 on M.
(b) Deduce that a point $p \in M$ is a fixed point of the action, i.e. satisfies $p\cdot\lambda = p$ for all $\lambda \in S^1$, if and only if p is a zero of V, i.e. satisfies $V_p = 0$.
(c) Assume now that the action is free i.e. that it has no fixed points (or, cf. the previous point, V has no zeroes). Show that for each $p \in M$, the orbit through p:

$$\mathscr{O}_p := \{p\cdot\lambda : \lambda \in S^1\} \tag{6.9.2}$$

is an embedded submanifold of M diffeomorphic to S^1.

Exercise 6.38. Consider again an S^1-manifold M. A differential form $\omega \in \Omega^\bullet(M)$ is called:

- equivariant: if $\mathrm{m}_\lambda^*(\omega) = \omega$ for all $\lambda \in S^1$, where $\mathrm{m}_\lambda : M \to M$ is the map $p \mapsto p\cdot\lambda$.
- horizontal: if $i_V(\omega) = 0$.
- basic form: if it is both horizontal as well as equivariant.

Show that:

(a) if ω is equivariant, then so is $d\omega$.
(b) ω is equivariant if and only if $L_V(\omega) = 0$.
(c) if ω is basic, then so is $d\omega$.
(d) in general, it is not true that if ω is horizontal than so is $d\omega$ (provide a counter-example, e.g. using the last homework).

Exercise 6.39. Assume now that M is a principal S^1-bundle, i.e. an S^1-manifold with the property that the action is free and that the orbits (6.9.2) of the action are precisely the fibers of a submersion

$$h : M \to B,$$

with values in some other manifold B (usually called the base of the bundle). Show that

(a) The pull-back map $h^* : \Omega^\bullet(B) \longrightarrow \Omega^\bullet(M)$ is injective.
(b) For any differential form $\eta \in \Omega^k(B)$, its pull-back $\omega := h^*(\eta)$ is a basic form on M.

(c) h^* is a linear isomorphism between $\Omega^\bullet(B)$ and the space of basic forms on M.

Exercise 6.40. Assume again that M is a principal S^1-bundle, with corresponding submersion $h : M \to B$, and corresponding vector field (6.9.1).

We choose a 1-form $\omega \in \Omega^1(M)$ which is equivariant and satisfies $\omega(V) = 1$. Show that

(a) $d\omega \in \Omega^2(M)$ is a basic 2-form.
(b) denoting by $\eta \in \Omega^2(B)$ the 2-form on characterized by

$$d\omega = h^*(\eta)$$

(which exists and is unique by the previous exercise), η is a closed 2-form on B.
(c) while η is built using any ω which was equivariant and satisfied $\omega(V) = 1$, prove that the cohomology class $[\eta] \in H^2(B)$ does not depend on the choice of ω.

(this cohomology class is known as "the first Chern class" of the principal S^1-bundle).

Exercise 6.41. Returning now to the Hopf fibration and the last homework(s),

$$h : S^3 \to S^2,$$

we are in the case of a principal S^1-bundle with the action given by $(z_0, z_1) \cdot \lambda = (\lambda z_0, \lambda z_1)$, and with corresponding vector field (as in the last homework):

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}.$$

Show that

(a) $\omega = -y \cdot dx + x \cdot dy - t \cdot dz + z \cdot dt \in \Omega^1(S^3)$ is equivariant and satisfies $\omega(V) = 1$.
(b) compute the resulting form $\eta \in \Omega^2(S^2)$ (cf. the previous exercise).
(c) show that $[\eta] \in H^2(S^2)$ is non-zero and try to draw conclusions on the Hopf fibration (e.g.: can it be isomorphic to the product fibration $S^1 \times S^2$?)

6.10 Exam 2017/2018 (November 8th, 2017)

Exercise 6.42. (*1 pt*) Show that, for a vector field X on a manifold M and $f \in C^\infty(M)$, one has $L_X(f) = 0$ if and only if f is constant on the integral curves of X.

Exercise 6.43. Consider the sphere $S^2 \subset \mathbb{R}^3$ and we use (x,y,z) to denote the standard coordinates in $\mathbb{R}^3$. We consider the following vector field tangent to the sphere

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(S^2)$$

as well as the volume form on the sphere:

$$\theta = x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy \in \Omega^2(S^2)$$

(as before, while the previous formula defines a 2-form on $\mathbb{R}^3$, θ is the restriction to S^2).

(a) (*0.5 pt*) Compute $i_X(\theta)$ and $d(i_X(\theta))$.
(b) (*0.5 pt*) Compute $d\theta$ and $i_X(d\theta))$.
(c) (*0.5 pt*) Compute $L_X(\theta)$ in two ways: one using the Cartan formula, and one using the properties of L_X (being a derivation, and commuting with d).
(d) (*0.5 pt*) Compute the flow ϕ^t of X.
(e) (*0.5 pt*) Show that $(\phi^t)^*\theta = \theta$ for all $t \in \mathbb{R}$.

Exercise 6.44. Consider

$$M := \{(x,y,z) \in \mathbb{R}^3 : (x^2+y^2+z^2-5)^2 + 16z^2 = 16\} \subset \mathbb{R}^3.$$

(a) (*1 pt*) Show that M is a submanifold of $\mathbb{R}^3$.
(b) (*1 pt*) Compute the tangent space of M at the point $p = (3,0,0)$; more precisely, show that it is spanned by

$$\left(\frac{\partial}{\partial y}\right)_p \quad \text{and} \quad \left(\frac{\partial}{\partial z}\right)_p \in T_pM.$$

Similarly, compute the tangent space at the point $q = (2,0,1)$.
(c) (*0.5 pt*) Draw a picture of M (hint: if you do not see how M looks like, maybe compute the tangent space at one more point).
(d) (*0.5 pt*) Prove that M is diffeomorphic to $S^1 \times S^1$.

Exercise 6.45. Assume that M is a connected 3-dimensional manifold, and $V^1, V^2, V^3 \in \mathfrak{X}(M)$ are vector fields on M with the property that V^1_p, V^2_p, V^3_p form a basis of T_pM for all $p \in M$ and let $\theta_1, \theta_2, \theta_3 \in \Omega^1(M)$ be the 1-forms that are dual to V^1, V^2, V^3, i.e. satisfying

$$\theta_i(V^j) = \delta^i_j \text{ (1 if } i = j \text{ and 0 otherwise)}.$$

(a) (*0.5 pt*) Show that, for any $f \in \mathscr{C}^\infty(M)$ one has

$$df = L_{V^1}(f) \cdot \theta_1 + L_{V^2}(f) \cdot \theta_2 + L_{V^3}(f) \cdot \theta_3.$$

(b) (*1 pt*) Show that the vector fields V^i satisfy

$$[V^1, V^2] = 2V^3, \quad [V^2, V^3] = 2V^1, \quad [V^3, V^1] = 2V^2.$$

if and only if the 1-forms θ_i satisfy

$$d\theta_1 = -2\theta_2 \wedge \theta_3, \quad d\theta_2 = -2\theta_3 \wedge \theta_1, \quad d\theta_3 = -2\theta_1 \wedge \theta_2.$$

From now on we assume that all these are satisfied. Assume furthermore that the 1-forms are invariant with respect to a vector field $V \in \mathfrak{X}(M)$ in the sense that

$$L_V(\theta_1) = L_V(\theta_2) = L_V(\theta_3) = 0.$$

Introduce the following real-valued functions on M:

$$h_1 = i_V(\theta_1), \quad h_2 = i_V(\theta_2), \quad h_3 = i_V(\theta_3).$$

(c) (*0.5 pt*) Prove that

$$dh_1 = 2h_2 \cdot \theta_3 - 2h_3 \cdot \theta_2,$$
$$dh_2 = 2h_3 \cdot \theta_1 - 2h_1 \cdot \theta_3,$$
$$dh_3 = 2h_1 \cdot \theta_2 - 2h_2 \cdot \theta_1$$

(d) (*0.5 pt*) Deduce that

$$h = (h_1, h_2, h_3) : M \to \mathbb{R}^3$$

takes values in a sphere S^2_r (of some radius $r \geq 0$).

From now on we assume that $r = 1$ (i.e. h takes values in S^2).

(e) (*0.5 pt*) Show that h is constant on each integral curve of V.
(f) (*0.5 pt*) Show that V^1 is h-projectable to the following tangent vector on S^2:

$$E^1 = 2\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right) \in \mathfrak{X}(S^2),$$

i.e. $(dh)_p(V^1_p) = E^1_{h(p)}$ for all $p \in M$. And similarly for V^2 and V^3.
(g) (*0.5 pt*) Show that if M is compact then h is surjective submersion onto S^2 and any fiber $h^{-1}(q)$ (with $q \in S^2$) which is connected is diffeomorphic to a circle.
(h) (*0.5 pt*) The homeworks show that the previous scenario can arise on $M = S^3$. Find another example, with M still connected, but for which the fibers of h are not connected.