The *tangent* space of M at p $\ni$ tangent vectors	$T_pM \ni v$ means function $v: \text{Charts}_p(M) \longrightarrow \mathbb{R}^m$, $v \longmapsto v^x$ satisfying (*) $v^{x'} = (dc)_{x(p)}(v^x)$ $(c = x'\circ x^{-1})$	$T_pM \ni \partial$ means linear map $\partial: C^\infty(M) \longrightarrow \mathbb{R}$. satisfying derivation rule $\partial(fg) = f(p)\cdot\partial(g) + g(p)\cdot\partial(f)$
Speeds of $\gamma \in \text{Curves}_p(M)$ $\in$ tangent space	$T_pM \ni \frac{d\gamma}{dt}(0)$: the function $x \longmapsto \frac{d\gamma^x}{dt}(0)$	$T_pM \ni \frac{d\gamma}{dt}(0)$: the derivation $f \longmapsto (f\circ\gamma)'(0)$
Differentials of smooth maps $p \in M \xrightarrow{F} N$	$\frac{d\gamma}{dt}(0) \in T_pM \xrightarrow{(dF)_p} \frac{dF\circ\gamma}{dt}(0) \in T_{F(p)}N$ OR: $\left((dF)_p(v)\right)^{x'} := \left(dF_x^{x'}\right)_{x(p)}(v^x)$	$T_pM \ni \frac{d\gamma}{dt}(0) \xrightarrow{(dF)_p} T_{F(p)}N \ni \frac{dF\circ\gamma}{dt}(0)$ OR: $(dF)_p(\partial)(f) := \partial(f\circ F)$
Basis induced by charts $(U, x) \ni p$	$\left(\frac{\partial}{\partial x_i}\right)_p \in T_pM$, $1 \le i \le p$ $x \longmapsto e_i$	$\left(\frac{\partial}{\partial x_i}\right)_p \in T_pM$, $1 \le i \le p$ $f \longmapsto \frac{\partial f}{\partial x_i}(p) := \frac{\partial f_x}{\partial x_i}(x(p))$
$M = \mathbb{R}^m$: use identity chart $x = Id_{\mathbb{R}^n}$	$\left(\frac{\partial}{\partial x_i}\right)_p \in T_p\mathbb{R}^m \xrightarrow[\text{Standard}_p]{\sim} \mathbb{R}^m \ni e_i$	$\mathbb{R}^m \xrightarrow{\sim} T_p\mathbb{R}^m \ni \left(\frac{\partial}{\partial x_i}\right)_p$

	$T_pM \xrightarrow{\quad I_p \quad}$	T_pM
The **tangent** space of M at p Ψ tangent vectors	$\boxed{T_pM}$ Ψ v means function $v : Charts_p(M) \to R^m$ $x \longmapsto v^x$ satisfying $(*)\quad v^{x'} = (dc)_{x(p)}(v^x)$ $(c = x'\circ x^{-1})$	$\boxed{T_pM}$ ∂ means linear map $\partial : C^\infty(M) \longrightarrow R$. satisfying derivation rule $\partial(fg) = f(p)\cdot\partial(g) + g(p)\cdot\partial(f)$
Speeds of $\gamma \in Curves_p(M)$ $\in$ tangent space	$T_pM \ni \boxed{\frac{d\gamma}{dt}(0)}$: the function $\xrightarrow{\quad I_p \quad}$ $x \longmapsto \frac{d\gamma^x}{dt}(0)$	$T_pM \ni \boxed{\frac{d\gamma}{dt}(0)}$ the derivation $f \longmapsto (f\circ\gamma)'(0)$
Differentials of smooth maps $p \in M$ $\downarrow F$ N	$\frac{d\gamma}{dt}(0) \in T_pM \xrightarrow{\quad I_p \quad}$ $\downarrow \boxed{(dF)_p}$ $\frac{dF\circ\gamma}{dt}(0) \in T_{F(p)}N \xrightarrow{\quad I_{F(p)} \quad}$ OR: $\left((dF)_p(v)\right)^{x'} := (dF_x^{x'})_{x(p)}(v^x)$	$T_pM \ni \frac{d\gamma}{dt}(0)$ $\downarrow \boxed{(dF)_p}$ $T_{F(p)}N \ni \frac{dF\circ\gamma}{dt}(0)$ OR: $(dF)_p(\partial)(f) := \partial(f\circ F)$
Basis induced by charts (U,x) Ψ p	$\boxed{\left(\frac{\partial}{\partial x_i}\right)_p} \in T_pM,\ 1\le i\le p$ $\xrightarrow{\quad I_p \quad}$ $x \longmapsto e_i$	$\boxed{\left(\frac{\partial}{\partial x_i}\right)_p} \in T_pM,\ 1\le i\le p$ $f \longmapsto \frac{\partial f}{\partial x_i}(p) := \frac{\partial f_x}{\partial x_i}(x(p))$
$M = R^m$: use identity chart $x = Id_{R^m}$	$\left(\frac{\partial}{\partial x_i}\right)_p \in T_pR^m \xrightarrow{\quad I_p \quad}$ $T_pR^m \xrightarrow[\sim]{Standard_p} R^m$ Ψ e_i	$T_pR^m \ni \left(\frac{\partial}{\partial x_i}\right)_p$ $R^m \xrightarrow[\sim]{\partial} T_pR^m$

$T_pM \ni v \longmapsto \partial_v \in T_pM$

defined by

$\partial_v : C^\infty(M) \longrightarrow R$

$(x \in Charts_p(M)\ a/any)$ $\quad f \longmapsto \boxed{\partial_v(f) := \partial_{v^x}(f_x)}$

$\boxed{\text{For } M \subseteq \mathbb{R}^n}$

$$\sum_{i=1}^{m} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p \in T_p\mathbb{R}^n \qquad \sum_{i=1}^{m} \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p \in T_p\mathbb{R}^n \quad (\lambda_i \in \mathbb{R}).$$

$$e_i \in \mathbb{R}^n$$

$$\begin{tikzcd}
T_p\mathbb{R}^n \arrow[rr, "I_p"] \arrow[dr, "\text{Standard}_p", "\sim"'] & & T_p\mathbb{R}^n \\
& \mathbb{R}^n \arrow[ur, "\sim"] & \\
T_pM \arrow[uu, hook, "\text{inclusion}"] \arrow[rr, "I_p"] \arrow[dr, "\text{Standard}_p", "\sim"'] & & T_pM \arrow[uu, hook, "\text{inclusion}"'] \\
& T_p^{class}M \arrow[uu, hook, "\text{inclusion}"] \arrow[ur, "\sim"] &
\end{tikzcd}$$

$$T_pM = \left\{\frac{d\gamma}{dt}(0) : \gamma \in Curves_p(M)\right\} \qquad T_pM \cong \left\{\frac{d\gamma}{dt}(0) : \gamma \in Curves_p(M)\right\}$$

$$T_p^{class}M = \left\{\frac{d\gamma}{dt}(0) : \gamma \in Curves_p(M)\right\}$$

Which expressions

$$\sum \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p \in T_p\mathbb{R}^n$$

are actually in **T_pM**?

It depends on M!

<u>Ex</u>: If M as in RVT:

$$M = F^{-1}(q) = \{x \in M : F(x) = q\}$$

where $F: \mathbb{R}^m \longrightarrow \mathbb{R}^k \ni q$ regular

then

$$T_pM = Ker(dF)_p = \{v \in T_pM : (dF)_p(v) = 0\}$$

$$\left((dF)_p : T_p\mathbb{R}^n \longrightarrow T_{F(p)}\mathbb{R}^k\right)$$

Which expressions

$$\sum \lambda_i \left(\frac{\partial}{\partial x_i}\right)_p \in T_p\mathbb{R}^n$$

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$$\left((dF)_p : T_p\mathbb{R}^n \longrightarrow T_{F(p)}\mathbb{R}^k\right).$$

GENERAL BASIC PROPERTIES

- $F: M \longrightarrow N$ SUBMERSION/ IMMERSION/ LOCAL DIFFEOMORPHISM at $p \iff (dF)_p : T_pM \to T_{F(p)}N$ SURJECTIVE/ INJECTIVE/ ISOMORPHISM

- $\boxed{M \subseteq N}$ SUBMANIFOLD $\Rightarrow$ $\boxed{T_pM \subseteq T_pN}$

... HOW TO RECOGNIZE WHEN A $v \in T_pN$ is in T_pM?

Trick 1: write $v = \frac{d\gamma}{dt}(0)$ with γ = path in M

OR

Trick 2: WHEN M is AS IN THE GENERAL RVT:

$$M = F^{-1}(q) = \{ x \in N : F(x) = q \} \quad \left(\begin{smallmatrix} q \\ \dots \end{smallmatrix} \right)$$

THEN

$$\boxed{T_pM = \mathrm{Ker}\,(dF)_p = \{ v \in T_pN : (dF)_p(v) = 0 \}.}$$

- Chain rule: $\underset{p}{\overset{M}{\cup\!\!\!\!\!\!\;\;}} \xrightarrow{G} N \xrightarrow{F} P \iff \boxed{(d\,G \circ F)_p = (dG)_{F(p)} \circ (dF)_p}$

Particular case: $M \subseteq N$ (& G = inclusion) $\Rightarrow$

$$\Rightarrow (dF)_p \Big|_{T_pM} = d\,(F|_M)_p .$$

$$\left(\begin{array}{ccc} (dF)_p : & T_pN & \longrightarrow T_{F(p)}P \\ & \cup\!\! \text{I} & \\ & T_pM & \end{array} \right)$$