

The *tangent* space of M at p
 \downarrow
 tangent vectors

$T_p M$
 \downarrow
 ψ means function
 $\psi: \text{Charts}_p(M) \rightarrow \mathbb{R}^m$
 $\psi \mapsto \psi^x$

$T_p M$

\Downarrow means linear map
 $\partial: C^\infty(M) \rightarrow \mathbb{R}$

satisfying derivation rule

$$\partial(fg) = f(p) \cdot \partial(g) + g(p) \cdot \partial(f)$$

satisfying

$$(x) \quad \psi^x = (dc)_{\psi(p)}(\psi^x) \quad (c = x \circ x^{-1})$$

Speeds of

$\gamma \in \text{Curves}_p(M)$
 \in tangent space

$T_p M \ni \frac{d\gamma}{dt}(0)$: the function
 $x \mapsto \frac{d\gamma^x}{dt}(0)$

$T_p M \ni \frac{d\gamma}{dt}(0)$: the derivation
 $f \mapsto (f \circ \gamma)'(0)$

Differentials

of smooth maps
 $p \in M$
 $\downarrow F$
 N

$\frac{d\gamma}{dt}(0) \in T_p M$
 \downarrow
 $\frac{dF \circ \gamma}{dt}(0) \in T_{F(p)} N$
 $(dF)_p$
 OR: $(dF)_p(\psi^x) = (dF_x^x)_{\psi(p)}(\psi^x)$

$T_p M \ni \frac{d\gamma}{dt}(0)$
 $\downarrow (dF)_p$
 $T_{F(p)} N \ni \frac{dF \circ \gamma}{dt}(0)$
 OR: $(dF)_p(\partial)(f) = \partial(f \circ F)$

Basis induced by charts (U, x)
 $\downarrow \psi$
 p

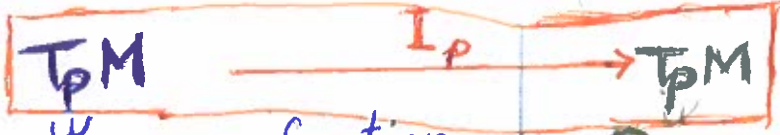
$(\frac{\partial}{\partial x_i})_p \in T_p M, 1 \leq i \leq p$
 $x \mapsto e_i$

$(\frac{\partial}{\partial x_i})_p \in T_p M, 1 \leq i \leq p$
 $f \mapsto \frac{\partial f}{\partial x_i}(p) = \frac{\partial f \circ x}{\partial x_i}(x(p))$

$M = \mathbb{R}^m$:
 use identity chart $x = \text{Id}_{\mathbb{R}^m}$

$T_p \mathbb{R}^m$
 \downarrow Standard p
 $\mathbb{R}^m \ni e_i$
 $T_p \mathbb{R}^m \ni (\frac{\partial}{\partial x_i})_p$

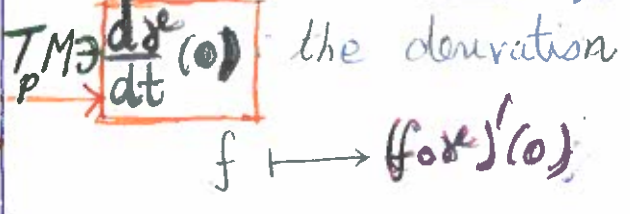
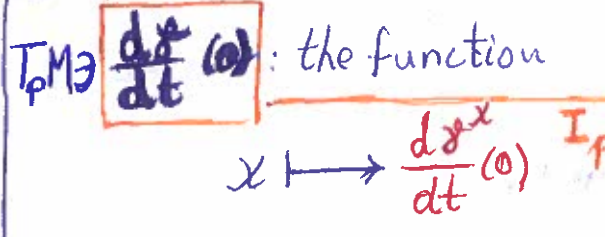
The **tangent space** of M at p
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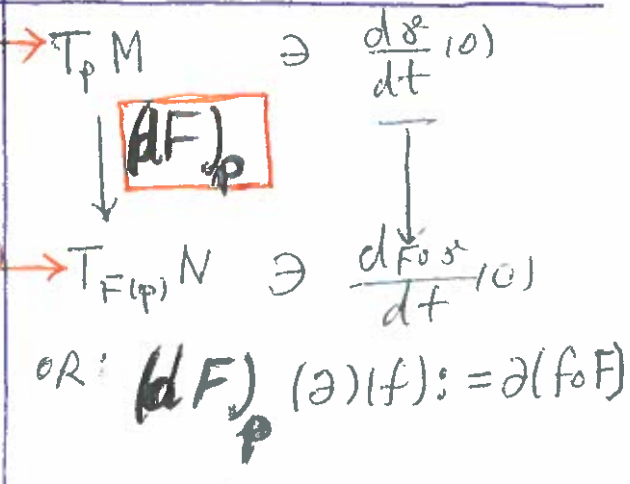
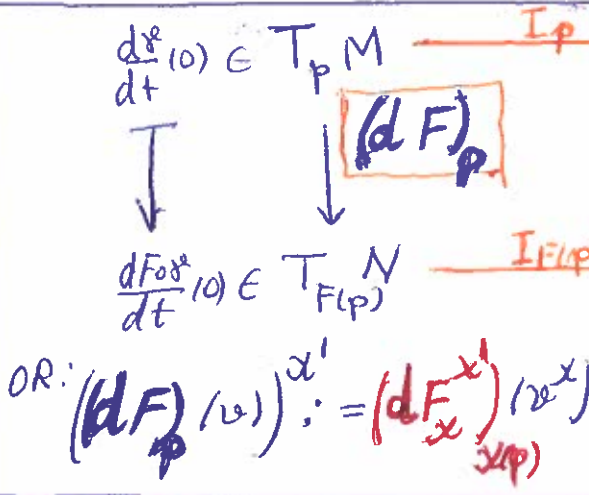
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 $x \mapsto x^x$
 Satisfying
 $(x) \quad \psi^{x'} = (dc)_{x(p)} (\psi^x)$
 $(c = x' \circ x^{-1})$

∂ means linear map
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 satisfying derivation rule
 $\partial(fg) = f(p) \cdot \partial(g) + g(p) \cdot \partial(f)$

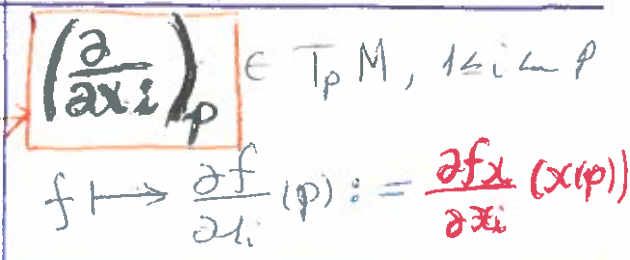
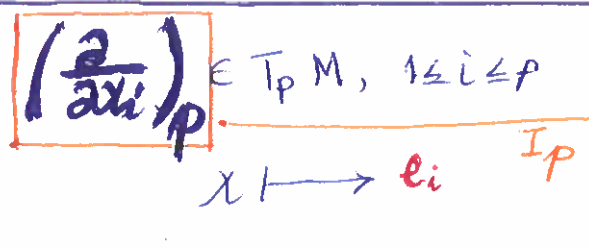
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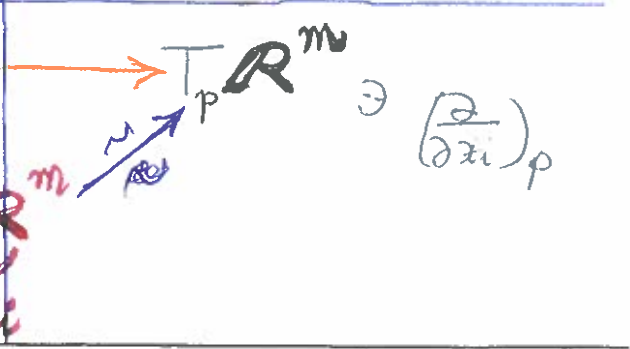
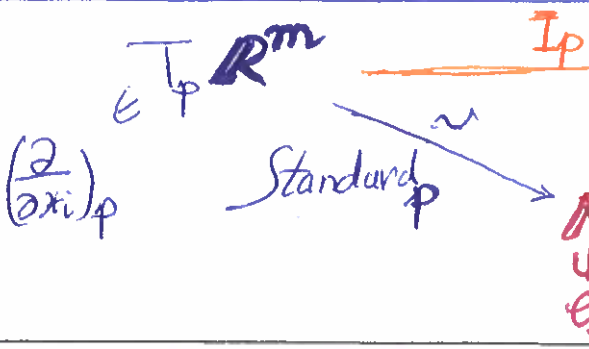
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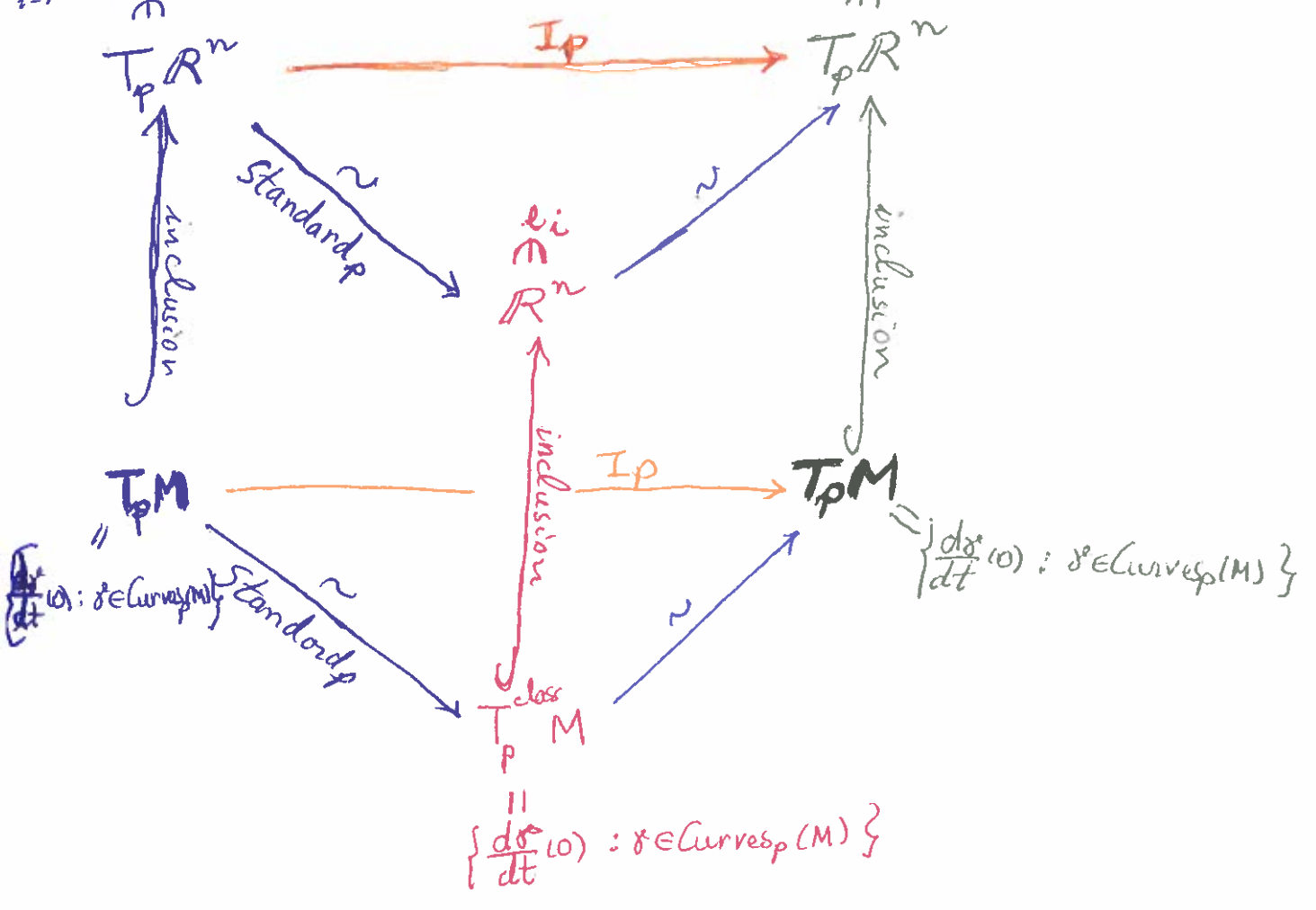
$T_p M \ni v \mapsto \partial_v \in T_p M$
 defined by
 $\partial_v: C^\infty(M) \rightarrow \mathbb{R}$
 $f \mapsto \partial_v(f) := \partial_{v^x}(f_x)$
 $(x \in \text{Chart}_p(M) \text{ a / any})$

For $M \subseteq \mathbb{R}^n$

$$\sum_{i=1}^m \lambda_i \left(\frac{\partial}{\partial x_i} \right)_p$$

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($\lambda_i \in \mathbb{R}$)



Which expressions

$$\sum \lambda_i \left(\frac{\partial}{\partial x_i} \right)_p \in T_p \mathbb{R}^n$$

are actually in $T_p M$?

It depends on M !

Ex: If M as in RVT:

$$M = F^{-1}(q) = \{x \in M : F(x) = q\}$$

where $F: \mathbb{R}^m \rightarrow \mathbb{R}^k \ni q$ regular

then

$$T_p M = \text{Ker}(dF)_p = \{v \in T_p M : (dF)_p(v) = 0\}$$

$$(dF)_p: T_p \mathbb{R}^m \rightarrow T_{F(p)} \mathbb{R}^k$$

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GENERAL BASIC PROPERTIES

$F: M \rightarrow N$ SUBMERSION/ IMMERSION/ LOCAL DIFFEOMORPHISM at $p \iff (dF)_p: T_p M \rightarrow T_p N$ SURJECTIVE/ INJECTIVE/ ISOMORPHISM

$M \subseteq N$ SUBMANIFOLD $\Rightarrow T_p M \subseteq T_p N$

... HOW TO RECOGNIZE WHEN A $v \in T_p N$ IS IN $T_p M$?

Trick 1: write $v = \frac{d\gamma}{dt}(0)$ with $\gamma =$ path in M

OR

Trick 2: WHEN M IS AS IN THE GENERAL RVT:

$$M = F^{-1}(q) = \{x \in M : F(x) = q\} \quad (q \dots)$$

THEN

$$T_p M = \text{Ker} (dF)_p = \{v \in T_p N : (dF)_p(v) = 0\}$$

Chain rule: $M \xrightarrow{G} N \xrightarrow{F} P \iff (dG \circ F)_p = (dG)_p \circ (dF)_p$

Particular case: $M \subseteq N$ (& $G =$ inclusion) \Rightarrow

$$\Rightarrow (dF)_p \Big|_{T_p M} = d(F|_M)_p$$

$$\left(\begin{array}{c} (dF)_p : T_p N \longrightarrow T_{F(p)} P \\ \cup \\ T_p M \end{array} \right)$$