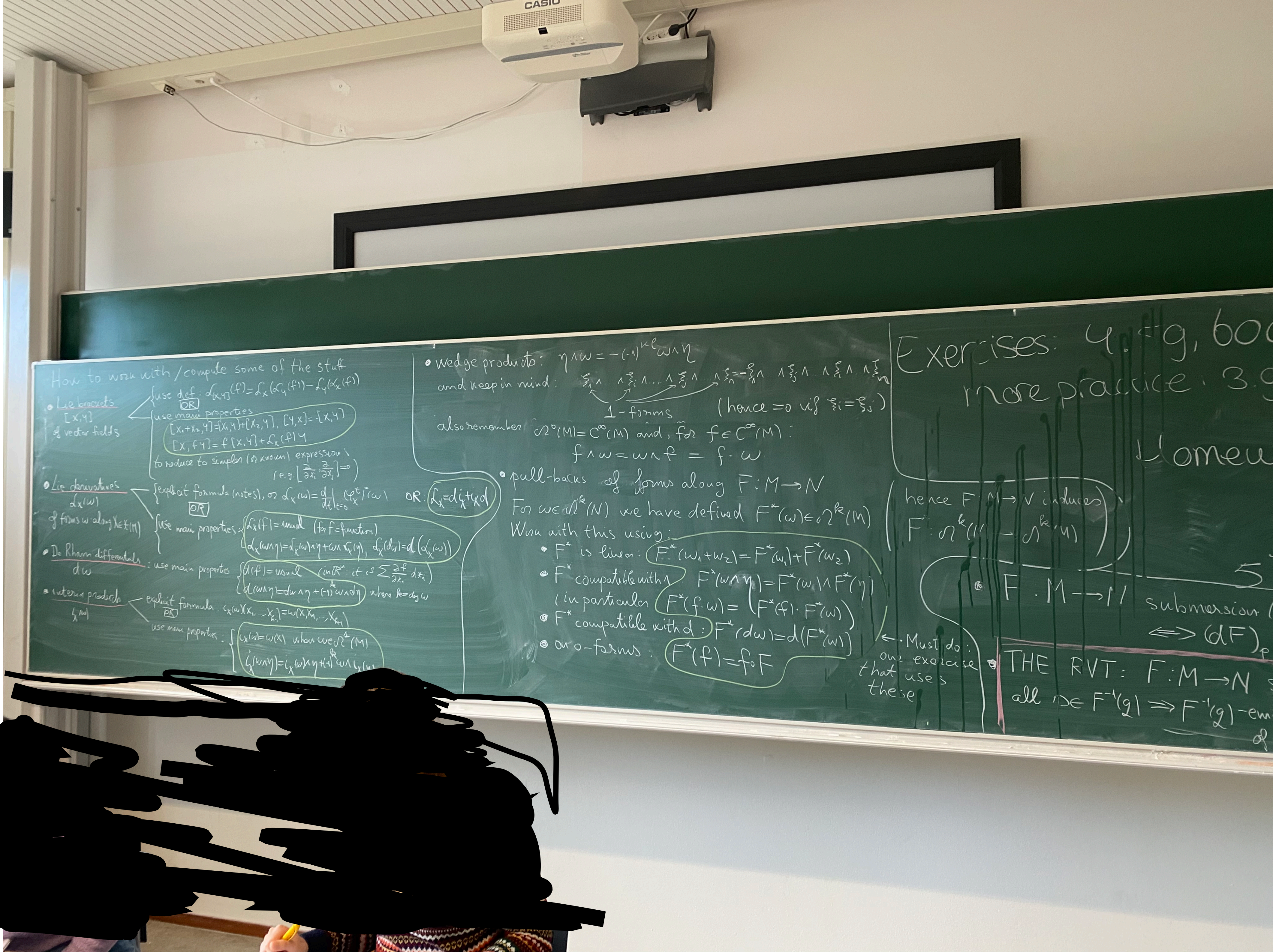

How to work with / compute some of the stuff
Lie brackets $[X,Y]$ of vector fields
use def: $\mathcal{L}_{[X,Y]}(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))$
OR
use main properties
$[X_1+X_2,Y]=[X_1,Y]+[X_2,Y]$, $[Y,X]=-[X,Y]$
$[X,fY]=f[X,Y]+\mathcal{L}_X(f)Y$
to reduce to simpler (or known) expressions (e.g. $[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}]=0$)
Lie derivatives $\mathcal{L}_X(\omega)$ of forms ω along $X \in \mathfrak{X}(M)$
explicit formula (notes), or $\mathcal{L}_X(\omega)=\frac{d}{dt}\big|_{t=0}(\varphi_X^t)^*(\omega)$
OR: $\mathcal{L}_X = d i_X + i_X d$
OR
use main properties: $\mathcal{L}_X(f)=$ usual (for f = function)
$\mathcal{L}_X(\omega\wedge\eta)=\mathcal{L}_X(\omega)\wedge\eta+\omega\wedge\mathcal{L}_X(\eta)$, $\mathcal{L}_X(d\omega)=d(\mathcal{L}_X(\omega))$
De Rham differentials $d\omega$: use main properties
$d(f)=$ usual (in $\mathbb{R}^n$: it is $\sum \frac{\partial f}{\partial x_i} dx_i$)
$d(\omega\wedge\eta)=d\omega\wedge\eta+(-1)^k\omega\wedge d\eta$ where $k=\deg\omega$
interior products $i_X(\omega)$
explicit formula: $i_X(\omega)(X_1,\dots,X_{k-1})=\omega(X,X_1,\dots,X_{k-1})$
OR
use main properties: $i_X(\omega)=\omega(X)$ when $\omega\in\Omega^1(M)$
$i_X(\omega\wedge\eta)=i_X(\omega)\wedge\eta+(-1)^k\omega\wedge i_X(\eta)$
wedge products: $\eta\wedge\omega=-(-1)^{kl}\omega\wedge\eta$
and keep in mind: $\xi_1\wedge\dots\wedge\xi_i\wedge\dots\wedge\xi_j\wedge\dots\wedge\xi_n=-\xi_1\wedge\dots\wedge\xi_j\wedge\dots\wedge\xi_i\wedge\dots\wedge\xi_n$
1-forms
(hence $=0$ if $\xi_i=\xi_j$)
also remember $\Omega^0(M)=C^\infty(M)$ and, for $f\in C^\infty(M)$:
$f\wedge\omega=\omega\wedge f=f\cdot\omega$
pull-backs of forms along $F: M\to N$
For $\omega\in\Omega^k(N)$ we have defined $F^*(\omega)\in\Omega^k(M)$
Work with this using:
F^* is linear: $F^*(\omega_1+\omega_2)=F^*(\omega_1)+F^*(\omega_2)$
F^* compatible with $\wedge$: $F^*(\omega\wedge\eta)=F^*(\omega)\wedge F^*(\eta)$
(in particular $F^*(f\cdot\omega)=F^*(f)\cdot F^*(\omega)$)
F^* compatible with d: $F^*(d\omega)=d(F^*(\omega))$
on 0-forms: $F^*(f)=f\circ F$
← Must do: one exercise that uses these
Exercises:
more practice: 3.9
Homew
hence $F: M\to N$ induces $F^*: \Omega^k(N)\to\Omega^k(M)$
$F: M\to N$ submersion
$\iff (dF)_p$
THE RVT: $F: M\to N$
all $p\in F^{-1}(q) \Rightarrow F^{-1}(q)$-em

How to work with / compute some of the stuff

- **Lie brackets** $[X,Y]$ of vector fields
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  - $\boxed{\text{OR}}$
  - use main properties:
    $[X_1+X_2, Y] = [X_1,Y]+[X_2,Y]$, $[Y,X] = -[X,Y]$
    $[X, fY] = f[X,Y] + \mathcal{L}_X(f)Y$
    to reduce to simpler (or known) expressions (e.g. $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]=0$)

- **Lie derivatives** $\mathcal{L}_X(\omega)$ of forms ω along $X \in \mathfrak{X}(M)$
  - explicit formula (notes), or $\mathcal{L}_X(\omega) = \frac{d}{dt}\Big|_{t=0} (\varphi_X^t)^*(\omega)$ OR: $\mathcal{L}_X = d i_X + i_X d$
  - $\boxed{\text{OR}}$
  - use main properties:
    $\mathcal{L}_X(f) = $ usual (for f = function)
    $\mathcal{L}_X(\omega\wedge\eta) = \mathcal{L}_X(\omega)\wedge\eta + \omega\wedge\mathcal{L}_X(\eta)$, $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X(\omega))$

- **De Rham differentials** $d\omega$: use main properties
  - $d(f) = $ usual (in $\mathbb{R}^n$: it is $\sum \frac{\partial f}{\partial x_i} dx_i$)
  - $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k \omega\wedge d\eta$ where $k = \deg \omega$

- **interior products** $i_X(\omega)$
  - explicit formula: $i_X(\omega)(X_1,\dots,X_{k-1}) = \omega(X, X_1,\dots,X_{k-1})$
  - $\boxed{\text{OR}}$
  - use main properties:
    $i_X(\omega) = \omega(X)$ when $\omega \in \Omega^1(M)$
    $i_X(\omega\wedge\eta) = i_X(\omega)\wedge\eta + (-1)^k \omega\wedge i_X(\eta)$

- wedge pro[...]
  and keep i[...]

  also rememb[...]

f the stuff

$_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))$

$[Y,X] = -[X,Y]$

$X(f)\, Y$

en) expression ;

(e.g. $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$)

$\mathcal{L}_X(\omega) = \frac{d}{dt}\Big|_{t=0} (\varphi_X^t)^*(\omega)$ OR: $\mathcal{L}_X = d\, i_X + i_X\, d$

= usual (for f = function)

$\wedge\eta) = \mathcal{L}_X(\omega)\wedge\eta + \omega\wedge\mathcal{L}_X(\eta)$, $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X(\omega))$

= usual (in $\mathbb{R}^n$: it is $\sum \frac{\partial f}{\partial x_i} dx_i$)

$\eta) = d\omega\wedge\eta + (-1)^k \omega\wedge d\eta$) where $k = \deg\omega$

$X_{k-1}) = \omega(X, X_1, \dots, X_{k-1})$

$\omega(X)$ when $\omega\in\Omega^1(M)$

$\eta) = i_X(\omega)\wedge\eta + (-1)^k \omega\wedge i_X(\eta)$

- wedge products: $\eta\wedge\omega = -(-1)^{kl}\,\omega\wedge\eta$

and keep in mind: $\xi_1\wedge\dots\wedge\xi_i\wedge\dots\wedge\xi_j\wedge\dots\wedge\xi_n = -\xi_1\wedge\dots\wedge\xi_j\wedge\dots\wedge\xi_i\wedge\dots\wedge\xi_n$

1-forms (hence $=0$ if $\xi_i = \xi_j$)

also remember: $\Omega^0(M) = C^\infty(M)$ and, for $f\in C^\infty(M)$:

$$f\wedge\omega = \omega\wedge f = f\cdot\omega$$

- pull-backs of forms along $F: M\to N$

For $\omega\in\Omega^k(N)$ we have defined $F^*(\omega)\in\Omega^k(M)$

Work with this using:

- F^* is linear: $F^*(\omega_1+\omega_2) = F^*(\omega_1)+F^*(\omega_2)$
- F^* compatible with $\wedge$: $F^*(\omega\wedge\eta) = F^*(\omega)\wedge F^*(\eta)$
  (in particular $F^*(f\cdot\omega) = F^*(f)\cdot F^*(\omega)$)
- F^* compatible with d: $F^*(d\omega) = d(F^*(\omega))$
- on 0-forms: $F^*(f) = f\circ F$

← Must do: one exercise that uses these

Exercises: 4,

more practi

hence $F: M\to N$ induces

$F^*: \Omega^k(N) \to \Omega^k(M)$

- $F: M \to N$

- THE RVT:

all $x\in F^{-1}(q)$ =

Exercises: 4.?9, 60c, 63c,

more practice: 3.93

Homework: d, e, i **two** points

hence $F: M \to N$ induces

$F^*: \Omega^k(N) \to \Omega^k(M)$

st ... $= \mathcal{L}_X i_X \cancel{\alpha}$

- $F: M \to N$ submersion (or immersion) at $p \in M \iff$

$\iff (dF)_p : T_p M \longrightarrow T_{F(p)} N$ is surjective (or injective)

- THE RVT: $F: M \to N$ smooth, $q \in N$ s.t. F is a submersion at all $x \in F^{-1}(q) \Rightarrow F^{-1}(q)$ - embedded submanifold of dimension $\dim M - \dim N$ and $T_p(F^{-1}(q)) = \{v \in T_p M : (dF)_p(v) = 0\}$

← Must do: one exercise that uses these