

How to work with / compute some of the stuff

• Lie brackets  
[X, Y]  
of vector fields

use def:  $d_{[X, Y]}(f) = \mathcal{L}_X(\mathcal{L}_Y(f)) - \mathcal{L}_Y(\mathcal{L}_X(f))$   
OR  
use main properties  
 $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y]$   
 $[X, fY] = f[X, Y] + \mathcal{L}_X(f)Y$   
to reduce to simpler (or known) expressions  
(e.g.  $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ )

• Lie derivatives  
 $\mathcal{L}_X(w)$   
of forms  $w$  along  $X \in \mathfrak{X}(M)$

explicit formula (notes), or  $\mathcal{L}_X(w) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^* w)$  OR:  $\mathcal{L}_X = d_X + X \lrcorner$   
OR  
use main properties:  $\mathcal{L}_X(f) = X(f)$  (for  $f$  function)  
 $\mathcal{L}_X(w \lrcorner \eta) = \mathcal{L}_X(w) \lrcorner \eta + w \lrcorner \mathcal{L}_X(\eta)$ ,  $\mathcal{L}_X(d\eta) = d(\mathcal{L}_X(\eta))$

• De Rham differential  
 $d$

use main properties:  $d(f) = X(f)$  (in  $\mathbb{R}^n$  it is  $\sum \frac{\partial f}{\partial x_i} dx_i$ )  
 $d(w \lrcorner \eta) = d(w) \lrcorner \eta + (-1)^k w \lrcorner d\eta$  where  $k = \text{deg } w$

• interior products  
 $\lrcorner$

explicit formula:  $\lrcorner_{X_i}(w) = w(x_i) - X_i \lrcorner w$   
OR  
use main properties:  $\lrcorner_X(w) = w(X)$  when  $w \in \mathfrak{X}^k(M)$   
 $\lrcorner_X(w \lrcorner \eta) = \lrcorner_X(w) \lrcorner \eta + (-1)^k w \lrcorner \lrcorner_X(\eta)$

• wedge products:  $\eta \wedge w = (-1)^{kl} w \wedge \eta$   
and keep in mind:  $\xi_1 \wedge \dots \wedge \xi_n = -\xi_n \wedge \dots \wedge \xi_1$   
1-forms (hence = 0 if  $\xi_i = \xi_j$ )  
also remember:  $\mathfrak{O}^k(M) = C^\infty(M)$  and, for  $f \in C^\infty(M)$ :  
 $f \wedge w = w \lrcorner f = f \cdot w$

• pull-backs of forms along  $F: M \rightarrow N$   
For  $w \in \mathfrak{O}^k(N)$  we have defined  $F^*(w) \in \mathfrak{O}^k(M)$   
Work with this using:

- $F^*$  is linear:  $F^*(w_1 + w_2) = F^*(w_1) + F^*(w_2)$
- $F^*$  compatible with  $\wedge$ :  $F^*(w \wedge \eta) = F^*(w) \wedge F^*(\eta)$   
(in particular  $F^*(f \cdot w) = F^*(f) \cdot F^*(w)$ )
- $F^*$  compatible with  $d$ :  $F^*(d w) = d(F^*(w))$
- on 0-forms:  $F^*(f) = f \circ F$

← Must do: one exercise that uses these

Exercises: 4.9, 6.0  
more practice: 3.9

hence  $F: M \rightarrow V$  induces  
 $F: \mathfrak{O}^k(M) \rightarrow \mathfrak{O}^k(V)$

•  $F: M \rightarrow N$  submersion  
 $\Leftrightarrow (dF)_p$

THE RVT:  $F: M \rightarrow N$   
all  $q \in F^{-1}(q) \Rightarrow F^{-1}(q)$  - em of



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• Lie brackets  
 $[X, Y]$   
of vector fields

use def:  $[X, Y](f) = L_X(L_Y(f)) - L_Y(L_X(f))$   
OR  
use main properties:  
 $[X_1 + X_2, Y] = [X_1, Y] + [X_2, Y], [Y, X] = -[X, Y]$   
 $[X, fY] = f[X, Y] + L_X(f)Y$   
to reduce to simpler (or known) expressions:  
(e.g.  $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ )

• Lie derivatives  
 $L_X(w)$   
of forms  $w$  along  $X \in \mathfrak{X}(M)$

explicit formula (notes), or  $L_X(w) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^X)^*(w)$  OR:  $L_X = d\iota_X + \iota_X d$

use main properties:  $L_X(f) = \text{usual}$  (for  $f = \text{function}$ )  
 $L_X(w \wedge \eta) = L_X(w) \wedge \eta + w \wedge L_X(\eta), L_X(dw) = d(L_X(w))$

• De Rham differentials  
 $d w$

use main properties:  $d(f) = \text{usual}$  (in  $\mathbb{R}^n$ : it is  $\sum \frac{\partial f}{\partial x_i} dx_i$ )  
 $d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$  where  $k = \text{deg } w$

• interior products  
 $\iota_X(w)$

explicit formula:  $\iota_X(w)(X_1, \dots, X_{k-1}) = w(X, X_1, \dots, X_{k-1})$   
OR  
use main properties:  $\iota_X(w) = w(X)$  when  $w \in \mathcal{O}^1(M)$   
 $\iota_X(w \wedge \eta) = \iota_X(w) \wedge \eta + (-1)^k w \wedge \iota_X(\eta)$

• wedge prod  
and keep i  
also remem



f the stuff  
 $L_Y(f) - L_Y(\alpha_X(f))$

$$[Y, X] = -[X, Y]$$

$\alpha_X(f) Y$

expression

$$\text{e.g. } \left[ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0$$

$$d_X(\omega) = \frac{d}{dt} \Big|_{t=0} (\varphi_t^X)^*(\omega) \quad \text{OR: } d_X = d_Y + \mathcal{L}_X$$

= usual (for  $f$ =function)

$$\varphi_t^X(\eta) = d_X(\omega) \wedge \eta + \omega \wedge \varphi_t^X(\eta), \quad d_X(d\omega) = d(d_X(\omega))$$

= usual (in  $\mathbb{R}^n$ : it is  $\sum \frac{\partial f}{\partial x_i} dx_i$ )

$$\eta = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \quad \text{where } k = \text{deg } \omega$$

$$X_{k_i} = \omega(x_1, \dots, x_{k-1})$$

$\omega(x)$  when  $\omega \in \Omega^1(M)$

$$\eta = L_X(\omega) \wedge \eta + (-1)^k \omega \wedge L_X(\eta)$$

• wedge products:  $\eta \wedge \omega = -(-1)^{kl} \omega \wedge \eta$

and keep in mind:  $\sum_{i_1 < \dots < i_k} \omega_{i_1, \dots, i_k} \wedge \dots \wedge \omega_{i_1, \dots, i_k} = 0$  (hence = 0 if  $i_i = i_j$ )

also remember  $\Omega^0(M) = C^\infty(M)$  and, for  $f \in C^\infty(M)$ :  
 $f \wedge \omega = \omega \wedge f = f \cdot \omega$

• pull-backs of forms along  $F: M \rightarrow N$

For  $\omega \in \Omega^k(N)$  we have defined  $F^*(\omega) \in \Omega^k(M)$

Work with this using:

•  $F^*$  is linear:  $F^*(\omega_1 + \omega_2) = F^*(\omega_1) + F^*(\omega_2)$

•  $F^*$  compatible with  $\wedge$ :  $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$

(in particular  $F^*(f \cdot \omega) = (F^*(f)) \cdot F^*(\omega)$ )

•  $F^*$  compatible with  $d$ :  $F^*(d\omega) = d(F^*(\omega))$

• on 0-forms:  $F^*(f) = f \circ F$

← Must do:  
one exercise  
that uses  
these

Exercises: 4.  
more practice

(hence  $F: M \rightarrow N$  induces  
 $F: \Omega^k(N) \rightarrow \Omega^k(M)$ )

$$F: M \rightarrow N$$

THE RVT:  $F^*(\eta) = \eta \circ F$   
all use  $F^*(\eta) =$



Exercises: 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100

more practice: 3, 9, 33

homework: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100

hence  $F: M \rightarrow V$  induces  $(F: \mathfrak{m}^k / \mathfrak{m}^{k+1} \rightarrow \mathfrak{m}^k / \mathfrak{m}^{k+1})$

$$f = f_1 \times \dots \times f_n$$

submodules (inclusion) at  $p \in M$

$$(dF)_p: T_p M \rightarrow T_p F_p, N \text{ is surjective for injective}$$

Must do:  $f$  that uses these

THE RVT:  $F: M \rightarrow N$  smooth,  $g \in N$  s.t.  $F$  is a submanifold of  $N$  and  $T_p(F(g)) = \{0\} \in T_p M$ .  
 all  $g \in F^{-1}(g) \Rightarrow F^{-1}(g)$  embedded submanifold of dimension  $\dim M - \dim N$