CHAPTER 6

Attaching cells

1. Cells

DEFINITION 6.1. *Let X be a Hausdorff topological space. An* <u>*open n-cell*</u> *in X is a subspace $e \subset X$ together with a homeomorphism*

$$\overset{\circ}{h}_e : \overset{\circ}{D}{}^n \longrightarrow e \subset X.$$

It is called an <u>*n-cell*</u> *if $\overset{\circ}{h}_e$ extends to a continuous map*

$$h_e : D^n \longrightarrow X.$$

We call h_e <u>*the defining map*</u> *of the n-cell, and we also say that e is an n-cell with defining map h_e, or that e is the image of the n-cell. The* <u>*cell boundary*</u> *of e is defined by:*

$$\partial_{cell}(e) := \overline{e} - e,$$

where $\overline{e}$ is the closure of e in X. The characteristic map χ_e of e is defined as the restriction

$$\chi_e := h_e|_{S^{n-1}} : S^{n-1} \longrightarrow X.$$

REMARK 6.2. Since $\overset{\circ}{D}{}^n$ is dense in D^n (hence each point $x \in D^n$ can be written as the limit of a sequence of points in the interior) and h_e must be continuous (hence preserves the limits), given $\overset{\circ}{h}_e$, the extension h_e will be unique.

In conclusion, an n-cell in the space X is just a subspace $e \subset X$ together woth a continuous map $h_e : D^n \longrightarrow X$ which, when restricted to $\overset{\circ}{D}{}^n$, is a homeomorphism between $\overset{\circ}{D}{}^n$ and e.

Here are some simple examples (more will come later).

EXAMPLE 6.3. Note that, for $n = 0$, a 0-cell in X is the same thing as a point of X.

An interesting 1-cell is $e = S^1 - \{(1, 0)\}$ which is a 1-cell in S^1 with defining map

$$h_e(t) = (cos(\pi t), sin(\pi t)).$$

EXAMPLE 6.4. Various cells inside the sphere are shown in Figure 1.

EXAMPLE 6.5. In general, an open n-cell may fail to be an n-cell. This is already clear when $n = 1$ and $X = \mathbb{R}$. A subspace $e \subset \mathbb{R}$ is an open n-cell (with some defining map) if and only if $e = (a, b)$ is an open interval (with a and b- real numbers or plus/minus infinity). Indeed, any such open cell will be a connected subspace of $\mathbb{R}$ hence it must be an interval. From the discussions inthe previous chapter, it must be an open interval.

On the other hand, if $e = (a, b)$ is a bounded interval (hence a, b are finite), then e (together with some defining map) is a 1-cell. However, un-bounded intervals cannot be made into 1-cells. This will also follow from the next proposition (which imply that the closure $\overline{e}$ of e in X must be compact), but let's check it directly here for the open 1-cell $e = (1, \infty)$ together with the defining map:

$$\overset{\circ}{h}_e : (-1, 1) \longrightarrow \mathbb{R}, t \mapsto \frac{2}{t+1}$$

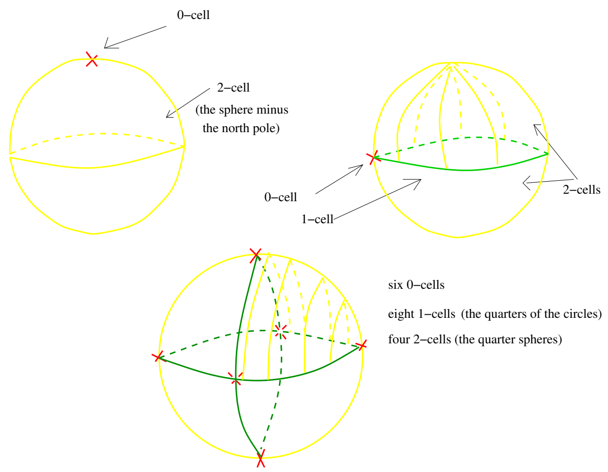


FIGURE 1.

This cannot have a continuous extension to $[-1, 1]$ because $x_n = -1 + \frac{1}{n}$ converges to -1, but $\mathring{h}_e(x_n) = 2n$ does not have a finite limit.

EXERCISE 6.1. *Let X be the one-point compactification of the space obtained from D^2 by removing two points on its boundary. Describe X in $\mathbb{R}^3$ and show that it is the closure of a 2-cell.*

Here are the main properties of n-cells.

PROPOSITION 6.6. *If $e \subset X$ is an n-cell with defining map $h_e : D^n \longrightarrow X$, then*

$$h_e(D^n) = \overline{e},\ \ h_e(S^{n-1}) = \partial_{cell}(e).$$

Moreover, as a map from D^n into $\overline{e}$, h_e is a topological quotient map.

PROOF. Since h_e is continuous, we have

$$h_e(\overline{B}) \subset \overline{h_e(B)}$$

for all $B \subset D^n$ (prove this!). Choosing $B = \mathring{D}^n$, we obtain $h_e(D^n) \subset \overline{e}$.

On the other hand,

$$e = h_e(\mathring{D}^n) \subset h_e(D^n),$$

and $h_e(D^n)$ is compact (as the image of a compact by a continuous map), hence closed in X (since X is Hausdorff). This implies $\overline{e} \subset h_e(D^n)$. Since the opposite inclusion has been proven, we get $h_e(D^n) = \overline{e}$. We now prove

$$h_e(S^{n-1}) = \overline{e} - e.$$

We first show the inverse inclusion: for $y \in \overline{e} - e$, by the first part, $y = h_e(x)$ for some $x \in D^n$ and, since $e = h_e(\mathring{D}^n)$, x cannot be in $\mathring{D}^n$; hence $x \in S^{n-1}$. We now prove the direct inclusion. So let $y = h_e(x)$ with $x \in S^{n-1}$, and we want to prove that $y \notin e$. Assume the contrary, i.e. $y = h_e(x')$ with $x' \in \mathring{D}^n$. Since x and x' are distinct, we find $U, V \subset D^n$ opens (in D^n) such that

$$x \in U, x' \in V, U \cap V = \emptyset.$$

We may assume that $V \subset \mathring{D}^n$. Since $\mathring{h}_e$ is a homeomorphism, $h_e(V)$ is open in e, hence also in $\overline{e}$. Since $h_e : D^n \longrightarrow \overline{e}$ is continuous, $h_e^{-1}(h_e(V))$ will be open in D^n; but it contains x, hence we find $\epsilon > 0$ such that

$$D^n \cap B(x,\epsilon) \subset h_e^{-1}(h_e(V))$$

(where the ball is with respect to the usual Euclidean metric). Since $x \in U$ and U is open in D^n, we may choose ϵ so small so that

$$D^n \cap B(x,\epsilon) \subset U.$$

Pick up an element $z \in \overset{\circ}{D}{}^n \cap B(x,\epsilon)$. By the inclusion above, we find $z' \in V$ such that $h_e(z) = h_e(z')$. But then both z and z' are in $\overset{\circ}{D}{}^n$ hence we must have $z = z'$. Hence $z \in V$. But $z \in U$ hence we obtain a contradiction with $U \cap V = \emptyset$. Finally, h_e is a topological quotient map as a continuous surjection from a compact space to a Hausdorff space. □

2. Attaching one n-cell

DEFINITION 6.7. *Let X be a Hausdorff space and $A \subset X$ closed.*

We say that X is obtained from A by attaching an n-cell if there exists an n-cell $e \subset X$ (with some defining map h_e) such that

$$X = A \cup e, A \cap e = \emptyset.$$

EXAMPLE 6.8. It is clear that the n-ball D^n is obtained from $\partial D^n = S^{n-1}$ by attaching an n-cell (the defining map being the identity map). From example 6.3, we see that S^1 can be obtained from a point (which is a 0-cell!) by attaching a 1-cell. In general, S^n can be obtained from a point by attaching an n-cell (see also example 6.13 below).

To treat examples treated in the previous lectures such as the torus, the Moebius band etc, the following lemma is very useful.

LEMMA 6.9. *Let $X = D^n$ or $X = [0,1]^n$, and let Y be a quotient of X obtained by gluing (certain) points on the boundary $\partial(X)$. Assume that Y is Hausdorff,*

Let $\pi : X \longrightarrow Y$ be the quotient map, and let $B = \pi(\partial(X))$ (i.e. the space obtained from $\partial(X)$ by the original gluing). Then Y is obtained from B by attaching an n-cell.

PROOF. Since D^n and $X = [0,1]^n$ are homeomorphic by a homeomorphism which preserves their boundary, we may assume that $X = D^n$. We then choose as defining map for the n-cell the quotient map

$$h : D^n \longrightarrow Y,$$

hence the n-cell will be $e := h(\overset{\circ}{D}{}^n)$. Note that $B = \pi(S^{n-1})$. Clearly, $Y = B \cup e$. Next, since no element on the boundary of D^n is equivalent (identified) with an interior element, we have $B \cap e = \emptyset$. Next, since no two interior points of D^n are equivalent, the restriction of h to $\overset{\circ}{D}{}^n$ is injective, hence

$$h|_{\overset{\circ}{D}{}^n} : \overset{\circ}{D}{}^n \longrightarrow e$$

is a continuous bijection. We still have to prove that this map is a homeomorphism, and for this it is enough to show that it sends closed sets to closed sets. So, let F be closed in $\overset{\circ}{D}{}^n$; write $F = \overset{\circ}{D}{}^n \cap K$ with $K \subset D^n$ closed. Then K is compact (as a closed inside a compact), hence $h(K)$ will be compact. Since Y was assumed to be Hausdorff, we deduce that $h(K)$ is closed in Y. So, to show that $h(F)$ is closed in e, it suffices to show that

$$h(\overset{\circ}{D}{}^n \cap K) = e \cap h(K).$$

The direct inclusion is clear. For the converse, note that if y belongs to the right hand side we have $y = h(x)$ and $y = h(x')$ with $x \in \overset{\circ}{D}^{n}$ and $x' \in K$, but since interior points are not being identified with any other points, we must have $x = x'$, hence y belongs to the left hand side. $\square$

Remark 6.10. The case $n = 2$ is particularly important: many spaces X can be obtained from D^2 by identifying certain parts of $\partial D^2 = S^1$, and the identification can be shown on the picture by labeling by letters the parts that are to be identified. In the quotient $B = S^1/\sim = \pi(S^1)$, each letter will appear only once (because we identified all the parts labeled by the same letter). When going one time around the circle, we will meet various labels that will give us a word whose letters are labels. Reading this word in the space B describes the characteristic map.

Example 6.11. Consider the torus as a quotient of $X = [0,1] \times [0,1]$ (see Section 5) of the first chapter, or of D^2. We can apply the previous lemma. The resulting space B is shown in the picture (Figure 2) and it consists of two circles on the torus touching each other in one point (a and b on the picture). This space can be drawn (is homeomorphic to) the space consisting of two circles in the plane touching in one point only (a bouquet of two circles). In conclusion, T^2 can be obtained from a bouquet of two spheres by attaching a 2-cell.

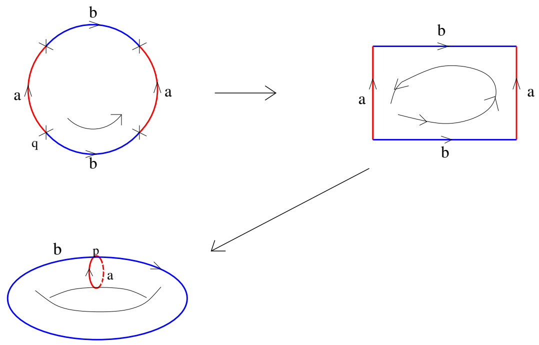


Figure 2.

Note that, according to the conventions from the previous remark, the characteristic map $\chi : S^1 \longrightarrow S^1 \vee S^1$ can be described symbolically as:

$$\chi = bab^{-1}a^{-1}$$

and which is further pictured in Figure 3.

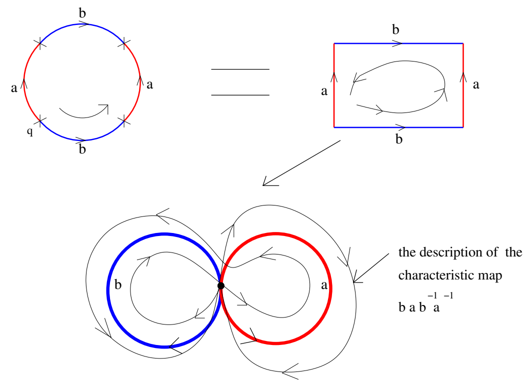


Figure 3.

EXAMPLE 6.12. A similar discussion applies to the Moebius band (section 4 in the first chapetr). The resulting space B, shown in Figure 4) is made of the three segments a, b and c on the Moebius band (hence B is the "boundary" of the Moebius band plus the segment a). The space B alone (as a topological space itself) is homeomorphic to the space consitsting of the circle S^1 together with a segment joining the north and the south pole (see the picture). In conclusion, the Moebius band can be obtained from this space by attaching a 2-cell.

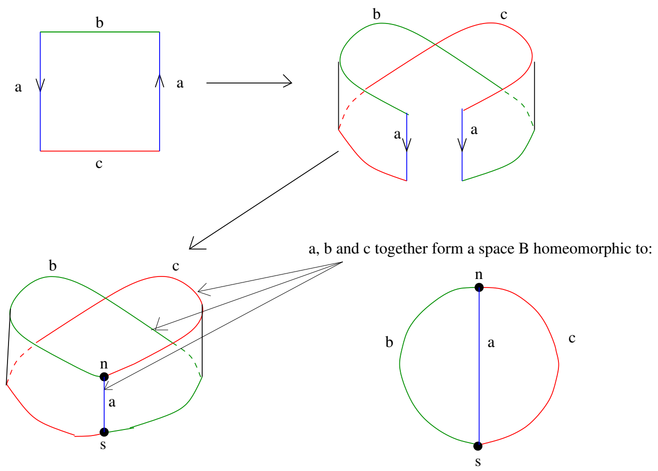


FIGURE 4.

The characteristic map can be described in symbols as:

$$\chi = caba$$

(on the picture, put the directions for b and c so that this formula is correct!). This, as a map defined on S^1, is further pictured in Figure 5.

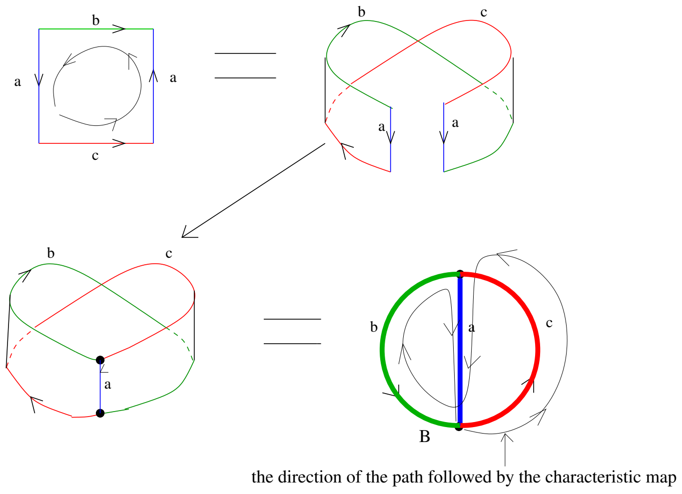


FIGURE 5.

EXERCISE 6.2. *Show that S^2 can be obtained by attaching a 2-cell to the interval $[0, 1]$. Describe at least two different ways of realizing this (and explain the characteristic maps).*

(Hint: see Figure 12 and Figure 13 in Chapter 1).

Exercise 6.3. *Do the same for $\mathbb{P}^2$ (see Section 7 of the first chapter).*

Exercise 6.4. *Do the Klein bottle.*

Exercise 6.5. *Show that $\mathbb{P}^2$ can be obtained by attaching a 2-cell to the the Moebius band.*

Example 6.13. Consider the sphere S^n. We already know that $S^n = D^n/\partial D^n$, i.e. S^n can be obtained from D^n by gluing all the points of $\partial D^n = S^{n-1}$ together into a single point (See Section 3 and Figure 11 in the first chapter and Example 3.24.iii in the previous chapter). Hence we can apply the previous lemma. The resulting space B clearly consists of one point only, hence S^n can be obtained from one point by adjoining an n-cell (the attaching map is, of course, just the constant map).

Example 6.14. Consider the projective space $\mathbb{P}^n$. We recall that $\mathbb{P}^n$ "is equal to" the space $D^n/\sim$ obtained by identifying (gluing) the antipodal points on $\partial D^n = S^{n-1}$ (see e.g. Exercise 1.27 in the first chapter). Hence, we can apply the previous lemma. The space B resulting from the lemma will be $S^{n-1}/\sim$- the space obtained from S^{n-1} by gluing its antipodal points. This is just another description of the projective space and we see that B "is equal to" $\mathbb{P}^{n-1}$. In conclusion, $\mathbb{P}^n$ can be obtained from $\mathbb{P}^{n-1}$ by adjoining an n-cell.

Exercise 6.6. *In the previous example, "is equal to" really means that "is homeomorphic to". Via the sequence of "is equal to" that is used in the example, it appears that $\mathbb{P}^{n-1}$ is a subspace of $\mathbb{P}^n$. Write out the "equalities" (i.e. homeomorphisms) that we used to conclude that the way we see $\mathbb{P}^{n-1}$ as a subspace of $\mathbb{P}^n$ is via the canonical inclusion*

$$\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$$

which associates to a line l inside $\mathbb{R}^n$ the line inside $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ given by

$$l \times \{0\} = \{(x, 0) : x \in l\} \subset \mathbb{R}^{n+1}.$$

3. The characteristic map

Note that, using Proposition 6.6, we see that the characteristic map of e can be viewed as a map

$$\chi_e : S^{n-1} \longrightarrow A.$$

The role of the characteristic map is that "it describes the way that A and e interact inside X", or, equivalently, that it describes "the relationship between the inclusion $i : A \hookrightarrow X$ and the defining map $h : D^n \longrightarrow X$. This relationship is $h|_{S^{n-1}} = i \circ \chi_e$ which should be interpreted as a commutative diagram:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\text{inclusion}"] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_e"] \\ A \arrow[r, "i"] & X \end{tikzcd}$$

But the real reason that the characteristic map χ_e is important comes from the fact that X can be reconstructed from the subspace A and the characteristic map χ_e. The aim of this section is to prove the following:

Theorem 6.15. *For any Hausdorff space A and any continuous map $\chi : S^{n-1} \longrightarrow A$, there is a space X which is unique up to homeomorphism, with the property that X is obtained from A by adjoining an n-cell with characteristic map equal to χ.*

We first estabilish the following universal property from which we will be able to deduce unqueness by a formal argument.

Proposition 6.16. *(the universal property) Assume that X is obtained from A by adjoining an n-cell e with defining map $h_e : D^n \longrightarrow X$. Let Y be another topological space. Then a map*

$$f : X \longrightarrow Y$$

is continuous if and only if

$$f|_A : A \longrightarrow Y,\ f \circ h_e : D^n \longrightarrow Y$$

are continuous. Moreover, the correspondence

$$f \mapsto (f|_A, f \circ h_e)$$

defines a 1-1 correspondence between

- *continuous maps $f : X \longrightarrow Y$*
- *pairs (f_A, f_e) with $f_A : A \longrightarrow Y$ and $f_e : D^n \longrightarrow Y$ continuous satisfying $f_A \circ \chi_e = f_e|_{S^{n-1}}$.*

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_e"'] \arrow[ddr, "f_e"] & \\
A \arrow[r, "i"] \arrow[drr, "f_A"'] & X \arrow[dr, "f"] & \\
& & Y
\end{tikzcd}$$

Proof. If f is continuous, it is clear that $f|_A$ and $f \circ h_e$ are continuous. For the converse, assume that $f|_A$ and $f \circ h_e$ are continuous. To show that f is continuous, we will show that, for $B \subset Y$ closed, $f^{-1}(B)$ is closed in X. Note that

$$f^{-1}(B) \cap A = (f|_A)^{-1}(B), f^{-1}(B) \cap \overline{e} = h_e((f \circ h_e)^{-1}(B)).$$

From the first equality and the continuity of $f|_A$, we deduce that $f^{-1}(B) \cap A$ is closed in A, hence (since A is closed in X), also in X. On the other hand, $(f \circ h_e)^{-1}(B)$ is closed in D^n, hence compact. Then $h_e((f \circ h_e)^{-1}(B))$ is a compact inside the Hausdorff X, hence it is closed in X. Since $f^{-1}(B)$ is the union of two closed subspaces of X (namely $f^{-1}(B) \cap A$ and $f^{-1}(B) \cap \overline{e}$), it is itself closed in X. This concludes the proof of the equivalence.

For the second part we remark that, given f_A and f_e, the conditions

$$f_A = f|_A, f_e = f \circ h_e$$

determine f uniquely because $X = A \cup \mathrm{Im}(h_e)$:

$$f(x) = f_A(x) \text{ if } x \in A, f(x) = f_e(v) \text{ if } x = h_e(v) \in \overline{e}.$$

Moreover, under the assumption $f_A \circ \chi_e = f_e|_{S^{n-1}}$, the previous formulas define f un-ambiguously: if x is both in A and of type $h_e(v)$ with $v \in D^n$, then $f_A(x) = f_e(v)$. Indeed, since $h_e(v) \in A$, v must be in S^{n-1}, hence we can write $x = \chi_e(v)$ and we can use the assumption. The continuity of the maps involved follows from the first part. □

The fact that X only depends on A and χ_e is even stronger indicated by the following:

Corollary 6.17. *Let A be a Hausdorff space and let $\chi : S^{n-1} \longrightarrow A$ be a continuous map. For $i \in \{1, 2\}$, assume that X_i is a space which is obtained from A by adjoining an n-cell e_i, and let $\chi_i : S^{n-1} \longrightarrow A$ be the characteristic map of e_i. If $\chi_1 = \chi_2$, then X_1 and X_2 are homeomorphic.*

Proof. Let $\chi = \chi_1 = \chi_2$, and let h_i be the maps defining the n-cell e_i. We apply the last part of the previous proposition to $X = X_1$, $Y = X_2$ and to the pair (i_2, h_2), where $i_2 : A \longrightarrow X_2$ is the inclusion. We find that there is one and only one continuous map $f_{1,2} : X_1 \longrightarrow X_2$ continuous such that

$$f_{1,2} i_1 = i_2, f_{2,1} h_1 = h_2.$$

These conditions mean that: for $a \in A$, $f_{1,2}(a) = a$, while for $y \in h_1(D^n)$, writing $y = h_1(x)$, $f_{1,2}(y) = h_2(x)$. Since $X_1 = A \cup h_{e_1}(D^n)$, these formulas do define $f_{1,2}$ uniquely, but what the universal property is telling us is that $f_{1,2}$ is well defined and it is continuous. Of course, this could have been checked directly (and that is precisely what we have done when proving the universal property), but the hole point of the "universal property philosophy" is that everything can be done by using this property alone (and not the fact that X_i is obtained from A by adjoining and n-cell).

Leaving aside the "philosophical arguments", and choosing one of the two (rather equivalent) ways of defining the map $f_{1,2}$, we end up with our continuous map $f_{1,2} : X_1 \longrightarrow X_2$. Similarly (by exchanging the role of X_1 and X_2), we obtain a map $f_{2,1} : X_1 \longrightarrow X_2$. Our maps fit into the following diagram:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "\chi_e"'] & D^n \arrow[d, "h_1"'] \arrow[ddr, "h_2"] \arrow[dddrr, "h_1"] & & \\
A \arrow[r, "i_1"] \arrow[drr, "i_2"] \arrow[ddrrr, "i_1"'] & X_1 \arrow[dr, "f_{1,2}"] & & \\
& & X_2 \arrow[dr, "f_{2,1}"] & \\
& & & X_1
\end{tikzcd}$$

We claim that $f_{1,2}$ and $f_{2,1}$ are inverse to each other, i.e. $f_{1,2}f_{2,1} = \mathrm{Id}_{X_1}$ and $f_{2,1}f_{1,2} = Id_{X_2}$. If we choose to use the explicit description of $f_{1,2}$ and $f_{2,1}$, this is immediate and concludes the proof. Alternatively (but equivalently!), these relations are, again, a consequence of the universal property. For instance, using the formulas that $f_{1,2}$ and $f_{2,1}$ satisfy (by their definition via the universal property), we deduce that $f := f_{1,2}f_{2,1} : X_1 \longrightarrow X_2$ satisfies:

$$f \circ i_1 = f_{2,1} \circ i_2 = i_1 = \mathrm{Id}_{X_1} \circ i_1,$$

$$f \circ h_{e_1} = f_{2,1} \circ h_{e_2} = h_{e_1} = \mathrm{Id}_{X_1} \circ h_{e_1}.$$

Using now the universal property for $X = X_1$, $Y = X_1$ (the uniqueness part!), we deduce that $f = \mathrm{Id}_{X_1}$. Similarly, $f_{2,1}f_{1,2} = \mathrm{Id}_{X_2}$. This proves that X_1 and X_2 are homeomorphic (... for any two spaces X_1 and X_2 which satisfy the universal property). $\square$

Finally, we show how to reconstruct X from the subspace A and the characteristic map χ_e. So, let's start with a Hausdorff topological space A and a continuous map

$$\chi : S^{n-1} \longrightarrow A.$$

We construct our space in two steps.

- We consider the disjoint union
$$A \coprod D^n.$$
As a set, it is the union of disjoint copies of A and D^n. Any subset of $A \coprod D^n$ is of type $U \coprod V$ with $U \subset A$ and $V \subset D^n$. On $A \coprod D^n$ we consider the topology consisting of those subsets $U \coprod V$ with U open in A and V open in D^n.
- We consider the quotient of $A \coprod D^n$ obtained by identifying each point $x \in S^{n-1}$ with $\chi(x) \in A$; we denote by $A \cup_\chi D^n$ the result. Hence
$$A \cup_\chi D^n = A \coprod D^n / \sim,$$
where $\sim$ is the smallest equivalence relation in $A \coprod D^n$ with the property that
$$x \sim \chi(x), \quad \forall\, x \in S^{n-1}.$$

Remark 6.18. Note that the construction of X is inspired by the universal property; actually, X is constructed by "brute force", forcing the universal property to hold. To explain this, we remind that maps out of X (i.e. defined on X) should correspond to certain pairs of maps, one out of A and one out of D^n. The first step (taking the disjoint union of A and D^n) takes care of this property. However, the inclusion of A into the disjoint union, and the inclusion of D^n into the disjoint union (a map out of A together with a map out of D^n!) do not satisfy the formula appearing in the universal property (for

$x \in S^{n-1}$, x itself and $\chi(x) \in A$ are distinct elements in the disjoint union!). The second step in our construction forces the desired formula (by identifying x with $\chi(x)$).

Some more explanations now.

- First about A. In the first step, A is a subspace of $A \coprod D^n$. Passing to the quotient $A \cup_\chi D^n$, note that the equivalence class of an element a coming from A can be safely denoted by the same symbol a (two elements coming from A are equivalent only if they are equal!). In this way we view A as a *subset* of $A \cup_\chi D^n$.
- Now D^n. As before, D^n is a subspace of $A \coprod D^n$. Passing to the quotient, we obtain a map which associates to x the equivalence class of x, and which we denote by
$$h : D^n \longrightarrow A \cup_\chi D^n.$$
In other words, h is the restriction of the quotient map $\pi : A \coprod D^n \longrightarrow A \cup_\chi D^n$ to D^n. The map h will be the defining map for our n-cell.
- In particular, we define the n-cell
$$e := h(\overset{\circ}{D}{}^n) \subset A \cup_\chi D^n.$$

Exercise 6.7. Show that A is a subspace of $A \cup_\chi D^n$, i.e. the topology induced on A (the restricted topology) coincides with the original topology of A.

Exercise 6.8. Show that the restriction of h to $\overset{\circ}{D}{}^n$ defines a homeomorphism between $\overset{\circ}{D}{}^n$ and e.

Exercise 6.9. We now know that $A \cup_\chi D^n$ is obtained from A by adjoining the n-cell e. Check that the characteristic map of e coincides with χ.

Exercise 6.10. Haven't we forgotten something? Prove that $A \cup_\chi D^n$ is Hausdorff.

Putting together everything, we deduce Theorem 6.15.

4. Adjoining more cells; cell complexes

Definition 6.19. *Let X be a Hausdorff space and let $A \subset X$ be closed. We say that X is obtained from A by adjoining n-cells if we are given n-cells $e_i \subset X$ (with i running in an index set I) such that*
$$X = A \cup (\cup_{i \in I} e_i)$$
and the following conditions are satisfied:

1. *$A \cap e_i = \emptyset$ for all i, $\partial_{cell}(e) \subset A$ and $e_i \cap e_j = \emptyset$ for all $i \neq j$.*
2. *$F \subset X$ is closed if and only if $F \cap A$ is closed, and $F \cap \overline{e_i}$-closed for all $i \in I$.*

Remark 6.20. It follows that

- e_i is open in X (for this one uses the last condition for $F = X - e_i$).
- since e_i's are open and $e_i \cap e_j = \emptyset$ for $i \neq j$, we deduce that $\overline{e}_i \cap e_j = \emptyset$ for all $i \neq j$ (indeed, $X - e_j$ will be a closed set containing e_i, hence also its closure).

Hence the characteristic maps χ_i of e_i will be continuous maps
$$\chi_i : S^{n-1} \longrightarrow A.$$
As in the case of one n-cell, the characteristic maps together with A determine X uniquely (the arguments we have presented before extend to an arbitrary number of cells without much trouble).

Definition 6.21. *Let X be a Hausdorff space. A <u>cell decomposition</u> of X is a sequence of closed subspaces*
$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \dots$$
with:

- *$\cup_n X_n = X$.*
- *For each $n \geq 0$ integer, X_n is obtained from X_{n-1} by attaching n-cells (the n-cells, together with the defining maps, are part of the structure!).*
- *A subset $F \subset X$ is closed if and only if $F \cap X_n$ is closed for all $n \geq 0$.*

The space X together with a cell decomposition is called a cellular space (note that the cells themselves, together with their defining maps, are viewed as part of the structure). The subspace X^n is called n-skeleton of X.

Given a cell complex X which is compact, define the Euler number of the cell complex X as

$$\chi(X) = \#(0-cells) - \#(1-cells) + \#(2-cells) - \dots .$$

REMARK 6.22. Hence, for each n, X_n is obtained from X_{n-1} by adjoining a family of n-cells:

$$\{e_i^n : i \in I_n\}$$

(I_n- a set indexing the n-cells). Note that

$$X_0 = \{e_i^0 : i \in I_0\},$$

is a closed set of points in X,

$$X = \cup_{n, i \in I_n} e_i^n,$$

and each e_i^n is open in X_n, but not in X.

The following exercise explains the compactness condition we added when defining the Euler number of a cell complex.

EXERCISE 6.11. *Let X be a cell complex. Show that the total number of cells is finite if and only if X is compact.*

EXAMPLE 6.23. Given a topological space X, it may admit many different cell decompositions. For instance, for $X = S^1$, we know that S^1 can be obtained from a point by adjoining a 1-cell, i.e. it has a cell decomposition with one 0-cell and one 1-cell. However, it can also has a cell decomposition with two 0-cells and two 1-cells, or one with three 0-cells and three 1-cells, etc (see Figure 6).

We would also like to mention here that, although the definition of $\chi(X)$ uses a cell-decomposition of X, it does not depend on the cell-decomposition one uses, but only X as a topological space X. This is a non-trivial result of algebraic topology which will not be proven in this course.

EXAMPLE 6.24. Here is a cell-decomposition of $\mathbb{R}$ (figure 7) the zero cells are the integers, while the one cells are

$$e_n = (n, n+1),$$

one for each integer n, with defining map

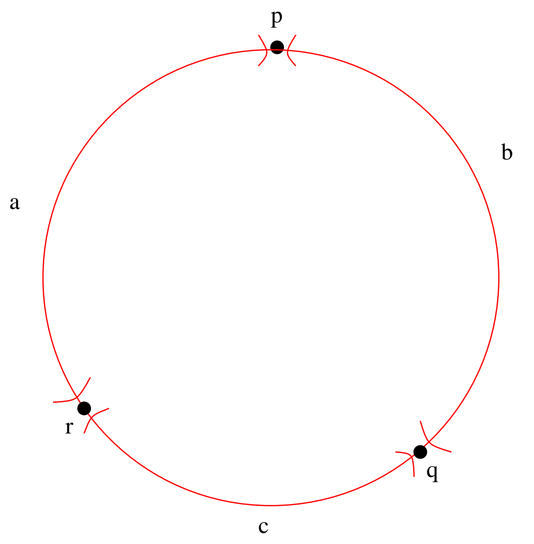
$$h_n : D^1 = [-1, 1] \longrightarrow \mathbb{R}, t \mapsto \frac{t + 2n + 1}{2}.$$

EXERCISE 6.12. *Deduce Euler's formula: for any polyhedra, $V - E + F = 2$ where $V =$ number of vertices, $E =$ number of edges, $F =$ number of faces.*

EXERCISE 6.13. *Describe a cell-decomposition of the plane $\mathbb{R}^2$.*

EXERCISE 6.14. *Describe a cell decomposition and compute the Euler number of the space X consisting of two circles joined by a line (Figure 8). Do the same for the space drawn in Figure 9.*

And also for the space consisting of the sides and the diagonal of a square (the left hand side of Figure 10).



A cell decomposition of the circle with three 0–cells (p, q and r) and three 1–cells (the arches a, b and c)

FIGURE 6.

Cell decomposition of the real line

FIGURE 7.

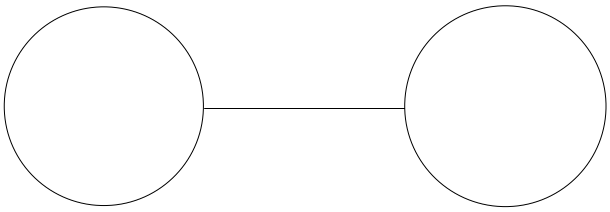


FIGURE 8.

EXERCISE 6.15. *Describe a cell decomposition of the cylinder, $C = S^1 \times [0, 1]$.*

EXAMPLE 6.25. Consider B to be a bouquet of two circles (two unit circles touching each other in one point). It is obtained from a point (the common point) by attaching two 1-cells (the connected components of the space that remains after removing the point).

On the other hand, we know that by (suitably) attaching a 2-cell to X one ends up with the torus T^2 (see Example 6.11). Hence T^2 has a cell decomposition with one 0-cell (the point), two 1-cells, and a 2-cell (see Figure 11). Symbolically, we write

$$T^2 = c_0 \cup c_1 \cup c_1 \cup c_2.$$

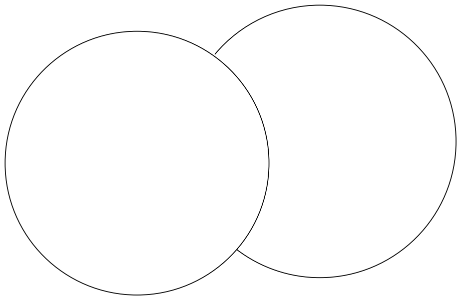


FIGURE 9.

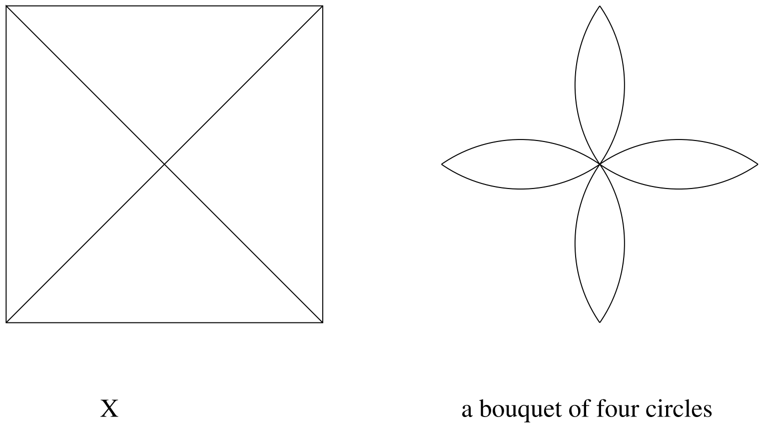


FIGURE 10.

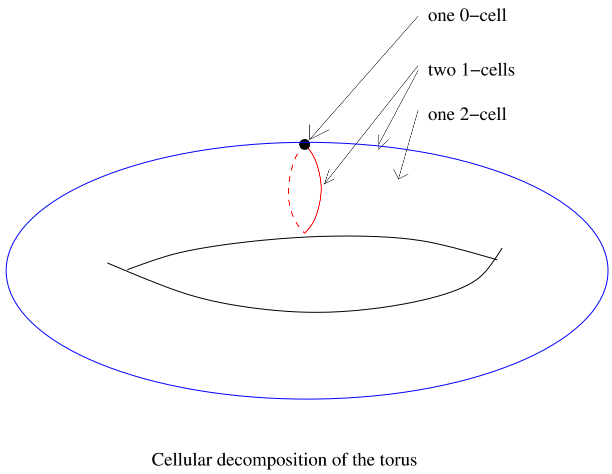
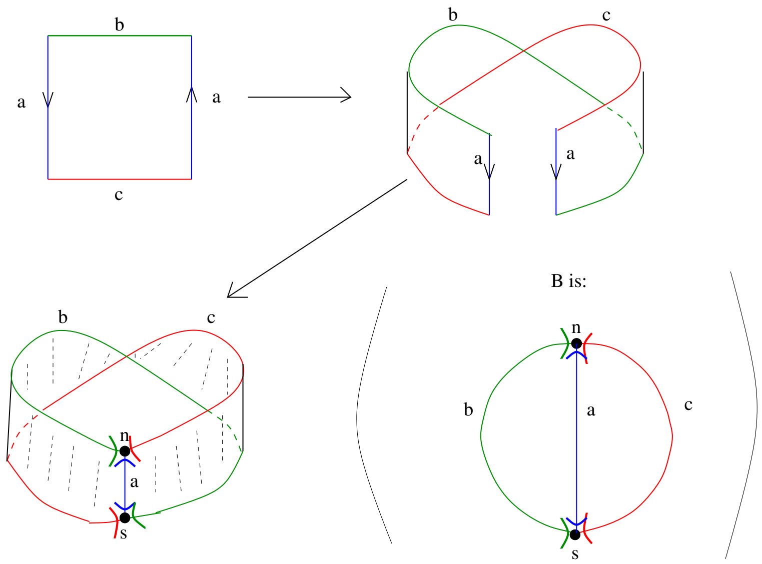


FIGURE 11.

EXAMPLE 6.26. Consider the space B which, after adjoining a 2-cell, gives the Moebius band M (see Example 6.12). This is obtained the south and the north pole by adjoining three 1-cells, as shown in Figure 12: the 0-cells are denoted n and s there, while the 1-cells are a, b and c. Hence the Moebius band has a cell decomposition with two 0-cells, three 1-cells and one 2-cell (see the picture):

$$M = c_0 \cup c_0 \cup c_1 \cup c_1 \cup c_1 \cup c_2.$$



Moebius band: two 0−cells (n and s), three 1−cells (a, b and c), one 2−cell.

FIGURE 12.

EXERCISE 6.16. *Do the Klein bottle?*

EXERCISE 6.17. *Show that T^2 and S^2 are not homeomorphic. Do the same for S^2 and S^3.*

EXERCISE 6.18. *For $\mathbb{P}^2$:*

(i) *describe a cell decomposition of type*

$$c_0 \cup c_0 \cup c_1 \cup c_1 \cup c_1 \cup c_2 \cup c_2.$$

(ii) *can you find one of type*

$$c_0 \cup c_0 \cup c_0 \cup c_1 \cup c_1 \cup c_1 \cup c_1 \cup c_2 \cup c_2 \cup c_2?$$

(Hint for (i): Moebius).

EXERCISE 6.19. *Find a cell decomposition of the annulus*

$$A(R, r) = \{(x, y) \in \mathbb{R}^2 : r^2 \leq x^2 + y^2 \leq R^2\}.$$

Then generalize this to arbitrary dimensions.

EXERCISE 6.20. *Show that the annulus $A(R, r)$ is not homeomorphic to the closed disk D^2.*

EXERCISE 6.21. *Consider the spaces from Figure 13.*

(a) *For each of these spaces, describe a cell decomposition (indicate it on the picture).*
(b) *Which of these spaces are homeomorphic among them? Explain your answer.*

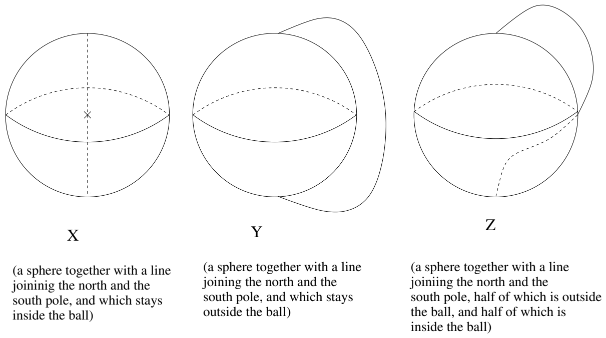


FIGURE 13.

EXAMPLE 6.27. From example 6.13 we know that

$$S^n = c_0 \cup c_n$$

(i.e. the n-sphere has a cell decomposition with one 0-cell and one n-cell). This can be pictured as shown in the first picture in Figure 14. However, there is yet another interesting cell decomposition of S^n. We already know that the two (open) semi-spheres S^n_+ and S^n_- are n-cells, hence S^n can be obtained from S^{n-1} by adjoining two n-cells (see Figure 14).

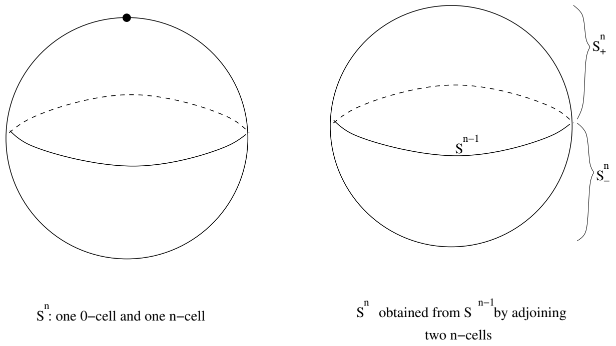


FIGURE 14.

One could now use the previous cell decomposition applied to S^{n-1} to deduce that

$$S^n = c_0 \cup c_{n-1} \cup c_n \cup c_n.$$

Alternatively, one could iterate our argument and deduce that S^n can be obtained from S^0 (two points!) by adjoining two 1-cells, and then two 2-cells, etc, and at the end two n-cells:

$$S^n = c_0 \cup c_0 \cup c_1 \cup c_1 \cup \ldots \cup c_n \cup c_n.$$

EXAMPLE 6.28. The way we expressed S^n as obtained from S^{n-1} by adjoining two n-cells (previous example) is interesting when looking at the projective space $\mathbb{P}^n$. Realizing $\mathbb{P}^n$ as $S^n/\mathbb{Z}_2$, note that the two cells will be identified in the quotient, and will define a single n-cell in $\mathbb{P}^n$. Since S^{n-1} goes in the quotient to a copy of $\mathbb{P}^{n-1}$, we deduce that $\mathbb{P}^n$ can be obtained from $\mathbb{P}^{n-1}$ by adjoining an n-cell. One can check that this coincides with the decomposition that we have discussed in Example 6.14- this is a matter of unraveling the definitions (exercise).

EXAMPLE 6.29. Since $\mathbb{P}^n$ can be obtained from $\mathbb{P}^{n-1}$ by adjoining and n-cell, continuing inductively, we find a cell decomposition of $\mathbb{P}^n$ of type

$$\mathbb{P}^n = c_0 \cup c_1 \cup \ldots \cup c_n.$$

EXERCISE 6.22. *Consider the square* $X = [0,1] \times [0,1]$ *and let* A *be the subset consisting of its corners:*

$$A = \{(0,0), (0,1), (1,0), (1,1)\}.$$

Consider the quotient space:

$$Y = X/A$$

(with the induced quotient topology).

(i) *Make a picture of* Y.
(ii) *PROVE that, indeed,* Y *can be embedded in* $\mathbb{R}^3$.
(iii) *Describe a cell-decomposition of* Y.

EXERCISE 6.23. *Draw a picture, describe a cell decomposition and compute the Euler number of the space* X *which is the one point compactification of the space consisting of the* xOy *plane together with the upper half plane of the unit sphere (Figure 15).*

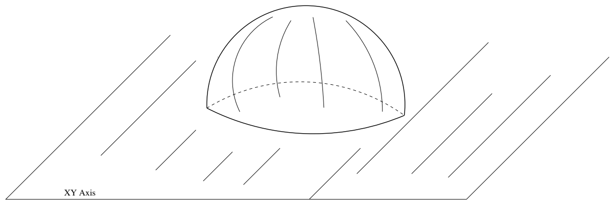


FIGURE 15.

EXERCISE 6.24. *Draw a picture, describe a cell decomposition and compute the Euler number of the space* X *which is the one point compactification of a plane with a handle (i.e. the* XOY *plane together with a segment whose end points belong to the plane, but which does not intersect the plane in any other point). (Figure 16).*

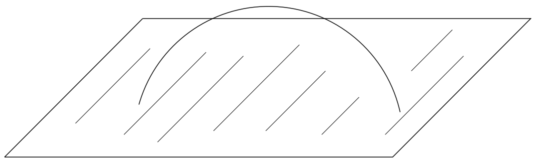


FIGURE 16.

EXERCISE 6.25. *Draw a picture, describe a cell decompositionand compute the Euler number of the space* X *which is the one point compactification of the space consisting of the* xOy *plane together with the upper half of the* Z *axis (Figure 17).*

EXERCISE 6.26. *Consider an octogon* X *in the plane and the equivalence relation* $\sim$ *on* X *which identifies the sides of* ∂X *as shown in the Figure 18.*

(1) *Show that* $\partial X/\sim$ *is homeomorohic to a bouquet* $S^1 \vee S^1 \vee S^1 \vee S^1$ *of four circles.*
(2) *Show that* $X/\sim$ *us homeomorphic to the double torus* T_2. *(Figure 19).*
(3) *Describe a cell decomposition of the double torus* T_2.
(4) *Compute the Euler number of the double torus.*

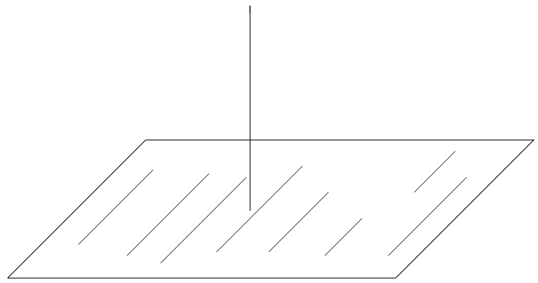


FIGURE 17.

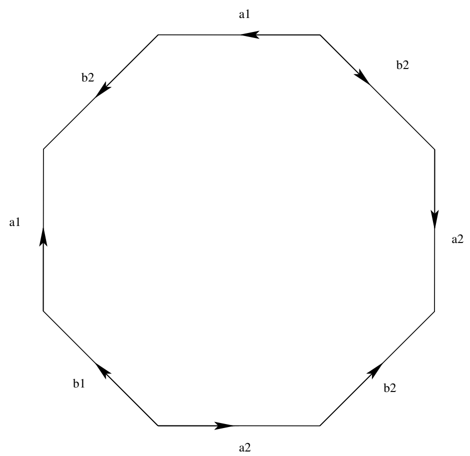


FIGURE 18.

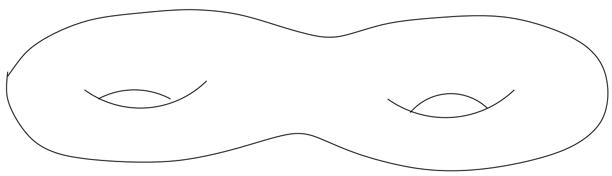


FIGURE 19.

EXERCISE 6.27. *Consider the unit circle*

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$

Let $n \geq 1$ *integer and define*

$$f_n : S^1 \to S^1, f_n(z) = z^n.$$

Let X_n *be the space obtained from* S^1 *by attaching a two cell with characteristic map* f_n*. Describe* X_n *as a quotient of* D^2*. Do you recognize* X_2*?*