(final) version

- received on December 6, 2017 -

POINCARÉ AND GEOMETRIZATION CONJECTURE

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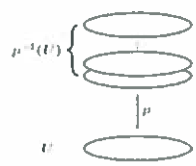
1. Introductory notions

Before looking at the 3 dimensional equivalent of the classification of surfaces, we will need to introduce some general notions.

First we ~~define~~ recall some purely topological notions[11]:

Definition 1. For a continuous surjective map $p : E \to B$ between topological spaces, we say that an open set $U \subset B$ is evenly covered by p if $p^{-1}(U)$ is the union of disjoint open sets in E, say $\{V_i\}$, such that p restricted to each of these open sets is a homeomorphism onto U. $\{V_i\}$ is called a partition of $p^{-1}(U)$ into slices.

This can be pictured as a collection of "slices" of E each having the same size and shape as U covering U. The map p simply squeezes all of these slices together and pushes them onto U. We can now define an important concept in algebraic topology:



Definition 2. If $p : E \to B$ is a continuous surjective map with the property that for all $x \in B$ there exists a $U \in \mathcal{N}_x$ that is evenly covered by p then p is called a **covering map** for B with associated covering space E. If E is simply connected then we say it is the **universal cover** of B.

We should note that universal coverings of a space are equivalent under a certain equivalence relation on covering spaces[11, Section 79 & 80], which justifies the use of the definite article in the definition.

Definition 3. If $p : E \to B$ is a covering map, then the group of all automorphisms ϕ on E that are lifts of p is called the **deck group**. By automorphisms ϕ that are lifts of p, we mean that $p \circ \phi = p$.

While on the topic of covering spaces, we introduce here a small technical lemma, which will be needed later on:

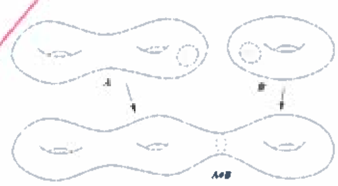
Lemma 1.1. *If $\pi : E \to B$ is a covering map with E path connected and B simply connected,* then *~~them~~ π is a homeomorphism.*

Proof. By definition of π it is continuous, surjective, and open. Hence we only need to prove that it is injective. To that end consider points $x, y \in E$ such that $\pi(x) = \pi(y)$. As E is path connected we have some path $c : I \to E$ such that $c(0) = x$ and $c(1) = y$. Hence $\pi \circ c$ is a loop based at $\pi(x)$. As B is simply connected it follows that $\pi \circ c$ is path homotopic to the constant path $\phi : I \to B$, $t \mapsto \pi(x)$. Now the constant path $\psi : I \to E$ given by $t \mapsto x$ is an obvious lift of ϕ, and by uniqueness of lifts under covering maps it is the lift of ϕ [11, Lemma 54.1]. Similarly c is the lift of $\pi \circ c$. As covering maps obey the homotopy lifting property [11, Theorem 54.3], it follows that $c(1) = \psi(1) = x$. Hence $x = y$ and π is indeed injective. $\square$

We can now turn our attention to concepts more directly associated with manifolds:

Definition 4. The connected sum of two manifolds is the manifold obtained by deleting a ball from both manifolds and glueing the boundaries together.

For example the connected sum of A, the 2-~~genus~~ torus, and B, the torus, is often denoted by $A\#B$ which is displayed in the picture.



still not clear!

While this definition may not seem very rigorous, for path-connected, oriented manifolds this construction is unique by letting the glueing map reverse orientation, furthermore the resulting manifold is unique.

Definition 5. A manifold is called prime if it cannot be represented as a connected sum of two manifolds, where neither is the n-dimensional sphere.

Notice that the n-dimensional sphere is excluded, as taking the connected sum of an n-manifold M with S^n is M.

1.0.1. *Metric.*

Definition 6. Let M be a manifold. ~~a~~ A Riemannian metric~~, or metric~~ on M is a function $g : TM \times_M TM \to R$. Meaning that at each point $p \in M$ $g_p : T_pM \times T_pM \to R$. With the following properties

(1) g is symmetric, i.e. $g(X, Y) = g(Y, X)$ for all $X, Y \in T_pM, p \in M$
(2) g is positive definite, i.e. $g(X, X) \geq 0$ for all $X \in T_pM, p \in M$

write properly please!

The pair (M, g) is called a **Riemannian manifold.**

Theorem 1.2. *If M is a smooth manifold, then there exists a smooth Riemannian metric g on M.*

Proof. First note that $\mathbb{R}^n$ can be given a Riemannian metric. For any set of bounded smooth functions $\{f_{ij} | i, j = 1, ..., n\}$, with $f_{ij} = f_{ji}$, the functions $g_{ij} = C\delta_{ij} + f_{ij}$ determine a metric, when C is sufficiently large. This metric g can be defined as the 2-tensor such that $g(\partial_i, \partial_j) = g_{ij}$ for the usual local frame $(\partial_1, ..., \partial_n)$ for $T\mathbb{R}^n$. Since for C sufficiently large each g_{ij} is positive definite and symmetric, g is a metric.

why don't you take $f_{ij} = 0$ and $C = 1$?

Let $\{U_\alpha, \varphi_\alpha\}$ be an atlas for M, and let $\{\rho_\alpha\}$ be a subordinate partition of unity. On each $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ a Riemannian metric g^α can be chosen. Then for all $X, Y \in T_pM$ define

$$g(X,Y) = \sum_\alpha \rho_\alpha g^\alpha(D\varphi_\alpha(X), D\varphi_\alpha(Y))$$

Note that g is positive definite since $g(X,X) \geq 0$ for all $X \in T_pM$ and also symmetric because each g^α is symmetric, thus g is a metric on M. □

A metric allows us to define the length of a smooth path $\gamma : [0,1] \to M$, denoted by $l(\gamma)$, as

$$l(\gamma) = \int_a^b g(\dot\gamma(t), \dot\gamma(t))^{1/2} dt$$

not nice

With this definition we can define a distance between two points $a, b \in M$, denoted by $d(a,b)$, as the minimum of the length of all paths between a and b. It can be shown that this length is independent of the parameterization of the path [15].

From the definition it is clear that d makes M into a metric space, so any smooth manifold can be made into a metric space.

Definition 7. A Riemannian metric is called complete when it is complete as a metric space.

Remark: If our manifold M happens to be a regular submanifold of $\mathbb{R}^n$, then there is a simpler way of defining a metric on our space, that agrees with the previous more general definition. In $\mathbb{R}^n$ we define the arc length, ℓ' of a curve γ to be

$$\ell'(\gamma) = \int_\gamma ds$$

where $ds^2 = dx_1^2 + \cdots + dx_n^2$ (ds being commonly referred to as the line element). We can take the definition of distance between two points in $\mathbb{R}^n$ to simply be the infimum over the arc length of all curves between the two points as defined above, and show that this ends up defining the same metric on $\mathbb{R}^n$ as if we used the general definition given earlier, with the Euclidean inner product as a Riemannian metric on $\mathbb{R}^n$. As M is a regular submanifold of $\mathbb{R}^n$, we can simply define the distance between points on M to be the infimum of the arc length of all curves lying on M between the given points. OK. It can be shown that if M is S^{n-1} then this metric gives rise to great circles giving the shortest paths between two points.

We can generalise this one step further, by changing the intuitive Euclidean notion of distance on $\mathbb{R}^n$ by altering the line element ds. We allow ourselves to consider all line elements of the form $ds^2 = \sum_{i,j=1}^n g_{ij} dx_i dx_j$, where $g : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ varies smoothly with $x \in \mathbb{R}^n$, and $g(x)$ is a positive definite matrix for all $x \in \mathbb{R}^n$. This implies that each $g_{ij} \in \mathbb{C}^\infty(\mathbb{R}^n)$. It is now an appropriate place to elaborate on what exactly the line element is. We can view it in two different lights depending on where it is used. Under an integral sign one can view this line element as simply defining how we calculate arc lengths, as is taught in a calculus course. I.e if $c : [a,b] \to \mathbb{R}^n$, $t \mapsto (x_1(t), \dots, x_n(t))$ is the parameterization of a curve γ then

$$\int_\gamma ds := \int_a^b \sqrt{\sum_{i,j=1}^{n} g_{ij}(c(t))\frac{dx_i(t)}{dt}\frac{dx_j(t)}{dt}}dt.$$

We take ds^2 by itself to be a metric tensor

whose
their ?

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx_i \otimes dx_j,$$

a linear combination of tensor products of differential forms over $C^\infty(\mathbb{R}^n)$. Our previous restrictions on g means that ds^2 is in fact a Riemannian metric. We can see via their definitions that this Riemannian metric will induce exactly the same metric as the one induced via the line element. Hence specifying a line element for R^n is in fact the same as giving R^n a Riemannian metric, and so giving any regular submanifold of R^n a Riemannian metric via restriction. With distances on regular submanifolds of $\mathbb{R}^n$ defined as before, different line elements give rise to very different "straight paths" on the manifold M.

$\mathbb{R}^n$

1.0.2. *Volume.* The additional structure of a metric enables a coordinate invariant definition of volume ~~on~~ of oriented Riemannian manifolds. For any Riemannian manifold there exists a smooth orthonormal frame $(E_1, ..., E_n)$ (see [5] Proposition 13.6), by changing E_1 to $-E_1$ if necessary the orthonormal frame induces an oriented orthonormal frame.

that is not true globally (only at each point)

Definition 8. Suppose (M, g) is an oriented Riemannian $n-$manifold. There is a unique smooth orientation form $\omega_g \in \Omega^n(M)$, called the Riemannian volume form. Such that

$$\omega_g(E_1, ..., E_n) = 1$$

?

f ~~F~~or every oriented orthonormal frame $(E_1, ..., E_n)$ for M

If M is compact, we ~~can~~ define the volume of M as

$$\text{Vol}(M) = \int_M \omega_g$$

2. The Poincaré conjecture

The Poincare conjecture is a conjecture about the characterization of the 3-sphere S^3 in terms of homotopical properties. The 3-sphere is defined as follows:

$$S^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\}$$

Here $\|.\|$ denotes the Euclidean norm given by $\|x\| = \sqrt{\sum_{i=1}^4 x_i^2}$. The first observation about S^3 is that it is path connected. It can also be proven that S^3 has a trivial fundamental group. Hence S^3 is simply connected. Another example of a simply connected 3-manifold is $\mathbb{R}^3$. However, an important difference is that S^3 is compact while $\mathbb{R}^3$ is not. The Poincaré conjecture states that these two properties are sufficient to characterize S^3. In a more concise manner, the Poincaré conjecture states the following:

for a nicer exposition:
use ittalico here

Conjecture 1. Every compact simply connected 3-manifold is homeomorphic to S^3.

Historically this conjecture didn't come out of nowhere. Before Poincaré stated this conjecture, he tried to characterize S^3 in terms of homology. At the time of his first conjecture, homotopy wasn't even around. The homology of S^3 is given by:

are you sure ? ?

$$H_0(S^3, \mathbb{Z}) \simeq H_3(S^3, \mathbb{Z}) \simeq \mathbb{Z}$$

reference ?

and

$$H_1(S^3, \mathbb{Z}) = H_2(S^3, \mathbb{Z}) = \{0\}$$

Poincaré conjectured [12] that these homology groups were sufficient to show that a 3-manifold is homeomorphic to S^3. The original conjecture stated the following:

Conjecture 2. Let X be a compact 3-manifold for which the homology groups are given by

$$H_0(X, \mathbb{Z}) \simeq H_3(X, \mathbb{Z}) \simeq \mathbb{Z} \quad (1)$$

and

$$H_1(X, \mathbb{Z}) = H_2(X, \mathbb{Z}) = \{0\} \quad (2)$$

which ones ? (1), (2) and (3) ? Or just (1) and (2)

Then X is homeomorhpic to S^3. (3)

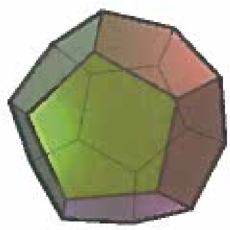
Definition: Any X that satisfies these conditions is called a homology sphere.

This iteration of the conjecture turned out to be false. Poincaré found a counterexample which is now called the Poincaré homology sphere. This manifold was first constructed by Poincaré in [12].

2.1. The Poincaré homology sphere

The Poincaré homology sphere can be constructed in three different ways. Each of the descriptions is usefull for different computations.

2.1.1. *Construction 1: As a gluing of the dodecahedron.* Recall that the dodecahedron is the Platonic solid with 12 pentagons as faces. At each vertex three faces meet. The dodecahedron is pictured at the right. As a topological space it is homeomorphic to the closed 3-dimensional disc.



Each face can be identified with the opposite face by the minimal clockwise rotation. The Poincaré homology sphere can be obtained by glueing each face with its opposite face according to this identification.

2.1.2. *Construction 2: As a quotient of* $SO(3)$. The special orthogonal group $SO(3)$ is defined as follows:

$$SO(3) = \{A \in GL(3, \mathbb{R}) \mid A^T = A^{-1}, det(A) = 1\}$$

where $GL(3, \mathbb{R})$ is the group of invertible 3×3 matrices (see definition 9 on page 9 for more details). These correspond to the rotations of $\mathbb{R}^3$ around the origin. The subgroup $I < SO(3)$ consists of all rotations that leave the dodecahedron invariant. The second construction of the Poincaré homology sphere is as the quotient $SO(3)/I$.

2.1.3. *Construction 3: As a quotient of S^3.* We can identify the set of unit quaternions with S^3. Each unit quaternion can be written as $q = cos(\frac{\theta}{2}) + (u_x i + u_y j + u_z k) sin(\frac{\theta}{2})$ where $u \in \mathbb{R}^3$ is a unit vector. If we map $p \mapsto p_x i + p_y j + p_z k = \tilde{p}$, then the conjugation $q\tilde{p}q^{-1}$ rotates p around u with an angle θ. The map which sends every unit quaternion q to the rotation it defines is a covering map $S^3 \to SO(3)$ of order two. Since S^3 is simply connected, it is the universal cover of $SO(3)$. The third and last description of the Poincaré homology sphere is as the quotient $S^3/\tilde{I}$, where $\tilde{I}$ is the binary icosahedral group.
The binary icosahedral group is a double cover of the icosahedral group. The binary icosahedral group is the subgroup of the unit quaternions which consists of the following elements:

- Elements of the form $\{\pm 1, \pm i, \pm j, \pm k\}$
- Elements of the form $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$
- Even permutations of $\frac{1}{2}(0 \pm i \pm \varphi^{-1} j \pm \varphi k)$ in the coordinates $0, 1, \varphi^{-1}, \varphi$

The group is generated by the elements $s = \frac{1}{2}(1 + i + j + k)$ and $t = \frac{1}{2}(\varphi + \varphi^{-1} i + j)$ with the relations $(st)^2 = s^3 = t^5$.

The third description will be the most useful description for computing homology. Since the Poincaré homology sphere is connected, the zeroth homology group is isomorphic to $\mathbb{Z}$. By the Poincaré duality theorem the same holds for the third homology group. Using Poincaré duality again on the first homology group shows that the Poincaré sphere satisfies conjecture 2 if the first homology group is trivial. To prove this Poincaré defined the fundamental group. The relation between the fundamental group and the first homology group is given by an application of the Hurewicz theorem. ?

Theorem 1. Let X be a topological space. Then

$$H_1(X, \mathbb{Z}) \simeq \pi_1(X)/[\pi_1(X), \pi_1(X)]$$

Proof. Let x_0 be any point in X. Define $h : \pi_1(X, x_0) \to H_1(X, \mathbb{Z})$ to be the map which sends the homotopy class $[a]$ of the loop a based at x_0 to the homology class $[a]$ of the cycle a. This is a well-defined surjective group homomorphism. why ?
For every $x \in X$ we choose a path λ_x from x_0 to x. Define $\lambda_x^* : t \mapsto \lambda_x(1 - t)$. Let σ be a singular 1-simplex, and define $\Phi(\sigma) = \lambda_{\sigma(1)}^* \cdot \sigma \cdot \lambda_{\sigma(0)}$, where the dot means juxtaposition. This gives a loop based at x_0. We can extend Φ to a group homomorphism $C_1(X, \mathbb{Z}) \to \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$ by defining $\Phi(\sigma + \tau) = \Phi(\sigma) + \Phi(\tau)$. Since the boundaries of singular 2-simplices vanish under Φ, Φ induces a homomorphism $\Phi_* : H(X, \mathbb{Z}) \to \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$. Now for any loop σ based at x_0 we get $\Phi(h(\sigma)) = [\lambda_{x_0}^* \cdot \sigma \cdot \lambda_{x_0}] = [\sigma \cdot \lambda_{x_0}^* \cdot \lambda_{x_0}] = [\sigma]$, so $\Phi o h$ is the quotient map $\pi_1(X, x_0) \to \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$. Hence h induces a monomorphism $h_* : \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \to H_1(X, \mathbb{Z})$. This proves the existence of an isomorphism $\pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \to H_1(X, \mathbb{Z})$. □

This theorem shows that computing the fundamental group of a topological space is sufficient for determining the first homology group. Now we need a tool to easily compute the fundamental group of the Poincaré homology sphere. Such a tool is provided by the following lemma: reference ?

Lemma 1. Let X be a simply connected topological ~~group~~ Space and Γ a finite subgroup. Then the fundamental group of the quotient is given by $\pi_1(X/\Gamma) \simeq \Gamma$.

Proof. The proof can be found in [3]. It relies on the universal lifting property of paths. □

From this lemma we conclude that the fundamental group of the Poincaré homology sphere is isomorphic to $\tilde{I}$. A group is perfect if it is equal to its commutator subgroup. Hence to show that the Poincaré homology sphere is indeed a homology sphere it suffices to show that $\tilde{I}$ is perfect.

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Proposition 1. The binary icosahedral group $\tilde{I}$ is perfect.

Proof. We use the abstract definition of the binary icosahedral group: $\tilde{I} =< s,t \mid (st)^2 = s^3 = t^5 >$.
First we see that the commutator subgroup $[\tilde{I},\tilde{I}]$ contains the element $[t^{-1},s] = t^{-1}sts^{-1} = t^{-1}st(stt^{-1}s^{-1})s^{-1} = t^{-1}(st)^2t^{-1}s^{-1}s^{-1} = t^3s^{-2} = t^3s^{-3}s = t^{-2}s$. Moreover it also contains the element $[s^{-1},t^{-1}] = s^{-1}t^{-1}st = s^{-1}t^{-1}st(stt^{-1}s^{-1}) = s^{-1}t^{-1}(st)^2t^{-1}s^{-1} = s^{-1}t^3s^{-1}$. If we multiply the two we get ts^{-1}. Therefore also $s^{-1}t \in [\tilde{I},\tilde{I}]$. Now $t^{-2}ss^{-1}t = t^{-1}$, which means that $t \in [\tilde{I},\tilde{I}]$ and therefore also $s \in [\tilde{I},\tilde{I}]$. Since $[\tilde{I},\tilde{I}]$ contains all the generators of $\tilde{I}$, we conclude that $\tilde{I}$ is a perfect group. □

?

The Poincaré homology sphere is an example of a spherical manifold. Such manifolds will be discussed later on.

2.2. Generalizations

The second conjecture was the predecessor of the Poincaré conjecture. The Poincaré conjecture also has some generalizations and ~~different~~ several versions. The obvious generalization is the Poincaré conjecture for arbitrary dimensions. In dimensions higher than 4 there exist simply connected manifolds which are not homotopy equivalent to the n-sphere. Hence the generalized Poincaré conjecture states the following:

Conjecture 3. Let X be a compact n-manifold which is homotopy equivalent to S^n. Then X is homeomorphic to S^n.

In dimension 1 this statement is trivial. In dimension 2 it follows directly from the classification of compact surfaces. It has been proven to be true in all higher dimensions as well. A different way to alter the conjecture is to look at manifolds equipped with additional structures. Such additional structures ~~include a~~ are! piecewise linear structures ~~and a~~ or smooth structures. For piecewise linear manifolds in all dimensions other than 4 the conjecture is proven to be true. The smooth case is much different. In dimension 7 there are counterexamples constructed which are called exotic spheres. Such smooth manifolds are homeomorphic to S^7, but they are not diffeomorphic to S^7. This means that there exist smooth manifolds which are homotopy equivalent to S^7 but not diffeomorphic to S^7. These exotic spheres were first constructed by Milnor in [8].

3. Geometrization Conjecture

3.1. Classifications in 2 dimensions

The classification of surfaces has been a well understood problem for some time, and we will use this as an introduction to the Geometrization conjecture. As the Geometrization conjecture focuses on the classification of certain 3-manifolds, the classification of surfaces is a clear and straightforward introduction of the geometrization conjecture. The classification of surfaces is a result for compact 2-dimensional manifolds (without boundary). It says that such a manifold is homeomorphic to one of the the following structures:

(1) The sphere S^2
(2) The connected sum of g tori T
(3) The connected sum of g projective planes RP^n

and much more...)

The actual classification of surfaces uses the notion of orientability. We will not define orientability in general, as that goes beyond the scope of this paper. The following is an outline of orientability in arbitrary dimensions. Whenever you choose an ordered set of n tangent vectors at a point, and then move the point continuously along the manifold the set of vectors stays the same. The Möbius band is the easiest example of a non-orientable manifold when you travel along it with two tangent vectors, at least one of them changes when you arrive back at the original point. too vague for a math text.

A 2-dimensional manifold is said to be orientable if it is impossible to embed a Möbius band inside the manifold. This definition can be proven to be equivalent with the general definition of orientability. Ref ?

We can now classify compact surfaces. Oriented surfaces are homeomorphic to either the sphere or the connected sum of tori. And non-orientable manifolds are homeomorphic to the connected sum of projective spaces. It should also be noted that the surfaces in these 3 categories can be distinguished by their Euler characteristic, which is $2 - 2g$ for connected sum of g tori and $2 - k$ for the connected sum of k projective planes. Note that the sphere may be viewed as the connected sum of 0 tori. So we can fully classify connected closed surfaces by two pieces of information, their orientability and Euler characteristic.

3.2. Classifications in 3 dimensions

2-manifolds had been well understood for many decades, but characterizing 3-manifolds proved a much more challenging task for 20th century mathematicians. It took until $1980's$, when, after proving that a large class of manifolds (the Haken manifolds, for the interested reader) are hyperbolic, i.e that they admit a metric of constant negative curvature, Thurston proposed the following geometric characterization of 3-manifolds [10]:

Conjecture 3.1. *Let M be a closed, orientable, prime 3-manifold. Then there exists an embedding of a disjoint union of incompressible 2-tori and Klein bottles in M such that* *every component of their complement admits a locally homogeneous Riemannian metric of finite volume, isometric to one of the* 8 *model geometries listed below.*

In fact Thurston himself did not include the word "incompressible" in his conjecture, but due to a technical result (see Morgan [10, Cor. 1.2.1]) speaking about incompressible embedded surfaces in the conjecture is equivalent to the original, and convenient for our purposes. We shall take as our definition of an **incompressible surface** of a 3-manifold M to be an embedding $i : S \to M$ such that $i_* : \pi_1(S) \to \pi_1(M)$ is injective. We recall that a Riemannian metric on a manifold M is said to be **homogeneous** if for all $x, y \in M$ there exists an isometry $f : M \to M$ such that $f(x) = y$. If instead of isometries between the manifold we only have isometries between neighbourhoods of each pair of points we call the metric **locally homogeneous.**

As the geometrization conjecture is ostensibly about categorizing all closed and orientable 3-manifolds as in the 2 dimensional case, it is natural to wonder why Thurston limits his conjecture to prime manifolds. The following theorem, roughly stated and proved in 1928 by Hellmuth Knesner but refined and put in its modern formulation by John Milnor in 1962, fully motivates this restriction[6]:

Theorem 3.2. *Every non-trivial compact* 3*-manifold M is isomorphic to a sum $P_1\# \dots \# P_k$ of prime manifolds. The summands P_i are uniquely determined up to order and isomorphisms thereof. This decomposition is essentially unique.*

To a non-expert in the field of geometry, Thurston's conjecture as he originally stated is fairly non-intuitive. Essentially what the conjecture is claiming, is that any prime manifold can either be given a complete Riemannian metric, which is locally isometric to one of 8 model structures (listed below), or can be split by incompressible tori and Klein-bottles into open pieces of the manifold, each of which can be endowed with a complete Riemannian metric [7]. Splitting by incompressible surfaces simply means that there are embedded surfaces contained in the manifold in question, the complement of which is are open submanifolds.

Before listing the 8 model structures referred to above we need to first talk about a class of manifolds with an almost ubiquitous presence in mathematics:

Definition 9. A Lie group is a topological group G together with a smooth structure such that group multiplication $\cdot : G \times G \to G$, $(a, b) \mapsto a \cdot b$ and the inverse map $\mathbb{I} : G \to G$, $a \mapsto a^{-1}$ are C^∞.

We can now talk about some important examples of Lie groups. We define the general linear group $GL_n(\mathbb{R})$ to be the set of all $n \times n$ real matrices with non-zero determinant equipped with matrix multiplication and the subspace topology inherited from $\mathbb{R}^{n^2}$. The fact that $\mathbb{R}\backslash\{0\}$ is open, and the determinant is continuous means that $GL_n(\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ and so is a smooth manifold, which means that $GL_n(\mathbb{R})$ a Lie group. The properties of the determinant under matrix multiplication and inversion mean that if we restrict our attention to all matrices in $GL_n(\mathbb{R})$ with determinant 1 we get a subgroup $SL_n(\mathbb{R})$, called the special linear group. The regular level set theorem [14, Theorem 9.9] shows that $SL_n(\mathbb{R})$ is a regular submanifold of $GL_n(\mathbb{R})$ of dimension $n^2 - 1$. It is also possible to prove that the multiplication and inverse maps restricted to $SL_n(\mathbb{R})$ are smooth, meaning that $SL_n(\mathbb{R})$ is a Lie group in its own right (the interested reader is referred to [14, Example 15.5] for details.) In two dimensions we can also define the projective special linear group over $\mathbb{R}$ as the quotient group $PSL_n(\mathbb{R}) := SL_2(\mathbb{R})/\{-I, I\}$.

Thurston Geometries

With the definitions behind us, we can now introduce the 8 model structures as promised, which are commonly referred to as the 8 Thurston geometries.

(1) Euclidean space $\mathbb{E}^3$. This is the manifold $\mathbb{R}^3$ equipped with the Euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2$$

(2) The 3 sphere S^3, which is the unit sphere in $\mathbb{R}^4$ equipped with the restricted Euclidean metric from $\mathbb{R}^4$.

(3) Hyperbolic 3 space $\mathbb{H}^3$. This can be thought of as the open upper half plane in $\mathbb{R}^3$ equipped with the metric $ds^2 = \frac{1}{z^2}dx^2 + dy^2 + dz^2$.

(4) $S^2 \times \mathbb{E}$

(5) $\mathbb{H}^2 \times \mathbb{E}$

(6) The Lie group Solv, which is the set $\mathbb{R}^3$ equipped with a multiplication operator $\star$ defined by

$$(x_1, y_1, z_1) \star (x_2, y_2, z_2) = (x_1 + e^{-z}x_2, y_1 + e^z y_2, z_1 + z_2).$$

This manifold is equipped with the left invariant metric $ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$.

(7) The Lie group Nil, consisting of all matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

We can identify Nil with $\mathbb{R}^3$ in the natural way, with multiplication in $\mathbb{R}^3$ defined via the corresponding matrix multiplication, which induces the metric

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2$$

on Nil.

(8) The universal covering of $SL_2(\mathbb{R})$, denoted by $\widetilde{SL_2(\mathbb{R})}$, which is also a Lie group as the universal cover of any Lie group is again a Lie group. Unfortunately there is no easy explicit formula for the metric we want on $\widetilde{SL_2(\mathbb{R})}$, so we need to do a bit of work. We first note that if M is a Riemannian manifold, then there is a natural Riemannian metric ~~on TM~~. To define what we mean by a natural metric here would take us too far afield unfortunately, but it suffices to say that there is an induced metric on the tangent bundle, which agrees in some sense with the Riemannian metric on M. The interested reader is directed to [4] for a detailed discussion about natural metrics.

on the manifold TM.

Thus we have a natural metric on the tangent bundle of the hyperbolic plane $T\mathbb{H}^2$, which induces a metric, g say, on the unit tangent bundle

$$U\mathbb{H}^2 := \coprod_{p \in M} \{v \in T_p\mathbb{H}^2 : g_p(v, v) = 1\}$$

via restriction. As $PSL_2(\mathbb{R})$ can be thought of as the orientation preserving isometry group of $\mathbb{H}^2$ [13] we can identify it diffeomorphically with $U\mathbb{H}^2$ as follows: We map I to $(i, i) \in U\mathbb{H}^2$, and map any möbius transformation $f(z) = \frac{az+b}{cz+d} \in PSL_2(\mathbb{R})$ to $(f(i), f'(i))$. Proving that this is in fact a diffeomorphism would take us too far afield. Now the projection $\pi : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$ is a double covering map, meaning that the universal cover of $PSL_2(\mathbb{R})$ is $\widetilde{SL_2(\mathbb{R})}$ (as composition of covering maps $f \circ g$ is itself a covering map if $g^{-1}(b_0)$ is finite. See [11, Exercise 54.4]). Thus we finally arrive at the Riemannian metric on $\widetilde{SL_2(\mathbb{R})}$, which is the pullback of the Riemannian metric on $U\mathbb{H}^2$ by the universal covering map $p : \widetilde{SL_2(\mathbb{R})} \to PSL_2(\mathbb{R})$.

Knowing what these model geometries are is just the beginning of the story. In light of Thurston's Geometrization theorem we know that all compact 3-manifolds can be decomposed into components, which are manifolds themselves, that can each be given a locally homogeneous Riemannian metric isometric to one of the model geometries listed above. An interesting question, and one that is not trivial to answer, is asking whether there exists some sort of classification of all the different 3-manifolds which serve as the components in the above decomposition (besides the incompressible tori and Klein-bottles). Colloquially speaking, we are interested in understanding the building blocks of manifolds. To better ask this question we need to reformulate what we mean by a 3-manifold being modeled on a Thurston geometry.

Lemma 3.3 ([2, pg. 4]). *If M is a Riemannian Manifold, with universal cover X, which admits a complete locally homogeneous Riemannian metric then M is isometric to X/Γ, where Γ is the deck group (definition 2) of X.*

In light of this lemma we can unambiguously give another definition:

Definition 10. A 3 manifold M has a **geometric structure** modeled on X if M is isometric to X/Γ where Γ is a subgroup of the group of isometries from X to itself, $\mathrm{Isom}(X)$.

This definition gives a concrete approach to classifying the components forming the building blocks of all manifolds, the atoms so to speak (note that this is not standard terminology, but due to the nature of this essay perfect rigour is unfortunately not possible). We know that a 3-manifold serves as such an atom precisely when is modeled on one of the Thurston geometries, which happens precisely when it is, up to isometric isomorphism, of the form X/Γ. Thus we will be able to classify all of these atomic 3-manifolds if we can find all subgroups Γ of $\mathrm{Isom}(X)$ such that X/Γ is a manifold. To this end we introduce the following concepts:

Definition 11. We say that a group Γ acts **properly discontinuously** on a topological space X if for every compact set $K \subset X$ we have that $\{\gamma \in \Gamma : \gamma K \cap K \neq \emptyset\}$ is finite.

on

We quickly recall a definition from group theory. If $G \times X \to X$ is a group action a space X, we define the stabilizer of a point $x \in X$ to be $\mathrm{Stab}(x) := \{g \in G : gx = x\}$.

Definition 12. If G is a group acting on a topological space X, we say that G acts **freely** on X if the stabilizer of every point is trivial.

We are now in a position to state a basic theorem allowing us to find some atomic 3-manifolds.

Theorem 3.4 ([5, pg. 549]). *If X is a smooth manifold and Γ is a countably infinite group with the discrete topology that acts freely and properly discontinuously on X, then X/Γ is a topological manifold with a unique smooth structure such that $\pi : X \to X/\Gamma$ is a smooth and regular covering map.*

Thus by identifying the subgroups of $\mathrm{Isom}(X)$ acting freely and properly discontinously on X for each model geometry X will identify atomic 3-manifolds. Unfortunately such an analysis of each $\mathrm{Isom}(X)$ requires a significant amount of work, and there is still the question of whether the subgroups acting freely and properly discontinuously are the only ones that need to be considered. Hence identifying all 3-manifolds modeled on the Thurston geometries is out of the scope of this essay, but we refer the interested reader to Peter Scott's extensive survey of the topic [13].

Having said that, it is still interesting to note that there are only seven 3-manifolds, up to isometry, modeled on $S^2 \times \mathbb{R}$, whereas there are infinitely many geometries modeled on $\mathbb{H}^3$, which is a far less intuitive geometry [13].

\vspace which which?

3.3. Proof Of The Geometrization Conjecture

The proof of the geometrization conjecture relies on the Ricci flow, which tries to deform the metric in a way that is similar to heat diffusion. This results in a 'smoother' smooth manifold and this evolves the geomery of the manifold into one of the eight Thurston geometries.

3.3.1. *Flows On Riemannian Manifolds.*

Definition 13. A flow on a Riemannian manifold (M, g) is a smooth mapping $\varphi : \mathbb{R} \to (M, g(t))$. Then, since g takes values as sections in a vector bundle, we can define $\partial_t g(t)$ as

$$\partial_t g(t) = \lim_{dt \to 0} \frac{g(t + dt) - g(t)}{dt}$$?

We can look at flows for which $(\partial_t g(t))$ satisfies certain equations, an example of this is the Ricci flow. A

Definition 14. Given a Riemannian manifold (M, g), with Ricci-tensor R_{ij}, the Ricci flow is defined as the equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

And the normalized Ricci flow is defined by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2}{n} R_{avg} g_{ij}$$

Where $R_{avg} = Tr(R_{ij})$

The normalized Ricci flow is defined for compact manifolds, for which it preserves the volume. ?

Theorem 2 (Short time existence). Assume (M, g) is a closed Riemannian manifold. There is a unique $\tau > 0$ and a unique Ricci flow $(g_t)_{t\in[0,\tau)}$ such that $g_0 = g$.

For the proof, see for example [1, page 7]. This theorem was first proven by Hamilton, who was also the first person to write down the Ricci flow equation. So at least for small times the metric can be evolved by the Ricci flow. Recently Perelman showed that this can be used as a tool to proof the geometrization conjecture, in 3 papers he published.

Relation between the Geometrization and Poincaré Conjectures.

Grigori Perelman became publicly known (as publicly known as any mathematician can be) for rejecting the Millenium Prize, offered to him by the Clay Math Institute, worth one million dollars. As this prize was offered to any one who resolved the Poincaré conjecture one way or another, it is a fairly common misconception, even amongst mathematics students, that Grigori Perelman focused his energies on directly proving the Poincaré conjecture. In fact Perelman was more focused on understanding how 3-manifolds evolved under Ricci flow with surgery. The understanding that he gained in fact proved the Thurston Geometrization conjecture, and as a sidelight proved the Poincaré conjecture, which is in fact, quite amazingly, just a rather simple corollary of the Geometrization theorem (conjecture no longer) as we shall now see [9, Pg. 27].

Theorem 3.5. *The Geometrization theorem implies the Poincaré conjecture.*

Proof. Let us consider a compact, simply connected 3 manifold M. As it has a trivial fundamental group, the decomposition referred to in Conjecture 3.1 must be trivial (because Tori and Klein bottles have non trivial fundamental groups, so there can be no such incompressible embedded surfaces.) Thus M must have a complete locally homogeneous metric. Definition 11 thus implies that $M \simeq X/\Gamma$, where X is one of the 8 Thurston geometries, and Γ a group of isometries of X acting freely and properly discontinuously on X. Thus X/Γ is also simply connected. The quotient map $\pi : X \to X/\Gamma$ is a regular covering map, because the action of Γ is properly discontinuous, and Γ is in fact the deck group of X [11, Thm 81.5]. As each of the model geometries is simply connected, so too is X/Γ, which means X is homeomorphic to X/Γ by Lemma 1.1. Hence M is homeomorphic to X, and as M is compact so too is X, but S^3 is the only compact model geometry. □

References

[1] Richard Bamler. Lecture notes. 2014.
[2] Noella Grady. The eight geometries of the geometrization conjecture. Master's thesis, Oregon State University, 2011.
[3] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[4] Elias Kappos. Natural metrics on tangent bundles. Master's thesis, Lund University, 2001.
[5] John M. Lee. *Introduction to Smooth Manifolds.* Springer, 2nd edition, 2003.
[6] J. Milnor. A unique decomposition theorem for 3-manifolds. *American Journal of Mathematics*, 84(1):1–7, 1962.
[7] J. Milnor. Towards the poincaré conjecture and the classification of 3-manifolds. *Notices of the AMS*, 50(10):1226–1233, 2003.
[8] John W. Milnor. On manifolds homeomorphic to the 7-sphere. *Annals of Mathematics*, 64:399–405, 1956.
[9] J. Morgan. 100 years of topology: Work stimulated by poincaré's approach to classifying manifolds. *Clay Mathematics Proceedings*, 19:7–30, 2014.
[10] John W. Morgan. Recent progress of the poincaré conjecture and the classification of 3 manifolds. *Bulletin of the American Mathematical Society*, 42, October 2004.
[11] James R. Munkres. *Topology.* Pearson Education, 2nd edition, 2000.
[12] Henri Poincaré. Cinquième complément à l'analysis situs. 1904.
[13] Peter Scott. The geometries of 3-manifolds. *Bulletin of the London Mathematical Society*, 15, September 1983.
[14] Loring W. Tu. *Introduction to Manifolds.* Springer, 2nd edition, 2011.
[15] Loring W. Tu. *Differential Geometry.* Springer, 1st edition, 2017.